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A New Method to Robust H_∞ Control of Uncertain Switched Systems

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Abstract—**This paper develops a new method to robust H_∞ control problem for a class of uncertain switched systems by constructing *single* robust H_∞ sliding surface. The method consists of two phases. One is to construct a single sliding surface which the reduced-order equivalent sliding motion is forced into, and to have the sliding motion robustly stabilized with H_∞ disturbance attenuation level γ under a hysteresis switching law to be designed; the other phase is to design variable structure controllers of the subsystems to thus drive the state of the switched system to reach the single sliding surface in finite time and remain on it thereafter. A numerical example is given to illustrate the effectiveness of the proposed method.**

I. Introduction

Switched systems consist of a family of continuous-time or discrete-time systems and certain rules of logic specifying at each instant of time which subsystem is activated along the system trajectory, thus represent a rather important class of hybrid systems. As a result, switched systems and switching control have recently gained a great deal of attention [1-8] mainly because many real-world systems such as chemical processes and transportation systems can be modeled as switched systems under certain reasonable assumptions. In the literature, switched linear systems without uncertainties have been extensively investigated; for instance, see [3-5] and references therein. Since uncertainties are ubiquitous in system models due to the complexity of the system itself, exogenous disturbance and so on, from a practical point of view, the study of uncertain switched systems is relatively more important.

Among the existing results of switched systems with uncertainties, [6] considered quadratic stabilization of switched systems with norm-bounded time varying uncertainties. In [7], L_2 induced norm of switched systems with external disturbances was considered under the condition of large dwell time. Robust H_∞ control and stabilization of uncertain switched linear systems were addressed in [8] based on multiple Lyapunov functions approach.

On the other hand, the sliding mode control (SMC) is one of most important methods in robust control domain, since it possesses various attractive features such as robustness, fast response, and good transient response [9, 10]. Over the years, there are many available papers [11-16] on SMC. Among the results concerning SMC, they are mainly partitioned into two ways, one is to develop SMC theory [11-13]; the other is to fuse SMC technique into other methods or other systems rather than the traditional ones, where there are already some exciting and significant results [14-16].

Along the latter, we will apply SMC to switched system. There are very few results focusing this interest except [17-20]. The authors of [17] proposed a SMC method to make a class of switched systems exponentially stable. [18] addressed the sliding mode control for planar switched systems under an arbitrary switching sequence. In [19], the sliding motion of switched systems without control input was analyzed and an approach was proposed to estimate the domain in which the sliding motion may occur. A variable structure controller with sliding mode sector for a hybrid system was presented which switches the hybrid system among subsystems to ensure its quadratic stability in [20]. As for tackling H_∞ control problem with resort to SMC technique, to the best of our knowledge, there are almost no results in the current literature, which is indeed our motivation.

In this paper, we investigate and solve the robust H_∞ sliding mode variable structure control problem for a class of uncertain switched linear systems. The outline of this paper is as follows. Section II presents the problem formulation and the necessary preliminaries. In Section III, the novel design is theoretically developed. In Section IV, the developed control design is applied to an illustrative example and numerical and simulation results are given to illustrate the effectiveness of the proposed design. Conclusion and references follow thereafter.

Throughout this paper, $\|\bullet\|$ denotes the Euclidean norm for a vector or the matrix induced norm for a matrix.

II. Problem Formulation and Preliminaries

Consider the following uncertain switched system

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$$\dot{x}(t) = (A_\sigma + \Delta A_\sigma)x(t) + B(u_\sigma + f_\sigma(x,t)) + B_1\omega(t), \\ z(t) = Cx(t), \tag{1}$$

where $x(t) \in R^n$ is the system state, $\sigma(t):[0,\infty) \to \Xi = \{1,2,\ldots,l\}$ is the piecewise constant switching signal that may depend on either time t or state x, $u_i \in R^m$ is the control input of the $i-th$ subsystem, $z(t)$ is the controlled output, $\omega(t) \in L_2[0,\infty)$ is the external disturbance input, B, B_1, C and A_i are constant matrices with appropriate dimensions, ΔA_i denote system parameter uncertainties, $f_i(x,t)$ represent nonlinearities of the system. The following assumptions are introduced.

