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Fault tolerant H_∞ control for a class of nonlinear discrete-time systems: Using sum of squares optimization

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Abstract— **This paper studies the fault tolerant control (FTC) problem for a class of nonlinear discrete-time systems with guaranteed H_∞ performance objective in the presence of actuator faults. The mode of faults under consideration is typical aberration of actuator effectiveness. The novelty of this paper is that the effect of the nonlinear terms is described as an index in order to transform the FTC design problem into a semi-definite programming (SDP). The proposed optimization approach is to find zero optimum for this index. Combined with H_∞ performance index, the conceived multi-objective optimization problem is solved by using sum of squares method (SOS) in a reliable and efficient way. A numerical example is included to verify the applicability of this new approach for the nonlinear FTC synthesis.**

I. INTRODUCTION

The increasing demands for higher system performance, product quality, productivity and cost efficiency lead to a continuous growth of the complexity and automation degree of technical processes. Associated with these development trends, high reliability, availability and safety become an important system requirement which is included in many international standards and regulations. The objective of fault tolerant control (FTC) system is to maintain current performances closed to desirable performances and preserve stability conditions in the presence of component and/or instrument faults; in some circumstances reduced performance could be accepted as a trade-off. The routine analysis procedures always need be modulated when taking additional fault tolerance characteristic into account. Accommodation capability of a control system depends on many factors such as the severity of the failure, the robustness of the nominal system, and the actuators redundancy. FTC can be motivated by different goals depending on the application under consideration, for instance, safety in civil aviation or reliability or quality improvements in industrial processes. Various approaches for FTC have been suggested in the literature [4]-[6], [8], [15], [16] but often deal with linear systems or lack effective numerical methods for nonlinear systems. For nonlinear systems, the design of fault tolerant controllers is far more complicated. Analysis of FTC performance for nonlinear system always suffers from one drawback: When consider additional fault tolerance, accommodating the original control mechanism or reconstructing a control mechanism in similar way to remove the influence of faults is hard. The conservation of the original control strategy is to be confined to give enough space for consideration of additional fault tolerant performance.

In this paper, a passive fault tolerant approach is adopted, so no fault detection algorithm is needed. A computational fault tolerant strategy is developed so as to reduce actuator fault effect on nonlinear systems, the developed method preserves the system performances by passive FTC mechanism in faulty situation. To transform FTC design problem into a tractable semi-definite programming (SDP), the effect of the nonlinear terms is described as an index. The proposed optimization approach is to find zero optimum for this index. Combined with optimal cost performance index or H_∞ optimal performance index, the original FTC control problem is converted into a multiobjective optimization problem. It includes a constraint set of state dependent linear polynomial matrix inequalities. Then sum of squares method (SOS) is used to solve this kind of optimization problems in a reliable and efficient way. One key difference between the proposed design and some existing FTC control designs is that the controller is built through algorithmic construction of Lyapunov functions. This is meaningful because the SOS approach often provides less conservative results than other relaxation methods for nonlinear systems [3]. For nonlinear systems, the existing results from the standard robust control technique [1] cannot numerically handle the fault tolerant control problem with performance optimization. This paper attempts to develop a tractable computational method to solve high-order performance FTC problem.

The paper is organized as follows. In Section II, we present some preliminary results concerning the sum of squares decomposition and its application to solving state dependent linear polynomial inequalities. Then in Sections III, the state feedback H_∞ control problem is settled. The result is obtained under an assumption that the faults happen in actuator aberration mode. A numerical example is presented to illustrate the proposed method in Section IV, and finally, the paper is ended by some conclusions in Section V.

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The notation used throughout the paper is fairly standard. For matrices or vectors, $*^T$ indicates transposition, and Φ_{sos} is defined as the set of all SOS polynomials in n variables.

II. Preliminaries and Problem Statement

A. Preliminaries

The computational method used in this paper relies on the sum of squares decomposition of multivariate polynomials. A multivariate polynomial $f(x)$ (where $x \in R^n$) is a sum of squares if there exist polynomials $f_1(x), \ldots, f_m(x)$ such that $f(x) = \sum_{i=1}^{m} f_i^2(x)$. This can be shown equivalent to the existence of a special quadratic form stated in the following proposition.

