2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Average Dwell-Time Method to L_2-Gain Analysis and Control Synthesis for Uncertain Switched Nonlinear Systems

Min Wang, Georgi M. Dimirovski and Jun Zhao

***Abstract*— This paper addresses the L_2-gain analysis and control synthesis problem for a class of switched nonlinear systems affected by both time-varying uncertainties and external disturbances. Firstly, the L_2-gain for the autonomous switched disturbed uncertain system is analyzed. Then, a switched state feedback control law is designed and the L_2 -gain is analyzed for the disturbed uncertain non-autonomous switched system. Sufficient conditions for these two cases are obtained using average dwell-time method incorporated with piecewise Lyapunov functions. The corresponding closed-loop disturbed uncertain switched system and the disturbed uncertain autonomous switched system are globally exponentially stable with a weighted L_2-gain under the sufficient conditions. Both the piecewise Lyapunov functions and the average dwell-time based switching laws are constructed based on the structural characteristics of the uncertain switched system.**

I. INTRODUCTION

Switched systems are consist of a family of continuous-time and/or discrete-time processes interacting with a logical or decision-making process. Analysis and synthesis of this kind of system have attracted lots of attention in recent years. Interests that focus on switched systems are mainly: stability analysis [1-5], stabilization [6-8], controllability [9], observability [10], switching optimal control [11, 12], H_∞ control [13-15], L_2-gain analysis [16-18] and so on. Stability is of great importance in the analysis of switched systems, and lots of researches are devoted to the study of this property. Among these researches the common Lyapunov function technique was introduce to check the uniformly stability or stabilizability of switched systems [1-3, 8]. But a common Lyapunov function may not exist or is too different to find. In this case, the multiple Lyapunov functions technique [4, 6] and the average dwell-time technique [5] were generally proposed to analyze the stability property for the switched systems under some designed switching laws for the purpose of more flexibility in choosing a Lyapunov function.

On the other hand, switched systems with disturbances are commonly found in practice. Thus, the stability and L_2-gain analysis problem for the disturbed switched system becomes an interesting issue due to its value both in practical and in theoretical practice. But researches studying this problem are relatively few. [16] addressed the L_2-gain analysis and control synthesis problem with Linear matrices inequality method for a class of discrete-time disturbed uncertain switched linear systems. [17] investigated the disturbance attenuation problem for a class of disturbed autonomous switched linear system using the average dwell-time method, and a weighted L_2-gain is achieved. The L_2-gain analysis problem for a class of disturbed switched delay linear system was addressed in [18]. All the papers mentioned above are mainly about switched linear systems. But as we all know that nonlinear phenomenon exists in almost all the dynamics in practice, so analyzing the L_2-gain property for switched nonlinear disturbed systems deserves to be paid more attention.

In this paper, we investigate the L_2-gain analysis and control synthesis problem for a class of disturbed uncertain switched nonlinear systems using average dwell-time method incorporated with piecewise Lyapunov functions. The switched system under consideration is composed of a nonlinear part and a linear part. This research will address the L_2-gain analysis and control synthesis problem for this disturbed uncertain switched system in the case that both the linear and the nonlinear parts are stabilizable under some average dwell-time based switching laws. The average dwell-time for the switched system that the designed switching laws satisfied is designed recursively, and the relationship between the average dwell-time for the switched system and the average dwell-times for the linear and the nonlinear parts of the switched system is analyzed. Firstly, the L_2-gain is analyzed for the autonomous switched system. Then, the switched state feedback is synthesized for the non-autonomous switched system. Sufficient conditions are expressed in the form of Linear matrices inequalities under which both the autonomous switched system and the non-autonomous switched system are globally exponentially stable and have a weighted L_2-gain. Moreover, the state decay are calculated explicitly.

II. PROBLEM STATEMENT AND PRELIMINARIES

In this paper, we study the L_2-gain analysis and control synthesis problem for the following uncertain switched nonlinear system:

$$\begin{cases} \dot{x}_1(t) = \hat{A}_{1\sigma(t)}x_1(t) + A_{2\sigma(t)}x_2(t) + \hat{B}_{\sigma(t)}u_{\sigma(t)}(t) \\ \qquad\qquad + G_{\sigma(t)}w(t), \\ \dot{x}_2(t) = f_{2\sigma(t)}(x_2(t)), \\ \;\; y(t) = C_{\sigma(t)}x_1(t), \end{cases} \tag{1}$$

This work was supported by Dogus University Fund for Science and the NSF of China under grants 60574013

Min Wang and Jun Zhao are with the Key Laboratory of Integrated Automation of Process Industry, Ministry of Education, Northeastern University, Shenyang, P. R. China, and Jun Zhao is also with the Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia. W_min3@126.com, zhaojun@ise.neu.edu.cn