Assumption 1. The parameter uncertainties can be represented and emulated as

$$\Delta A_i = E\Sigma_i(t)F, i \in \Xi,$$

where E and F are known constant matrices with appropriate dimensions, $\Sigma_i(t)$ are unknown matrices with Lebesgue measurable elements and satisfy $\Sigma_i^{\mathrm{T}}(t)\Sigma_i(t) \le I$.

Assumption 2. There exist known nonnegative scalar-valued functions $\phi_i(x,t)$, $i \in \Xi$ such that $\|f_i(x,t)\| \le \phi_i(x,t)$ for all t.

Assumption 3. There exists a known nonnegative constant ϖ such that $\|\omega(t)\| \le \varpi$ for all t.

Assumption 4. The input matrix B has full rank m and $m < n$.

Remark 1. Assumptions 1~4 are standard assumptions in the study of variable structure control.

We now recall the concept of asymptotic stability with H_∞ disturbance attenuation level γ.

Definition 1 ([21]). Consider the following uncertain switched linear system

$$\dot{x} = A_\sigma x + B\omega, \\ z = Cx. \tag{2}$$

For a given positive constant $\gamma > 0$, if there exists a switching law $\sigma = \sigma(x)$ and a positive definite matrix P, such that the inequality

$$x^{\mathrm{T}}(A_\sigma^{\mathrm{T}}P + PA_\sigma + \gamma^{-2}PBB^{\mathrm{T}}P + C^{\mathrm{T}}C)x < 0 \tag{3}$$

holds, then system (2) is called asymptotically stable and satisfied H_∞ disturbance attenuation level γ.

Lemma 1 ([22]). Given real matrices R_1 and R_2 with appropriate dimensions and an unknown matrix $\Sigma(t)$ with Lebesgue measurable elements such that $\Sigma^{\mathrm{T}}(t)\Sigma(t) \le I$, then we have

$$R_1\Sigma R_2 + R_1^{\mathrm{T}}\Sigma^{\mathrm{T}}R_2^{\mathrm{T}} \le \beta R_1R_1^{\mathrm{T}} + \beta^{-1}R_2^{\mathrm{T}}R_2,$$

where $\beta > 0$.

Now, we introduce a convex combination of the system (1) without the matched uncertainties $f_i(x,t)$ as follows

$$x(t) = (\overline{A} + \Delta\overline{A})x(t) + Bu + B_1\omega(t), \\ z(t) = Cx(t), \tag{4}$$

where $\overline{A} = \sum_{i=1}^{l}\alpha_iA_i$, $\Delta\overline{A} = \sum_{i=1}^{l}\alpha_i\Delta A_i$, $\alpha_i \ge 0$, $\sum_{i=1}^{l}\alpha_i = 1$.

Lemma 2. Given a constant $\gamma > 0$, if there exist matrix $P > 0$, state feedback gain K, constant $\lambda > 0$ and scalars $\alpha_i \ge 0$, $\sum_{i=1}^{l}\alpha_i = 1$ satisfying

$$(\overline{A} - BK)^{\mathrm{T}}P + P(\overline{A} - BK) + P(\lambda^2EE^{\mathrm{T}} + \gamma^{-2}B_1B_1^{\mathrm{T}})P \\ + \frac{1}{\lambda^2}F^{\mathrm{T}}F + C^{\mathrm{T}}C < 0, \tag{5}$$

then the system (4) is robustly stabilized with H_∞ disturbance attenuation level γ.

Proof. Let

$$\begin{aligned} Q &= (\overline{A} + \Delta\overline{A} - BK)^{\mathrm{T}}P + P(\overline{A} + \Delta\overline{A} - BK) \\ &\quad + \gamma^{-2}PB_1B_1^{\mathrm{T}}P + C^{\mathrm{T}}C \\ &= (\overline{A} - BK)^{\mathrm{T}}P + P(\overline{A} - BK) + \gamma^{-2}PB_1B_1^{\mathrm{T}}P \\ &\quad + C^{\mathrm{T}}C + \Delta\overline{A}^{\mathrm{T}}P + P\Delta\overline{A}. \end{aligned}$$