Proposition 2.1. Let $f(x)$ be a polynomial in $x \in R^n$ of degree $2d$. In addition, let $Z(x)$ be a column vector whose entries are all monomials in x with degree no greater than d. Then $f(x)$ is a sum of squares iff there exists a positive semi-definite matrix Q such that

$$f(x) = Z^T(x) Q Z(x). \tag{1}$$

Proof: See [10]. ■

Remark 2.2. A sum of squares decomposition for $f(x)$ can be computed using semi-definite programming, since it amounts to searching for an element Q in the intersection of the cone of positive semi-definite matrices and a set defined by some affine constraints that arise from (1). What is even more instrumental for control applications is the fact that when the polynomial $f(x)$ is not exactly determined but its coefficients are otherwise affinely parameterized in terms of some unknowns, the search for coefficient values which render $f(x)$ a sum of squares can still be performed using semi-definite programming. This has been exploited for algorithmically constructing Lyapunov functions for nonlinear systems [10], [13].

In this paper, the above methodology is used to solve state dependent linear polynomial matrix inequalities. What is meant by state dependent linear polynomial matrix inequalities (LPMI) is an infinite dimensional convex optimization problem of the form

$$\begin{array}{c} \text{minimize } \sum_{i=1}^{m} a_i c_i \\ \text{subject to } F_0(x) + \sum_{i=1}^{m} c_i F_i(x) \geq 0 \end{array} \tag{2}$$

where the a_i's are some fixed real coefficients, the c_i's are the decision variables, and the $F_i(x)$ are some symmetric matrix functions of the indeterminate $x \in R^n$. The matrix inequality (2) basically means that the left hand side of the inequality is positive semi-definite for all $x \in R^n$. Solving the above optimization problem amounts to solving an infinite set of polynomial inequalities and hence is computationally hard. However, when the $F_i(x)$'s are symmetric polynomial matrices in x, the sum of squares decomposition can provide a computational relaxation for the condition (2). This relaxation is stated in the following proposition [14].

Proposition 2.3. Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in R^n$, furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with degree no greater than d, and consider the following conditions.

(1) *$F(x) \geq 0$ for all $x \in R^n$.*
(2) *There exists a real vector $v \in R^N$ such that $v^T F(x) v$ is a sum of squares.*
(3) *There exists a positive semidefinite matrix Q such that $v^T F(x) v = (v \otimes Z(x))^T Q (v \otimes Z(x))$,*

where $\otimes$ denotes the Kronecker product.
Then (1)$\Longleftarrow$(2) and (2) $\Longleftrightarrow$ (3).

Proof: see [2] ■

Remark 2.4. The converse implication (1) $\Longrightarrow$ (2) generally does not hold. A special case for which this implication holds is when $n = 1$ [7]. In the same reference, a symmetric polynomial matrix $F(x)$ such that $v^T F(x) v$ is a sum of squares is termed a sum of squares matrix. By Proposition 2.3, it is clear that any solution to the sum of squares optimization problem

$$\begin{array}{c} \text{minimize } \sum_{i=1}^{m} a_i c_i \\ \text{subject to } v^T (F_0(x) + \sum_{i=1}^{m} c_i F_i(x)) v \geq 0 \end{array} \tag{3}$$

is a sum of squares, (3) is also a solution to the state dependent linear polynomial matrix inequalities (2). However, (3) is much easier to solve than (2) from computational perspective. In particular, semi-definite programming can be used for this purpose, e.g. with the help of the software [12]. For nonlinear discrete-time system, the FTC design problem will be converted into state dependent linear polynomial matrix inequalities in the form of (3) where 'x ' is substituted by '$x(k)$'.