M. Dimirovski is with the Department of Computer Engineering, Dogus University, Kadikoy, TR-34722, Istanbul, Turkey gdimirovski@dogus.edu.tr

where $x_1(t) \in R^{n-d}$, $x_2(t) \in R^d$ are the states, $u(t) \in R^m$ is the control input, $w(t) \in L_2[0,\infty)$ is the external disturbance input, and $y(t) \in R^p$ is the controlled output. $\sigma(t) : [0,\infty] \to I_N = \{1,\ldots,N\}$ is the switching signal, which is a piecewise constant function of time and will be determined later. $\sigma(t) = i$ means that the ith subsystem is activated. $\hat{A}_{1i} = A_{1i} + \triangle A_{1i}$, $\hat{B}_i = B_i + \triangle B_i(t)$, A_{1i}, A_{2i}, B_i, G_i and $C_i (i \in I_N)$ are constant matrices of appropriate dimensions which describe the nominal systems. $f_{2i}(x_2(t))$ are smooth vector fields with $f_{2i}(0) = 0$. $\triangle A_{1i}(t)$ and $\triangle B_i(t)$ are uncertain time-varying matrices denoting the uncertainties in the system matrices and having the following form

$$[\triangle A_{1i}(t), \triangle B_i(t)] = E_i \Gamma(t) [F_{1i}, F_{2i}], \qquad i \in I_N. \tag{2}$$

where $E_i \in R^{(n-d)\times l}$, $F_{1i} \in R^{k\times(n-d)}$, and $F_{2i} \in R^{k\times m}$ are given constant matrices which characterize the structure of uncertainty, and F_{2i} is of full column rank. Γ is the norm-bounded time-varying uncertainty, i.e.

$$\Gamma = \Gamma(t) \in \{\Gamma(t) : \Gamma(t)^T \Gamma(t) = I, \Gamma(t) \in R^{l\times k}, \text{ the elements of } \Gamma(t) \text{ are Lebesgue measurable}\}.$$

There are several reasons for assuming that the system uncertainties have the structures given in (2), see [14] for details.

We are interested in L_2-gain analysis and control synthesis of uncertain switched nonlinear systems under some average dwell-time based switching law. This analysis is to establish sufficient conditions such that the switched system

$$\begin{cases} \dot{x}_1(t) = \hat{A}_{1\sigma(t)} x_1(t) + A_{2\sigma(t)} x_2(t) + G_{\sigma(t)} w(t), \\ \dot{x}_2(t) = f_{2\sigma(t)}(x_2(t)), \\ \ y(t) = C_{\sigma(t)} x_1(t), \end{cases} \tag{3}$$

is globally exponentially stable with a weighted L_2-gain (see Definition 3 below), whereas control synthesis is to design a switched state feedback control law

$$u_{\sigma(t)} = K_{\sigma(t)} x_1(t), \tag{4}$$

such that the corresponding closed-loop switched system (1) is globally exponentially stable with an weighted L_2-gain.

Consider the linear switched system described by equations of the form

$$\dot{x}(t) = A_{\sigma(t)} x(t). \tag{5}$$

where $x(t) \in R^n$ is the state, $\sigma(t) : [t_0, \infty] \to I_N = \{1,\ldots,N\}$ is the switching signal, which is a piecewise constant function of time. $A_i (i \in I_N)$ are constant matrices of appropriate dimension describing the subsystems, and $N > 1$ is the number of subsystems.

Definition 1. The switched system (5) is said to be globally exponentially stable with stability degree $\lambda > 0$ if $\|x(t)\| \le e^{\alpha - \lambda(t-t_0)}$ holds for all $t \ge t_0$ and a constant α.

Definition 2. System (1) is said to be globally exponentially stabilizable via switching if there exist a switching signal $\sigma(t)$ and an associate switched state feedback $u_{\sigma(t)}(t) = K_{\sigma(t)} x_1(t)$ such that the corresponding closed-loop system (1) with $w(t) \equiv 0$ is globally exponentially stable for all admissible uncertainties.

Consider the switched system

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} w(t), \\ y(t) = C_{\sigma(t)} x(t), \end{cases} \tag{6}$$

where $x(t) \in R^n$, $w(t)$, $y(t)$, $\sigma(t)$ are the same as stated in (1), A_i, B_i, C_i $(1 \le i \le N)$ are known constant matrices.

Definition 3. System (6) is said to have a $e^{-\lambda t}$-weighted L_2-gain over $\sigma(t)$, from the disturbance input $w(t)$ to the controlled output $y(t)$, if the following inequality holds for $\sigma(t)$ and some real-valued function $\beta(t)$ with $\beta(0) = 0$

$$\int_0^\infty e^{-\lambda t} y^T(t) y(t) dt \le \gamma^2 \int_0^\infty w^T(t) w(t) dt + \beta(x(0)), \tag{7}$$

along the solution to (6), where $w(t) \in L_2[0,+\infty)$, $x(0) \ne 0$ is the initial state.

The following lemmas will be used in the development of the main results.

Lemma 1 [1]. Consider the nonlinear switched system

$$\dot{x}(t) = f_i(x(t)), \qquad i \in I_N = \{1,\ldots,N\}. \tag{8}$$

Assume for each $i \in I_N$ there exists a Lyapunov function V_i such that

$$a_i \|x\|^2 \le V_i(x) \le b_i \|x\|^2, \tag{9}$$

and

$$\frac{\partial V_i(x)}{\partial x} f_i(x) \le -c\|x\|^2. \tag{10}$$

for some positive constants a_i, b_i, and c. Then, two positive constant μ, λ can be found such that the switched nonlinear system (8) is globally exponentially stable for any switching signal that has the average dwell time property with $\tau_a \ge \ln \mu / \lambda$.

Remark 1. It is not difficult to find from [1] that $\mu = \sup\{\frac{b_p}{a_q} : p, q \in I_N\}$, $\lambda \in [0, \lambda_0)$ and $\lambda_0 = \min\{\frac{c}{b_i} : i \in I_N\}$ in Lemma 1.