Using Lemma 1, one obtains

$$\begin{aligned} \Delta\overline{A}^{\mathrm{T}}P + P\Delta\overline{A} &= (\textstyle\sum_{i=1}^{l}\alpha_i\Delta A_i)^{\mathrm{T}}P + P(\sum_{i=1}^{l}\alpha_i\Delta A_i) \\ &= [E(\textstyle\sum_{i=1}^{l}\alpha_i\Sigma_i(t))F]^{\mathrm{T}}P + P[E(\sum_{i=1}^{l}\alpha_i\Sigma_i(t))F] \\ &\le \lambda^2PEE^{\mathrm{T}}P + \lambda^{-2}F^{\mathrm{T}}F. \end{aligned}$$

Hence, we have

$$Q \le (\overline{A} - BK)^{\mathrm{T}}P + P(\overline{A} - BK) + P(\lambda^2EE^{\mathrm{T}} + \gamma^{-2}B_1B_1^{\mathrm{T}})P \\ + \frac{1}{\lambda^2}F^{\mathrm{T}}F + C^{\mathrm{T}}C < 0,$$

which implies that the system (4) is robustly stabilized with H_∞ disturbance attenuation level γ. This completes the proof.

Remark 2. The inequality (5) can be converted into a linear matrix inequality (LMI) by using Schur complement and the change of variable such that $\hat{K} = KP^{-1}$. Hence, the feasible solutions can be globally found by the LMIs method [16].

To get a regular form of the system (1), we define a nonsingular matrix T and an associated vector ξ as follows

$$T = \begin{bmatrix} \widetilde{B}^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}, \tag{6}$$

and

$$\xi(t) = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = Tx(t) = \begin{bmatrix} \widetilde{B}^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} x(t), \tag{7}$$

with $\xi_1 \in R^{n-m}$, $\xi_2 \in R^m$, where $\widetilde{B}$ is an orthogonal complement of the matrix B. We can easily show

$$T^{-1} = \left[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} \quad B(B^{\mathrm{T}}B)^{-1}\right]. \tag{8}$$

By means of the state transformation $\xi(t) = Tx(t)$, the system (1) is transformed into the following regular form

$$\begin{aligned} &\dot{\xi} = (\widehat{A}_\sigma + \Delta\widehat{A}_\sigma)\xi + \widehat{B}(u_\sigma + f_\sigma(x,t)) + \widehat{B}_1\omega(t), \\ &z(t) = \widehat{C}\xi(t), \end{aligned} \tag{9}$$

where $\widehat{A}_\sigma = TA_\sigma T^{-1}$, $\Delta\widehat{A}_\sigma = T\Delta A_\sigma T^{-1}$, $\widehat{B} = TB$, $\widehat{B}_1 = TB_1$, $\widehat{C} = CT^{-1}$. The system (9) is equivalent to the following form

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \widehat{A}_{\sigma 11} & \widehat{A}_{\sigma 12} \\ \widehat{A}_{\sigma 21} & \widehat{A}_{\sigma 22} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B^{\mathrm{T}}B \end{bmatrix}(u_\sigma + f_\sigma(x,t)) + \begin{bmatrix} \widetilde{B}^{\mathrm{T}}B_1 \\ B^{\mathrm{T}}B_1 \end{bmatrix}\omega,$$

$$z(t) = C\left[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} \quad B(B^{\mathrm{T}}B)^{-1}\right]\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{10}$$

where

$$\widehat{A}_{\sigma 11} = \widetilde{B}^{\mathrm{T}}A_\sigma\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} + \widetilde{B}^{\mathrm{T}}E\Sigma_\sigma(t)F\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1},$$

$$\widehat{A}_{\sigma 12} = \widetilde{B}^{\mathrm{T}}A_\sigma B(B^{\mathrm{T}}B)^{-1} + \widetilde{B}^{\mathrm{T}}E\Sigma_\sigma(t)FB(B^{\mathrm{T}}B)^{-1},$$

$$\widehat{A}_{\sigma 21} = B^{\mathrm{T}}A_\sigma\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} + B^{\mathrm{T}}E\Sigma_\sigma(t)F\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1},$$

$$\widehat{A}_{\sigma 22} = B^{\mathrm{T}}A_\sigma B(B^{\mathrm{T}}B)^{-1} + B^{\mathrm{T}}E\Sigma_\sigma(t)FB(B^{\mathrm{T}}B)^{-1}.$$