B. Problem Statement

Consider the nonlinear system described as follow:

$$\begin{aligned} x(k+1) &= f\big(x(k)\big) + g_w\big(x(k)\big)\omega(k) + g_u\big(x(k)\big)u(k) \\ z(k) &= h_z\big(x(k)\big) + J_{zw}\big(x(k)\big)w(k) + J_{zu}\big(x(k)\big)u(k) \\ y(k) &= h_y\big(x(k)\big) + J_{yw}\big(x(k)\big)w(k) \end{aligned}$$

which can be approximated by the following state dependent linear-like representation:

$$\begin{aligned} x(k+1) &= A\big(x(k)\big)x(k) + B_w\big(x(k)\big)w(k) + B_u\big(x(k)\big)u(k) \\ z(k) &= C_z\big(x(k)\big)x(k) + D_{zw}\big(x(k)\big)w(k) + D_{zu}\big(x(k)\big)u(k) \\ y(k) &= C_y\big(x(k)\big)x(k) + D_{yw}\big(x(k)\big)w(k) \end{aligned} \tag{4}$$

where the state vector $x(k) \in R^n$ and other vectors are of monomials. All other matrices are of polynomials, and they have appropriate dimensions. $y(k)$ is the measured output, and $z(k)$ is a vector of output signals related to the performance of the control system.

To formulate the fault tolerant control problem, the following fault model from [11] is adopted in this paper:

$$u_{ij}^F(k) = \rho_i^j u_i(k), \rho_i^j \in [\underline{\rho}_i^j, \bar{\rho}_i^j], \bar{\rho}_i^j > \underline{\rho}_i^j \geq 0, \quad i = 1, \cdots, m, j = 1, \cdots, L \tag{5}$$

where $u_{ij}^F(t)$ represents the signal from the ith actuator that failed in the jth faulty mode, ρ_i^j is an unknown constant,

the index j denotes the jth faulty mode and L is the number of total faulty modes. For every faulty modes, $\underline{\rho}_i^j$ and $\overline{\rho}_i^j$ represent the lower and upper bounds of ρ_i^j, respectively. Note that, when $\underline{\rho}_i^j = \overline{\rho}_i^j = 1$, there is no fault for $i-$th actuator u_i is outrange in the jth faulty mode. When $0 \leq \underline{\rho}_i^j \leq \overline{\rho}_i^j < 1$, in the jth faulty mode the type of actuator faults is loss of effectiveness.
Denote

$$u_j^F(k) = [u_{1j}^F(k), u_{2j}^F(k), \cdots, u_{mj}^F(k)]^T = \rho^j u(k) \tag{6}$$

where $\rho^j = diag[\rho_1^j, \rho_2^j, \cdots, \rho_m^j], j = 1, \cdots, L$. Considering the lower and upper bounds $\underline{\rho}_i^j$ and $\overline{\rho}_i^j$, the following set can be defined

$$N_{\rho^j} = \{\rho^j | \rho^j = diag[\rho_1^j, \rho_2^j, \cdots, \rho_m^j], \rho_i^j = \underline{\rho}_i^j \ or \ \rho_i^j = \overline{\rho}_i^j\} \tag{7}$$

Thus, the set N_{ρ^j} contains a maximum of 2^m elements. For convenience in the following sections, for all possible faulty modes L, the following uniform actuator fault model is exploited:

$$u^F(k) = \rho u(k), \rho \in \{\rho^1, \cdots, \rho^L\} \tag{8}$$

and ρ can be described by $\rho = diag[\rho^1, \cdots, \rho^m]$.
For the system described by (4) with actuator faults (8), state-feedback and output-feedback FTC controllers are designed for the closed-loop system to guarantee H_∞ performance.