Lemma 2 [19]. Given a symmetric matrix G, and any nonzero matrices M, N of appropriate dimensions. Then

$$G + M\Gamma N + N^T \Gamma^T M^T \le 0$$

for all Γ satisfying $\Gamma^T \Gamma \le I$ if and only if there exists a constant $\varepsilon > 0$ such that

$$G + \varepsilon M M^T + \frac{1}{\varepsilon} N^T N^T \le 0.$$

III. L_2-GAIN ANALYSIS

This section gives the L_2-gain analysis for the uncertain switched nonlinear system (3).

Theorem 1 Given any constant $\gamma > 0$, suppose that switched system (3) satisfies the following conditions

(i) if there exist constants $\varepsilon_i > 0$, $\lambda_0 > 0$, $\mu \ge 1$, such that the following inequalities

$$\begin{aligned} A_{1i}^T P_i + P_i A_{1i} + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i G_i G_i^T P_i \\ + \varepsilon_i^2 F_{1i}^T F_{1i} + C_i^T C_i + \lambda_0 P_i + I < 0, \end{aligned} \tag{11}$$

$$P_i \le \mu P_j, \quad i,j = 1,\ldots,N. \tag{12}$$

have positive definite solutions P_i.
(ii) there exist proper, positive definite, and radially unbounded function $W_i(x_2(t))$ such that

$$\frac{dW_i(x_2)}{dx_2} f_{2i}(x_2(t)) \le -\beta_i \|x_2\|^2, \tag{13}$$

$$a_{1i}\|x_2\|^2 \le \|W_i(x_2)\| \le a_{2i}\|x_2\|^2. \tag{14}$$

for some constants $\beta_i > 0$, $a_{1i} > 0$, $a_{2i} > 0$, $i = 1,\ldots,N$.

Then, switched system (3) is globally exponentially stable when $w(t) = 0$ and achieves a weighted L_2-gain which is less than or equal to γ under arbitrary switching laws satisfying the average dwell time

$$\tau_a \ge \tau_a^* = \frac{\ln \hat{\mu}}{\lambda}, \tag{15}$$

where $\hat{\mu} = \max\{\mu, \frac{a_{2i}}{a_{1j}} : i,j \in I_N\}$, $\lambda \in [0, \hat{\lambda}_0)$, $\hat{\lambda}_0 = \min\{\lambda_0, \frac{\beta_i}{a_{2i}} : i \in I_N\}$.

Proof. Define the following piecewise Lyapunov function candidate for switched system (3)

$$V(x_1, x_2) = x_1^T P_{\sigma(t)} x_1 + k W_{\sigma(t)}(x_2). \tag{16}$$

where P_i are the solutions of (11) and (12).
Then, based on Lemma 2, when $\sigma(t) = i$, the time derivative of $V(x_1, x_2)$ along the trajectory of the switched system (3) is

$$\begin{aligned}
\dot{V} &= x_1^T(\hat{A}_{1i}^T P_i + \hat{A}_{1i} P_i)x_1 + 2x_1^T P_i A_{2i} x_2 + 2x_1^T P_i G_i w \\
&\quad + k\frac{dW_i(x_2)}{dx_2} f_{2i}(x_2) \\
&= x_1^T(A_{1i}^T P_i + A_{1i} P_i)x_1 + 2x_1^T P_i \Delta A_{1i} x_1 \\
&\quad + 2x_1^T P_i A_{2i} x_2 + 2x_1^T P_i G_i w + k\frac{dW_i(x_2)}{dx_2} f_{2i}(x_2) \\
&\le x_1^T(A_{1i}^T P_i + A_{1i} P_i)x_1 + 2x_1^T P_i E_i \Gamma F_{1i} x_1 \\
&\quad + 2x_1^T P_i A_{2i} x_2 + 2x_1^T P_i G_i w - k\beta_i \|x_2\|^2 \\
&\le x_1^T(A_{1i}^T P_i + A_{1i} P_i)x_1 + \varepsilon_i^{-2} x_1^T P_i E_i E_i^T P_i x_1 \\
&\quad + \varepsilon_i^2 x_1^T F_{1i}^T F_{1i} x_1 + 2x_1^T P_i A_{2i} x_2 + 2x_1^T P_i G_i w \\
&\quad - k\beta_i \|x_2\|^2 \\
&= x_1^T(A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i})x_1 \\
&\quad + 2x_1^T P_i A_{2i} x_2 + 2x_1^T P_i G_i w - k\beta_i \|x_2\|^2.
\end{aligned}$$

It is easy to see that there exist constants $l_i > 0$, $q_i > 0$, $i \in I_N$ such that

$$\|A_{2i} x_2\| \le l_i \|x_2\|, \qquad \|x_1^T P_i\| \le q_i \|x_1\|.$$

Let $l = \max\{l_i q_i : i \in I_N\}$, $b = \min\{\frac{\beta_i}{a_{2i}} : i \in I_N\}$, from (11) and (14), we can obtain