Without loss of generality, we assume that the single robust H_∞ sliding surface is given as follows

$$\zeta(t) = M\xi_1 + \xi_2 = 0, \tag{11}$$

where $M \in R^{n\times(n-m)}$ is a matrix to be chosen. Then it follows that $\zeta(t) = Sx(t) = (M\widetilde{B}^{\mathrm{T}} + B^{\mathrm{T}})x(t)$. Substituting $\xi_2 = -M\xi_1$ into (10) yields the following sliding motion

$$\begin{aligned} &\dot{\xi}_1 = (\widehat{A}_{\sigma 11} - \widehat{A}_{\sigma 12}M)\xi_1 + \widetilde{B}^{\mathrm{T}}B_1\omega, \\ &z(t) = C\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}\xi_1 - CB(B^{\mathrm{T}}B)^{-1}M\xi_1, \end{aligned} \tag{12}$$

The objective of this paper is to determine the matrix M, the switching law $\sigma(t)$, and the variable structure controllers $u_i, i \in \Xi$ such that

i). the sliding mode (12) restricted to the single sliding surface (11) is robustly stabilized with H_∞ disturbance attenuation level γ under the switching law $\sigma(t)$.

ii). the state of the system (1) can reach the single sliding surface (11) in finite time and subsequently remain on it.

Remark 3. The single sliding surface $\zeta(t) = Sx(t) = 0$ is designed such that the switched system (1) is asymptotically stable with an H_∞ norm bound based on the single Lyapunov function approach in the sliding surface. The purpose of designing the single sliding surface for the switched system is to reduce the reaching phase in which systems are sensitive to uncertainties and perturbations, and improve the transient performance and robustness.

Remark 4. We can see that the matched uncertainties $f_i(x,t)$ disappear in the sliding motion (12) and the order of the switched system (1) is reduced in the sliding surface (11). Therefore, we only need to study the stability of the $n-m$ dimensional switched system (12).

III. MAIN RESULT

In this section, the variable structure control design comprises two steps. Firstly, to construct the sliding surface, such that the controlled system yields the desired dynamic performance in the sliding surface. Secondly, design the variable structure controller to drive the trajectory of the system reaches the sliding surface and remains on it for all subsequent time.

The following theorem shows that the system (1) in the sliding surface (11) is robust asymptotically stabilized with H_∞ disturbance attenuation level γ via switching.

Theorem 1. Suppose that (5) is solvable i.e., the system (4) is robustly stabilized with H_∞ disturbance attenuation level γ. Then the sliding motion (12) with $M = ((B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}PB(B^{\mathrm{T}}B)^{-1})^{-1} \times (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}$ is robust stabilized with H_∞ disturbance attenuation level γ via switching and the single robust H_∞ sliding surface is

$$\begin{aligned} \zeta(t) &= Sx(t) \\ &= \{[(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}PB(B^{\mathrm{T}}B)^{-1}]^{-1}(B^{\mathrm{T}}B)^{-1} \\ &\quad \times B^{\mathrm{T}}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}\widetilde{B}^{\mathrm{T}} + B^{\mathrm{T}}\}x(t) \\ &= 0, \end{aligned} \tag{13}$$

where P satisfies (5) in Lemma 2.

Proof. The sliding motion (12) can be rewritten equivalently as

$$\begin{aligned} &\dot{\xi}_1 = (\hat{A}_{\sigma 11} - \hat{A}_{\sigma 12}M + \hat{E}\Sigma_\sigma(t)\hat{F})\xi_1 + \widetilde{B}^{\mathrm{T}}B_1\omega, \\ &z(t) = (C\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} - CB(B^{\mathrm{T}}B)^{-1}M)\xi_1, \end{aligned} \tag{14}$$

where $\hat{A}_{\sigma 11} = \widetilde{B}^{\mathrm{T}}A_\sigma\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}$, $\hat{A}_{\sigma 12} = \widetilde{B}^{\mathrm{T}}A_\sigma B(B^{\mathrm{T}}B)^{-1}$, $\hat{E} = \widetilde{B}^{\mathrm{T}}E$, $\hat{F} = F\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} - FB(B^{\mathrm{T}}B)^{-1}M$.

Denote

$$\begin{aligned} \overline{A}_c &= T(\overline{A} - BK)T^{-1} \\ &= \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} - B^{\mathrm{T}}BK\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} & \overline{A}_{22} - B^{\mathrm{T}}BKB(B^{\mathrm{T}}B)^{-1} \end{bmatrix} \end{aligned} \tag{15}$$

with $\overline{A}_{11} = \widetilde{B}^{\mathrm{T}}\overline{A}\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}$, $\overline{A}_{12} = \widetilde{B}^{\mathrm{T}}\overline{A}B(B^{\mathrm{T}}B)^{-1}$, and calculate

$$\begin{aligned} \overline{P} &= T^{-\mathrm{T}}PT^{-1} \\ &= \begin{bmatrix} (\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}\widetilde{B}^{\mathrm{T}}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} & (\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}\widetilde{B}^{\mathrm{T}}PB(B^{\mathrm{T}}B)^{-1} \\ (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1} & (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}PB(B^{\mathrm{T}}B)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{12}^{\mathrm{T}} & \overline{P}_{22} \end{bmatrix}. \end{aligned} \tag{16}$$

Then, the inequality (5) in Lemma 2 can be rewritten as

$$\begin{aligned} &\overline{A}_c^{\mathrm{T}}\overline{P} + \overline{P}A_c + \overline{P}T(\lambda^2 EE^{\mathrm{T}} + \gamma^{-2}B_1B_1^{\mathrm{T}})T^{\mathrm{T}}\overline{P} \\ &+ T^{-\mathrm{T}}(\frac{1}{\lambda^2}F^{\mathrm{T}}F + C^{\mathrm{T}}C)T^{-1} < 0. \end{aligned} \tag{17}$$

By multiplying (17) with $[I_{n-m}, -\overline{P}_{12}\overline{P}_{22}^{-1}]$ and $[I_{n-m}, -\overline{P}_{12}\overline{P}_{22}^{-1}]^{\mathrm{T}}$ from left and right, respectively, we have

$$\begin{aligned}&(\overline{A}_{11}-\overline{A}_{12}\overline{P}_{22}^{-1}\overline{P}_{12}^{\mathrm{T}})^{\mathrm{T}}\overline{P}_r+\overline{P}_r(\overline{A}_{11}-\overline{A}_{12}\overline{P}_{22}^{-1}\overline{P}_{12}^{\mathrm{T}})\\&+\overline{P}_r\widetilde{B}^{\mathrm{T}}(\lambda^2EE^{\mathrm{T}}+\gamma^{-2}B_1B_1^{\mathrm{T}})\widetilde{B}\overline{P}_r+[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}\\&-B(B^{\mathrm{T}}B)^{-1}\overline{P}_{22}^{-1}\overline{P}_{12}^{\mathrm{T}}]^{\mathrm{T}}(\frac{1}{\lambda^2}F^{\mathrm{T}}F+C^{\mathrm{T}}C)\\&\times[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}-B(B^{\mathrm{T}}B)^{-1}\overline{P}_{22}^{-1}\overline{P}_{12}^{\mathrm{T}}]<0,\end{aligned} \tag{18}$$

where $\overline{P}_r=\overline{P}_{11}-\overline{P}_{12}\overline{P}_{22}^{-1}\overline{P}_{12}^{\mathrm{T}}$. $\overline{P}_r>0$ since $\overline{P}>0$. Therefore, by setting $M=[(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}PB(B^{\mathrm{T}}B)^{-1}]^{-1}(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}P\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}$ $=\overline{P}_{22}^{-1}\overline{P}_{12}^{\mathrm{T}}$, (18) becomes

$$\begin{aligned}&(\overline{A}_{11}-\overline{A}_{12}M)^{\mathrm{T}}\overline{P}_r+\overline{P}_r(\overline{A}_{11}-\overline{A}_{12}M)+\overline{P}_r\widetilde{B}^{\mathrm{T}}(\lambda^2EE^{\mathrm{T}}\\&+\gamma^{-2}B_1B_1^{\mathrm{T}})\widetilde{B}\overline{P}_r+[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}-B(B^{\mathrm{T}}B)^{-1}M]^{\mathrm{T}}\\&\times(\frac{1}{\lambda^2}F^{\mathrm{T}}F+C^{\mathrm{T}}C)[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}-B(B^{\mathrm{T}}B)^{-1}M]<0.\end{aligned} \tag{19}$$