$$\begin{aligned}
&\dot{V} + y^T y - \gamma^2 w^T w \\
&\le x_1^T(A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i})x_1 \\
&\quad + 2l\|x_1\|\|x_2\| - k\beta_i\|x_2\|^2 + 2x_1^T P_i G_i w + x_1^T C_i^T C_i x_1 \\
&\quad - \gamma^2 w^T w \\
&\le x_1^T(A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i G_i G_i^T P_i \\
&\quad + \varepsilon_i^2 F_{1i}^T F_{1i} + C_i^T C_i)x_1 + 2l\|x_1\|\|x_2\| - k\beta_i\|x_2\|^2 \\
&\quad - (\gamma^{-1} G_i^T P_i x_1 - \gamma w)^T(\gamma^{-1} G_i^T P_i x_1 - \gamma w) \\
&\le -\lambda_0 x_1^T P_i x_1 - kbW_i(x_2) + kbW_i(x_2) - x_1^T x_1 \\
&\quad + 2l\|x_1\|\|x_2\| - k\beta_i\|x_2\|^2 \\
&\le -\hat{\lambda}_0 V + ka_{2i} b\|x_2\|^2 - x_1^T x_1 + 2l\|x_1\|\|x_2\| \\
&\quad - k\beta_i\|x_2\|^2 \\
&\le -\hat{\lambda}_0 V - (k\beta_i - ka_{2i} b - l^2)\|x_2\|^2.
\end{aligned}$$

where $\hat{\lambda}_0 = \min\{\lambda_0, b\} = \min\{\lambda_0, \frac{\beta_i}{a_{2i}} : i \in I_N\}$.
Choose $k \ge \max\{\frac{l^2}{\beta_i - a_{2i} b} : i \in I_N\}$, we have

$$\dot{V} + y^T y - \gamma^2 w^T w \le -\hat{\lambda}_0 V. \tag{17}$$

When $w(t) = 0$, from the above inequality, we obtain

$$\dot{V} \le -\hat{\lambda}_0 V. \tag{18}$$

Moreover, from (12), (14)and (16), it is easy to get

$$V_i(t) \le \hat{\mu} V_j(t), \qquad i,j \in I_N. \tag{19}$$

where $\hat{\mu} = \max\{\mu, \frac{a_{2i}}{a_{1j}} | i,j \in I_N\}$.

For arbitrary $t > 0$, denote $t_0 \le t_1 \le t_2 \le \ldots \le t_k \ldots \le t_{N_\sigma(0,t)}$ as the switching instants of $\sigma(t)$ over the interval $(0,t)$. Then, from (18) and (19), we have

$$\begin{aligned}
V(t) &\le e^{-\hat{\lambda}_0(t - t_{N_\sigma(0,t)})} V(t_{N_\sigma(0,t)}) \le \hat{\mu} e^{-\hat{\lambda}_0(t - t_{N_\sigma(0,t)})} \\
&\qquad V(t^-_{N_\sigma(0,t)}) \le \hat{\mu} e^{-\hat{\lambda}_0(t - t_{N_\sigma(0,t)-1})} V(t_{N_\sigma(0,t)-1}) \\
&\le \ldots \le \hat{\mu}^{N_\sigma(0,t)} e^{-\hat{\lambda}_0 t} V(0) = e^{N_\sigma(0,t)\ln\hat{\mu} - \hat{\lambda}_0 t} V(0).
\end{aligned}$$

Furthermore, based on $N_\sigma(0,\tau) \le \frac{\tau}{\tau_a^*}$, for $\forall \tau > 0$, (15) implies that

$$N_\sigma(0,\tau)\ln\hat{\mu} \le \lambda\tau, \tag{20}$$

Thus,

$$V(t) \le e^{-(\hat{\lambda}_0 - \lambda)t} V(0). \tag{21}$$

From (14), we know that there exist constants $\lambda_1 > 0$, $\lambda_2 > 0$, $a_1 > 0$, such that

$$\lambda_1\|x_1\|^2 + a_1\|x_2\|^2 \le V(t) \le \lambda_2\|x_1\|^2 + a_2\|x_2\|^2 \tag{22}$$

where $\lambda_1 = \min\{\lambda_{\min}(P_i) | i \in I_N\}$, $a_1 = \min\{a_{1i} | i \in I_N\}$, $\lambda_2 = \max\{\lambda_{\max}(P_i) | i \in I_N\}$.
Let $b_1 = \min\{\lambda_1, a_1\}$, $b_2 = \max\{\lambda_2, a_2\}$, we have

$$b_1(\|x_1\|^2 + \|x_2\|^2) \le V(t) \le b_2(\|x_1\|^2 + \|x_2\|^2) \tag{23}$$

In view of (21), (22) and (23), the following inequality follows

$$\|x(t)\| \le \sqrt{\frac{b_2}{b_1}} e^{-\frac{(\hat{\lambda}_0 - \lambda)}{2} t} \|x(0)\|. \tag{24}$$

Hence, the globally exponential stability of system (3) when $w(t) = 0$ follows.