Further, denoting

$$\begin{aligned}Q_i=&(\widetilde{B}^{\mathrm{T}}A_i\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}-\widetilde{B}^{\mathrm{T}}A_iB(B^{\mathrm{T}}B)^{-1}M)^{\mathrm{T}}\overline{P}_r\\&+\overline{P}_r(\widetilde{B}^{\mathrm{T}}A_i\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}-\widetilde{B}^{\mathrm{T}}A_iB(B^{\mathrm{T}}B)^{-1}M)\\&+\overline{P}_r\widetilde{B}^{\mathrm{T}}(\lambda^2EE^{\mathrm{T}}+\gamma^{-2}B_1B_1^{\mathrm{T}})\widetilde{B}\overline{P}_r+[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}\\&-B(B^{\mathrm{T}}B)^{-1}M]^{\mathrm{T}}(\frac{1}{\lambda^2}F^{\mathrm{T}}F+C^{\mathrm{T}}C)[\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}\\&-B(B^{\mathrm{T}}B)^{-1}M],\end{aligned}$$

and substituting $\overline{A}_{11}=\widetilde{B}^{\mathrm{T}}\overline{A}\widetilde{B}(\widetilde{B}^{\mathrm{T}}\widetilde{B})^{-1}$, $\overline{A}_{12}=\widetilde{B}^{\mathrm{T}}\overline{A}B(B^{\mathrm{T}}B)^{-1}$ and $\overline{A}=\sum_{i=1}^{l}\alpha_iA_i$ into inequality (19) gives

$$\alpha_1Q_1+\alpha_2Q_2+\ldots+\alpha_lQ_l<0. \tag{20}$$

We define the regions

$$\Omega_i=\{\xi_1|\xi_1^{\mathrm{T}}Q_i\xi_1<0, i\in\Xi\}. \tag{21}$$

Obviously, $\bigcup_{i\in\Xi}\Omega_i=R^{(n-m)}\setminus\{0\}$.

The hysteresis switching law for the sliding motion (12) is designed as follows

$$\begin{aligned}&\sigma(0)=\min\arg\{\Omega_i|\,\xi_1(0)\in\Omega_i\},\\&\text{for } t>0,\\&\sigma(t)=\begin{cases}i, & \text{if } \xi_1(t)\in\Omega_i \text{ and } \sigma(t^-)=i,\\ \min\arg\{\Omega_k|\xi_1(t)\in\Omega_k\}, & \text{if } \xi_1(t)\notin\Omega_i \text{ and } \sigma(t^-)=i.\end{cases}\end{aligned} \tag{22}$$

By virtue of Definition 1, we conclude that sliding motion (12) is robust stabilized with H_∞ disturbance attenuation level γ under the switching law (22). The proof is thus completed.

In the end, we give the following result.

Theorem 2. Assume that the conditions of Theorem 1 are satisfied and the sliding surface of system (1) is given by (13). Then under the control laws

$$\begin{aligned}u_i=&-(SB)^{-1}SA_ix-(SB)^{-1}(\|SE\|\|Fx\|+\|SB\|\phi_i(x,t)\\&+\varpi\|SB_1\|+\mu)sign(\zeta), i\in\Xi,\end{aligned} \tag{23}$$

the state of the system (1) can enters in finite time and subsequently remains on the sliding surface, where μ is a positive scalar to adjust the convergent rate.

Proof. The derivative of the sliding function $\zeta(t)=Sx(t)$ along the trajectory of the system (1) is

$$\dot{\zeta}(t)=S(A_i+\Delta A_i)x(t)+SBu_i+SBf_i+SB_1\omega(t). \tag{24}$$

With regard to Assumptions 1~3, substituting the control laws (23) into (24) yields $\zeta(t)\dot{\zeta}(t)\le-\mu\|\zeta(t)\|$, which implies that the state of the system (1) reaches the sliding surface (13) in finite time and thereafter remains on it. This completes the proof.

Remark 5. The single sliding surface is reached in finite time according to the reaching rate of sliding surface μ. The $\zeta(t)\dot{\zeta}(t)\le-\mu\|\zeta(t)\|$ implies the decay rate of the sliding surface is no less than $e^{-\mu t}$.

IV. AN ILLUSTRATIVE EXAMPLE

In this section, we present a numerical example to demonstrate the effectiveness of the proposed design method.

Consider the following uncertain switched linear system

$$\begin{aligned}&\dot{x}(t)=(A_\sigma+\Delta A_\sigma)x(t)+B(u_\sigma+f_\sigma)+B_1\omega(t),\\&z(t)=Cx(t),\end{aligned} \tag{25}$$

where $\sigma(t)\in\Xi=\{1,2\}$,

$$A_1=\begin{bmatrix}-3&-0.5&1\\1&-0.5&1\\0&1&-2\end{bmatrix},\ A_2=\begin{bmatrix}-1&-1&1\\2&1&-1\\1&0&-2\end{bmatrix},\ B=\begin{bmatrix}0\\-0.5\\1\end{bmatrix},$$

$B_1=\begin{bmatrix}0\\-0.1\\0.1\end{bmatrix}$, $C=\begin{bmatrix}1\\0\\0\end{bmatrix}^{\mathrm{T}}$, the parameter uncertainties $\Delta A_i=E\times\Sigma_i(t)F$ where $E=\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $F=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\Sigma_i=\eta_i\in[-1,1]$, and $f_1=f_2=0$.

We choose the convex combination coefficients $\alpha_1=\alpha_2=0.5$ and the constant $\lambda=\frac{1}{\sqrt{2}}$. The disturbance attenuation level is given by $\gamma=\frac{1}{\sqrt{2}}$.

Taking the matrix $K=B^{\mathrm{T}}P$, by solving the inequality (5), one can obtain the following solution

$$P = \begin{bmatrix} 9.2006 & 7.8894 & 3.7381 \\ 7.8894 & 8.1363 & 3.6193 \\ 3.7381 & 3.6193 & 3.2078 \end{bmatrix}.$$

Then we obtain the matrix

$$M = [0.0831, 0.2195].$$

The single robust H_∞ sliding function is given as follows

$$\zeta = Sx = [-0.1591, -0.3458, 1.0771]^{\mathrm{T}} x . \tag{26}$$

According to (23) the subsystem control laws are given by

$$\begin{aligned} u_1 &= -0.1054x_1 - 1.0637x_2 + 2.1273x_3 \\ &\quad - 0.8(0.1866\|x_2 + x_3\| + 1)sign(\zeta), \\ u_2 &= -0.4358x_1 + 0.1493x_2 + 1.5741x_3 \\ &\quad - 0.8(0.1866\|x_2 + x_3\| + 1)sign(\zeta). \end{aligned} \tag{27}$$

The simulation results for system state responses of the two subsystems alone with the initial state vector $x_0 = [1, 2, -1]^{\mathrm{T}}$ are shown in Fig. 1 and Fig. 2, respectively. We can easily see that both subsystems are unstable.

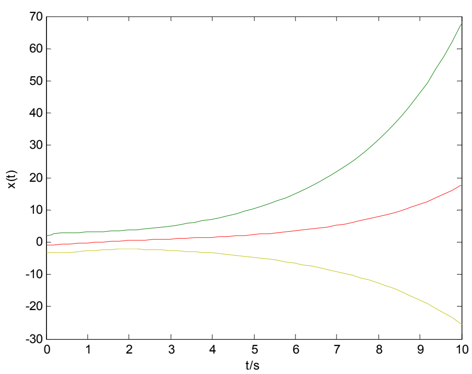

Fig. 1 The state response of the subsystem 1

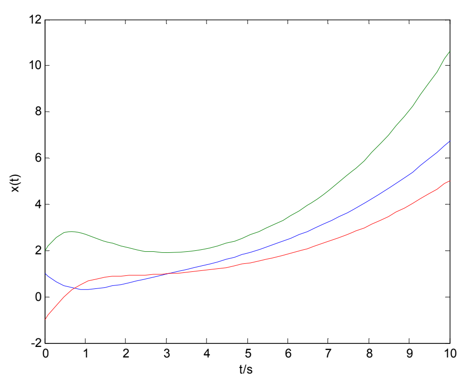

Fig. 2 The state response of the subsystem 2

It is easy to verify that the conditions of Theorem 1 and 2 are satisfied.