Integrating both sides of (17), and from (19), we can get

$$\begin{aligned}
V(t) \leq & \; V(t_{N_{\sigma(0,t)}})e^{-\hat{\lambda}_0(t-t_{N_{\sigma(0,t)}})} - \int_{t_{N_{\sigma(0,t)}}}^{t} e^{-\hat{\lambda}_0(t-\tau)} \\
& \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\
\leq & \; \hat{\mu}V(t^-_{N_{\sigma(0,t)}})e^{-\hat{\lambda}_0(t-t_{N_{\sigma(0,t)}})} - \int_{t_{N_{\sigma(0,t)}}}^{t} e^{-\hat{\lambda}_0(t-\tau)} \\
& \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\
\leq & \; \hat{\mu}\big[V(t_{N_{\sigma(0,t)}-1})e^{-\hat{\lambda}_0(t_{N_{\sigma(0,t)}}-t_{N_{\sigma(0,t)}-1})} \\
& - \int_{t_{N_{\sigma(0,t)}-1}}^{t_{N_{\sigma(0,t)}}} e^{-\hat{\lambda}_0(t_{N_{\sigma(0,t)}}-\tau)}[y^T(\tau)y(\tau) \\
& -\gamma^2 w^T(\tau)w(\tau)]d\tau\big]e^{-\hat{\lambda}_0(t-t_{N_{\sigma(0,t)}})} - \int_{t_{N_{\sigma(0,t)}}}^{t} e^{-\hat{\lambda}_0(t-\tau)} \\
& \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\
& \vdots \\
\leq & \; \hat{\mu}^{N_\sigma(0,t)}e^{-\hat{\lambda}_0 t}V(0) - \hat{\mu}^{N_\sigma(0,t)}\int_0^{t_1} e^{-\hat{\lambda}_0(t-\tau)} \\
& \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau - \hat{\mu}^{N_\sigma(0,t)-1} \\
& \cdot \int_{t_1}^{t_2} e^{-\hat{\lambda}_0(t-\tau)}[y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau - \ldots \\
& -\hat{\mu}^0 \int_{t_{N_{\sigma(0,t)}}}^{t} e^{-\hat{\lambda}_0(t-\tau)}[y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\
= & \; e^{-\hat{\lambda}_0 t + N_\sigma(0,t)\ln\hat{\mu}}V(0) - \int_0^t e^{-\hat{\lambda}_0(t-\tau)+N_\sigma(\tau,t)\ln\hat{\mu}} \\
& \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau.
\end{aligned}$$

Multiplying both sides of the above inequality by $e^{-N_\sigma(0,t)\ln\hat{\mu}}$, leads to

$$\begin{aligned}
e^{-N_\sigma(0,t)\ln\hat{\mu}}V(t) \leq & \; e^{-\hat{\lambda}_0 t}V(0) - \int_0^t e^{-\hat{\lambda}_0(t-\tau)-N_\sigma(0,\tau)\ln\hat{\mu}} \\
& [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau. \quad (25)
\end{aligned}$$

Thus, the following inequality follows from (20)

$$\begin{aligned}
& \int_0^t e^{-\hat{\lambda}_0(t-\tau)-\lambda\tau}y^T(\tau)y(\tau)d\tau \\
\leq & \quad e^{-\hat{\lambda}_0 t}V(0) + \gamma^2\int_0^t e^{-\hat{\lambda}_0(t-\tau)}w^T(\tau)w(\tau)d\tau. \quad (26)
\end{aligned}$$

Integrating both sides of the foregoing inequality from $t=0$ to ∞ and rearranging the double-integral area, we obtain

$$\int_0^\infty e^{-\lambda\tau}y^T(\tau)y(\tau)d\tau \leq \gamma^2\int_0^\infty w^T(\tau)w(\tau)d\tau + V(0). \quad (27)$$

From Definition 3, we know that system (3) has an weighted L_2-gain.

Remark 2. Applying Shur complement formula, the first matrix inequality of condition (i) can be easily transformed into the LIMs form. The second inequality of condition (i) is trivial, as long as we let $\mu = \sup_{i,j\in I_N}\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}$.

Remark 3. When $x_2(t)=0$, it is easy to verify that the x_1-subsystem of system (3) is exponentially stable with an weighted L_2-gain under arbitrary switching laws which satisfy the average dwell-time $\tau_1 \geq \tau^*_{a1} = \frac{\ln\mu_1}{\lambda_1}$, $\lambda_1 \in [0,\lambda_0)$, μ_1 is equivalent to μ in (12). And from Lemma 1, we know that the x_2-subsystem of system (3) is exponentially stable under arbitrary switching laws which satisfy the average dwell-time $\tau_2 \geq \tau^*_{a2} = \frac{\ln\mu_2}{\lambda_2}$, $\mu_2 = \max\{\frac{a_{2i}}{a_{1j}} : i,j \in I_N\}$, $\lambda_2 \in [0,\tilde{\lambda}_0)$, and $\tilde{\lambda}_0 = \min\{\frac{\beta_i}{a_{2i}} : i \in I_N\}$. From (15), we know that the average dwell-time for the whole cascade switched system satisfies $\tau_a \geq \tau^*_a \geq \max\{\tau^*_{a1},\tau^*_{a2}\}$. Consequently, the two parts of the switched system are all globally exponentially stable under arbitrary switching laws that satisfy the average dwell time τ_a. Based on the cascade system theory introduced in [20], we know that the whole switched system is exponentially stable under arbitrary switching laws that satisfy the average dwell time τ_a.

IV. CONTROL SYNTHESIS

In this section, we design an switched state feedback controller, such that the corresponding closed-loop system (1) is globally exponentially stable with an weighted L_2-gain under some switching laws satisfying an average dwell-time. By Theorem 1 this problem reduces to finding $u_{\sigma(t)} = K_{\sigma(t)}x_1$, such that

$$\begin{cases}
\dot{x}_1(t) = (\hat{A}_{1\sigma(t)} + \hat{B}_{\sigma(t)}K_{\sigma(t)})x_1(t) + A_{2\sigma(t)}x_2(t) \\
\qquad\quad + G_{\sigma(t)}w(t), \\
\dot{x}_2(t) = f_{2\sigma(t)}(x_2(t)), \\
\;\; y(t) = C_{\sigma(t)}x_1(t),
\end{cases} \quad (28)$$

is globally exponentially stable with an weighted L_2-gain.