The hysteresis switching law is

$$\sigma(t) = \begin{cases} 1, if\,(x(0) \in \Omega_1) or (x(t) \in \Omega_1 and\, \sigma(t^-) = 1) \\ \quad or (x(t) \notin \Omega_2 and\, \sigma(t^-) = 2), \\ 2, if\,(x(0) \notin \Omega_1) or (x(t) \in \Omega_2 and\, \sigma(t^-) = 2) \\ \quad or (x(t) \notin \Omega_1 and\, \sigma(t^-) = 1), \end{cases} \tag{28}$$

where

$$\Omega_1 = \{x(t) | x^{\mathrm{T}}(t) \begin{bmatrix} -32.5006, -13.8465, -6.9232 \\ -13.8465, -0.5939, -0.2969 \\ -6.9232, \ -0.2969, \ -0.1485 \end{bmatrix} x(t) < 0\},$$

$$\Omega_2 = \{x(t) | x^{\mathrm{T}}(t) \begin{bmatrix} 23.2917, \ 9.9988, 4.9994 \\ 9.9988, -2.3998, -1.1999 \\ 4.9994, -1.1999, -0.5999 \end{bmatrix} x(t) < 0\} .$$

The simulation results are depicted in Fig. 3-Fig. 6.

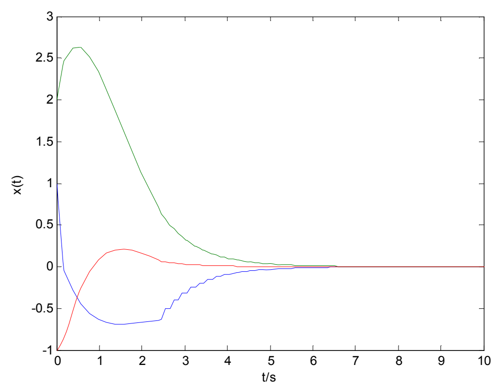

Fig. 3 The system state responses of the switched system (25)

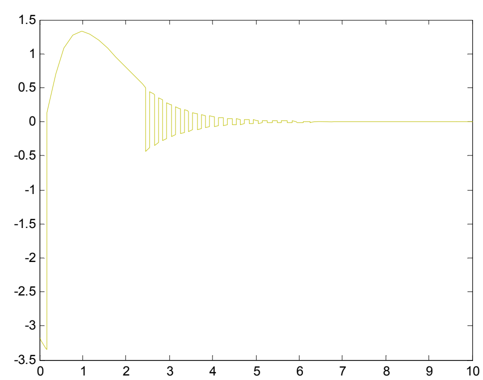

Fig. 4. The input signal of the switched system (25)

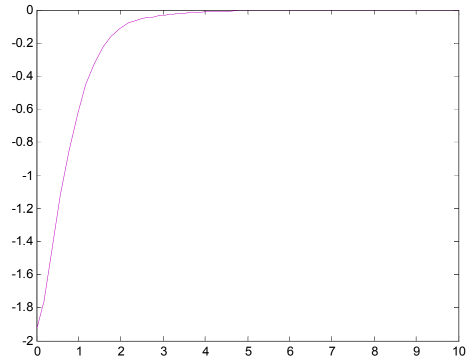

Fig. 5. The trajectory of the sliding function (26)

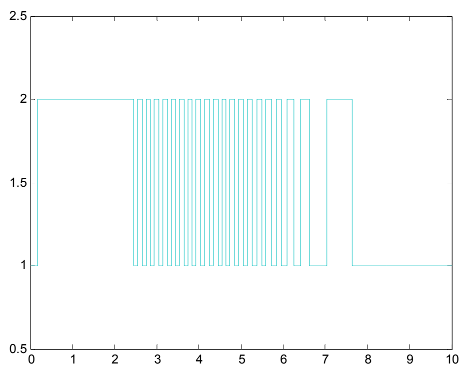

Fig. 6. The switching signal (28)

The simulation results for the system states in the closed-loop and with the same initial state vector $x_0 = [1, 2, -1]^{\mathrm{T}}$ are shown in Fig. 3. It is clearly seen that the closed-loop system of the switched system (25) with the designed controller (27) and the switching law (28) is asymptotically stable. Fig. 4 is the input signal of the switched system (25). The trajectory of the sliding function (26) is shown in Fig. 5. The switching signal is given by Fig. 6.

V. Conclusion

This paper has developed the new method to robust H_∞ control problem for a class of uncertain switched systems by constructing *single* robust H_∞ sliding surface. The sufficient condition for the existence of the single robust H_∞ sliding surface has been derived in terms of Riccati inequality associated with the convex combination of the switched system. The switching law has been constructed such that the $n-m$ dimensional sliding motion is robustly stabilized with H_∞ disturbance attenuation level γ. Variable structure controllers have been designed to drive the state of the switched system to reach the single robust H_∞ sliding surface in finite time.

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