Theorem 2 Given any constant $\gamma > 0$, suppose that switched system (1) satisfies the following conditions

(i) if there exist constants $\varepsilon_i > 0$, $\lambda_0 > 0$, $\mu \geq 1$, such that the following inequalities

$$\begin{aligned}
& A^T_{1i}P_i + P_iA_{1i} + \varepsilon_i^{-2}P_iE_iE_i^TP_i + \gamma^{-2}P_iG_iG_i^TP_i \\
& \quad +\varepsilon_i^2F_{1i}^TF_{1i} + C_i^TC_i + \lambda_0P_i + I - (\varepsilon_i^{-1}P_iB_i \\
& \quad +\varepsilon_iF_{1i}^TF_{2i})(F_{2i}^TF_{2i})^{-1}(\varepsilon_i^{-1}P_iB_i + \varepsilon_iF_{1i}^TF_{2i})^T < 0,
\end{aligned} \quad (29)$$

$$P_i \leq \mu P_j, \quad i,j = 1,\ldots,N. \quad (30)$$

have positive definite solutions P_i.

(ii) there exist proper, positive definite, and radially unbounded functions $W_i(x_2(t))$ such that

$$\frac{dW_i(x_2)}{dx_2}f_{2i}(x_2(t)) \leq -\beta_i\|x_2\|^2, \quad (31)$$

$$a_{1i}\|x_2\|^2 \leq \|W_i(x_2)\| \leq a_{2i}\|x_2\|^2. \quad (32)$$

for some constants $\beta_i > 0$, $a_{1i} > 0$, $a_{2i} > 0$, $i = 1,\ldots,N$.

Then, switched system (1) is globally exponentially stable with an weighted L_2-gain which is less than or equal to γ when $w(t) = 0$ with the switched state feedback

$$u_i = -(F_{2i}^TF_{2i})^{-1}(\varepsilon_i^{-2}B_i^TP_i + F_{2i}^TF_{1i})x_1(t). \quad (33)$$

under arbitrary switching laws satisfying the average dwell time

$$\tau_a \geq \tau_a^* = \frac{\ln \hat{\mu}}{\lambda}, \tag{34}$$

where $\hat{\mu} = \max\{\mu, \frac{a_{2i}}{a_{1j}} : i, j \in I_N\}$, $\lambda \in [0, \hat{\lambda}_0)$, $\hat{\lambda}_0 = \min\{\lambda_0, \frac{\beta_i}{a_{2i}} \mid i \in I_N\}$.

proof. For switched system (1), define the following piecewise Lyapunov function

$$V(x) = x_1^T P_{\sigma(t)} x_1 + \hat{k} W_{\sigma(t)}(x_2), \tag{35}$$

where P_i are the solutions of (29) and (30).

Then, based on Lemma 2, from (33), when the ith switched subsystem is activated, we can get

$$\begin{aligned}
\dot{V} =\ & x_1^T(A_{1i}^T P_i + A_{1i}P_i)x_1 + 2x_1^T P_i \Delta A_{1i} x_1 \\
& +2x_1^T P_i B_i u_i + 2x_1^T P_i \Delta B_i u_i + 2x_1^T P_i A_{2i} x_2 \\
& +\hat{k}\frac{dW_i(x_2)}{dx_2} f_{2i}(x_2) + 2x_1^T P_i G_i w \\
\leq\ & x_1^T(A_{1i}^T P_i + A_{1i}P_i)x_1 + 2x_1^T P_i E_i \Gamma(F_{1i}x_1 + F_{2i}u_i) \\
& +2x_1^T P_i B_i u_i + 2x_1^T P_i A_{2i} x_2 - \hat{k}\beta_i \|x_2\|^2 \\
& +2x_1^T P_i G_i w \\
\leq\ & x_1^T(A_{1i}^T P_i + A_{1i}P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i})x_1 \\
& +2x_1^T P_i A_{2i} x_2 - \hat{k}\beta_i \|x_2\|^2 + 2x_1^T P_i G_i w + \big[\varepsilon_i u_i \\
& +(F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-1} B_i^T P_i x_1 + \varepsilon_i F_{2i}^T F_{1i} x_1)\big]^T (F_{2i}^T F_{2i}) \\
& \big[\varepsilon_i u_i + (F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-1} B_i^T P_i x_1 + \varepsilon_i F_{2i}^T F_{1i} x_1)\big] \\
& -x_1^T \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-1} P_i B_i \\
& +\varepsilon_i F_{1i}^T F_{2i})^T\big] x_1, \\
\leq\ & x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\
& \quad -\big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-1} P_i B_i \\
& \quad +\varepsilon_i F_{1i}^T F_{2i})^T\big] \Big\} x_1 + 2x_1^T P_i A_{2i} x_2 - \hat{k}\beta_i \|x_2\|^2 \\
& \quad +2x_1^T P_i G_i w.
\end{aligned}$$

It is easy to know that there exist constants $m_i > 0$, $n_i > 0$, $i \in I_N$, such that

$$\|x_1^T P_i\| \leq m_i \|x_1\|, \qquad \|A_i x_1\| \leq n_i \|x_2\|.$$

Let $p = \max\{m_i n_i : i \in I_N\}$, and take (29) into consideration, it is easy to obtain that

$$\begin{aligned}
& \dot{V} + y^T y - \gamma^2 w^T w \\
\leq\ & x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\
& -\big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-1} P_i B_i \\
& +\varepsilon_i F_{1i}^T F_{2i})^T\big] \Big\} x_1 + 2p\|x_1\|\|x_2\| - \hat{k}\beta_i \|x_2\|^2 \\
& +2x_1^T P_i G_i w + x_1^T C_i^T C_i x_1 - \gamma^2 w^T w \\
\leq\ & x_1^T \Big\{ A_{1i}^T P_i + A_{1i} P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\
& +\gamma^{-2} P_i G_i G_i^T P_i + C_i^T C_i - \big[(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) \\
& (F_{2i}^T F_{2i})^{-1}(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T\big] \Big\} x_1 - \hat{k}\beta_i \|x_2\|^2 \\
& +2p\|x_1\|\|x_2\| \\
\leq\ & -\lambda_0 x_1^T P_i x_1 - \hat{k} b W(x_2) + \hat{k} b W(x_2) - x_1^T x_1 \\
& +2p\|x_1\|\|x_2\| - \hat{k}\beta_i \|x_2\|^2 \\
\leq\ & -\hat{\lambda}_0 V + \hat{k} a_{2i} b \|x_2\|^2 - x_1^T x_1 + 2p\|x_1\|\|x_2\| \\
& -\hat{k}\beta_i \|x_2\|^2 \\
\leq\ & -\hat{\lambda}_0 V - (\hat{k}\beta_i - \hat{k} a_{2i} b - p^2)\|x_2\|^2.
\end{aligned}$$

where $b = \min\{\frac{\beta_i}{a_{2i}} : i \in I_N\}$, $\hat{\lambda}_0 = \min\{\lambda_0, b\} = \min\{\lambda_0, \frac{\beta_i}{a_{2i}} : i \in I_N\}$.

Let $\hat{k} \geq \frac{p^2}{\beta_i - a_{2i} b : i \in I_N}$, we have

$$\dot{V} + y^T y - \gamma^2 w^T w \leq -\hat{\lambda}_0 V. \tag{36}$$

The remainder of the proof for exponential stability when $w(t) = 0$ and the weighted L_2-gain analysis for the closed-loop system (1), i.e. for switched system (28), is the same as that of Theorem 1.

V. EXAMPLE

Consider the switched system (3) with $I_N = \{1, 2\}$, $n - d = 2$, $d = 2$ and

$$A_{11} = \begin{bmatrix} -4 & 0 \\ 2 & 1 \end{bmatrix}, \ A_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ C_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad F_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, \ f_{21} = \begin{bmatrix} -x_3 \\ -3x_3 - 4.4x_4 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} -5 & -2 \\ 3 & -4 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad F_{12} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix},$$

$$F_{22} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \ f_{22} = \begin{bmatrix} -2x_3(1 + x_4^2) \\ -x_4 \end{bmatrix},$$

$$\Gamma(t) = \begin{bmatrix} \sin t & 0 \\ 0 & \cos t \end{bmatrix},$$

For $\gamma = 1$, let $\varepsilon_1 = \varepsilon_2 = 1$. Solving (29), gives

$$P_1 = \begin{bmatrix} 2.6179 & 0.3342 \\ 0.3342 & 2.3329 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 2.9598 & 0.1577 \\ 0.1577 & 2.0921 \end{bmatrix},$$

It is easy to verify that P_1 and P_2 are positive definite matrices, which indicate that condition (i) in Theorem 2 is satisfied. Choosing

$$W_1 = 1.5x_3^2 + 0.8x_3x_4 + 1.5x_4^2, \qquad W_2 = x_3^2 + 2x_4^2$$

We have $a_{11} = 1.1$, $a_{21} = 1.9$, $a_{12} = 1$, $a_{22} = 2$, $\dot{W}_1 \leq -1.6(x_3^2 + x_4^2)$, $\dot{W}_2 \leq -4(x_3^2 + x_4^2)$. This implies that condition (ii) is satisfied. Using Theorem 2, we design the average dwell-time and switched state feedback. Let $\mu = 1.4175$, $\lambda_0 = 0.8$, $\lambda = 0.7$, we can get $\hat{\mu} = 1.9$, $\tau_a^* = 0.9$, let $\tau_a^* \leq \tau_a = 1$. Design the switching law as

$$\sigma(t) = \begin{cases} 1, & k = 0, 2, 4, \ldots, \\ 2, & k = 1, 3, 5, \ldots, \end{cases} \qquad t_k = k. \tag{37}$$

and the switched state feedback is given as:

$$u_i = \begin{cases} -1.6432x_1 - 3.5000x_2, & i = 1, \\ -21.8392x_1 - 0.6308x_2, & i = 2. \end{cases} \quad (38)$$

A simple calculation shows that the average dwell time for the linear switched part of the system is $\tau_{a1} \geq \tau_{a1}^* = \frac{\ln \mu}{\lambda} = 0.5$, and the average dwell time for the nonlinear switched part is $\tau_{a2} \geq \tau_{a2}^* = 0.7$. Thus, $\tau_a \geq \tau_a^* \geq \max\{\tau_{a1}^*, \tau_{a2}^*\}$ is obvious. Let $x(0) = (-2, 1.5, 2.5, -2.5)^T$.

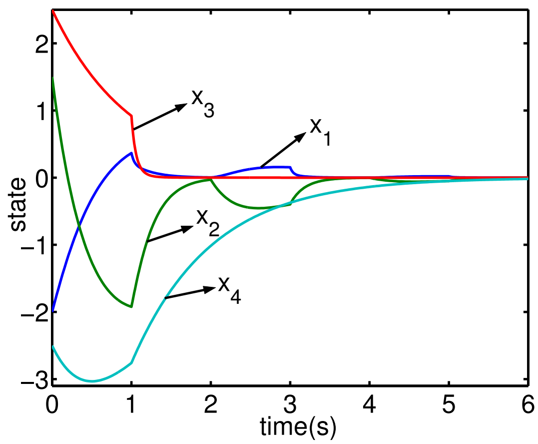

Fig. 1. The state response of the switched system

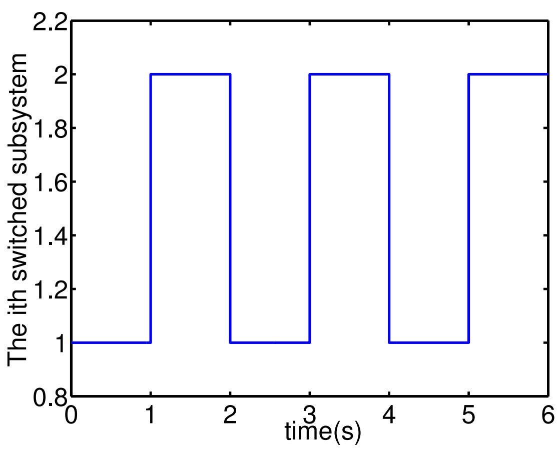

Fig. 2. The switching signal for the switched system

Fig. 1 and Fig. 2 are the state response and the switching signal of the whole switched system separately, which indicate the feasibility of our results.

VI. CONCLUSIONS

In this paper, we have studied the L_2-gain analysis and control synthesis problem for a class of uncertain switched nonlinear cascade systems with external disturbances input. Sufficient conditions for both the weighted L_2-gain analysis and the control synthesis have been expressed in the form of linear matrix inequalities. The disturbed uncertain autonomous switched system and the disturbed uncertain non-autonomous switched system with the designed switched state feedback are globally exponentially stable and achieved a weighted L_2-gain under arbitrary designed switching laws that satisfy some average dwell-time. Moreover, the average dwell-time and the state decay have been calculated explicitly.

REFERENCES

[1] D. Liberzon, A. S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag. vol. 19, 1999, pp. 59-70.

[2] J. L. Mancillla-Aguilar, A condition for the stability of switched nonlinear systems, IEEE Trans. Autom. Control, vol. 45, 2000, pp. 2077-2079.

[3] J. Zhao, Dimirovski G. M, Quadratic stability of a class of switched nonlinear systems, IEEE Trans. Autom. Control, 2004, vol. 49, pp. 574-578.

[4] M. S. Branicky, Multiple Lyapunov functions and other analysis tools for switched hybridsystems, IEEE Trans. Autom. Control, 1998, vol. 43, pp. 475-482.

[5] J. P. Hespanha, A. S. Morse, Stability of switched systems with average dwell-time, Proceedings of the 38th IEEE Conference on Decision and Control, 1999, pp. 2655-2660.

[6] S. Pettersson, Synthesis of switched linear systems, in proc. $42nd$ IEEE Conf. Decision Control, 2003, pp. 5283-5288.

[7] Z. Sun, S. S. Ge, Analysis and synthesis of switched linear control systems, Automatic, 2005, vol. 42, pp. 181-195.

[8] D. Cheng, Stabilization of planar switched systems, Syst. Control Lett. vol. 51, 2004, 79-88.

[9] Z. Sun, S. S. Ge, T. H. Lee, Controllability and reachablity criteria for switched linear systems, Automatic, 2002, vol. 38, pp. 775-786.

[10] J. P. Hespanha, D. Liberzon, D. Angeli, E. D. Sontag, Nonlinear norm-observability notions and stability of switched systems, IEEE Trans. Automa. Control. 2005, vol. 52, pp. 154-168.

[11] A. Bemporad, M. Morari, Control of systems integrating logic, dynamics, and constraints, Automatica, 1999, vol. 35, pp. 407-427.

[12] X. Xu and P. J. Antsaklis, Optimal control of switched systems based on parameterization of the switching instants, IEEE Trans. Autom. Control, 2004, vol. 49, pp. 2-16.

[13] J. Zhao, David J. Hill, On stability, and L_2-gain and H_∞ control for switched systems, Automatica (accepted).

[14] P. P. Khargonekar, I. R. Petersen, K. Zhou, Robust stabilization of uncertain linear systems: quadratic stabilizablity and H_∞ control theory, IEEE Trans. Autom. Control, vol. 35, 2001, pp. 356-361.

[15] G.H. Yang, J.L. Wang, T.C. Soh, Reliable H_∞ controller design for linear systems, Automatica, 2001, vol. 37, pp. 717-725.

[16] D. Xie, L. Wang, F. Hao, G. Xie, LMI approach to L2-gain analysis and control synthesis of uncertain switched systems, IEE Proceedings of Control Theory and Applications, 151, pp. 21-28, 2004.

[17] G. Zhai, B. Hu, K. Yasuda, A. N. Michel, Disturbance attenuation properties of time-contolled switched systems, Journal of the Franklin Institute, vol. 338, 2001, pp. 765-779.

[18] M. Sun, J. Zhao, and D. J. Hill, Stability and L_2-gain analysis for switched delay systems: A delay-dependent method, Automatica, vol. 42, 2006, pp. 1769-1744.

[19] F. Hao, T. Chu, L. Huang and L. Wang, Non-fragile controllers of peak gain minimization for uncertain systems via LMI approach, Dynamics of Continuous, Discrete and Impulsive Systems, vol.10, 2003, pp. 681-693.

[20] Sepulchre. R, Jankovic. M, P. V. Kokotovic, Constructive nonlinear control, Springer Verlag, 1997.