III. State Feedback H_∞ Control with Fault Tolerant Objective

A. Extended PMI Characterization of the H_∞ Specification

In this paper, a special Lyapunov function is defined to check the stability of system (4).

$$V\big(x(k)\big) = x^T(k)\mathscr{P}\big(x(k)\big)x(k) \tag{9}$$

where $\mathscr{P}\big(x(k)\big)$ is defined to be a $n \times n$ polynomial matrix whose $(i,j)-th$ entry is given by

$$p_{ij}(k) = p_{ij}^{(0)} + \sum_{l=1}^{n} p_{ij}^{(l)} x_l(k) = p_{i,j}^{(0)} + [p_{i,j}^{(1)}, \cdots, p_{i,j}^{(n)}]^T x(k) \tag{10}$$

$p_{ij}^{(l)}$ $i = 1, \cdots, n, \ j = 1, \cdots, n, \ l = 1, \cdots, n$ are scalars. So $V\big(x(k)\big)$ is linearly parameterized in n^2 one-degree polynomials $p_{i,j}$. The connection of the system (4) with the linear-like controller to be defined will provide the linear-like system with closed-loop state-space representation.

$$\begin{aligned} x(k+1) &= \mathscr{A}\big(x(k)\big)x(k) + \mathscr{B}\big(x(k)\big)w(k) \\ z(k) &= \mathscr{C}\big(x(k)\big)x(k) + \mathscr{D}\big(x(k)\big)w(k) \end{aligned} \tag{11}$$

Then the standard H_∞ analysis changes to the following form.
Lemma 3.1 (H_∞ norm) The inequality $\|H_{wz}(\zeta)\|_\infty^2 < \mu$ holds if there exists a symmetric polynomial matrix $\mathscr{P}\big(x(k)\big)$ such that

$$\begin{bmatrix} \mathscr{P}\big(x(k)\big) & \mathscr{A}\big(x(k)\big)\mathscr{P}\big(x(k+1)\big) & \mathscr{B}\big(x(k)\big) & \mathbf{0} \\ *^T & \mathscr{P}\big(x(k+1)\big) & \mathbf{0} & \mathscr{P}\big(x(k+1)\big)\mathscr{C}^T\big(x(k)\big) \\ *^T & *^T & I & \mathscr{D}^T\big(x(k)\big) \\ *^T & *^T & *^T & \mu I \end{bmatrix} > \mathbf{0} \tag{12}$$

is feasible.

Proof: The proof can be easily obtained from the literature of linear system [9]. ■

Remark 3.2 As expected, the condition requires that matrix $\mathscr{A}\big(x(k)\big)$ be Schur stable since the fundamental Lyapunov inequality

$$\begin{bmatrix} \mathscr{P}\big(x(k)\big) & \mathscr{A}\big(x(k)\big)\mathscr{P}\big(x(k+1)\big) \\ *^T & \mathscr{P}\big(x(k+1)\big) \end{bmatrix} > \mathbf{0} \tag{13}$$

appears as one of their diagonal blocks.
Remark 3.3 Unfortunately, $\mathscr{P}\big(x(k+1)\big)$ does not depend linearly on $p_{i,j}^{(l)}$ in general. It has been shown in (de Oliveira *et al*. 1999 a) that it is possible to extended the Lyapunov inequality (13) with the introduction of an additional instrumental matrix variable. This technique is used here to reduce the number of $\mathscr{P}\big(x(k+1)\big)$ in the form (12), and it is generalized in the next theorem to cope with the H_∞ norm calculation.
Theorem 3.4 (Extended H_∞ norm) The inequality $\|H_{wz}(\zeta)\|_\infty^2 < \mu$ holds if, and only if, there exists a polynomial matrix $\mathscr{G}\big(x(k)\big)$ and a symmetric polynomial matrix $\mathscr{P}\big(x(k)\big)$ such that

$$\begin{bmatrix} \mathscr{P}\big(x(k)\big) & \mathscr{A}\big(x(k)\big)\mathscr{G}\big(x(k)\big) & \mathscr{B}\big(x(k)\big) & \mathbf{0} \\ *^T & \mathscr{G}\big(x(k)\big) + \mathscr{G}^T\big(x(k)\big) - \mathscr{P}\big(x(k+1)\big) & \mathbf{0} & \mathscr{G}\big(x(k)\big)\mathscr{C}^T\big(x(k)\big) \\ *^T & *^T & I & \mathscr{D}^T\big(x(k)\big) \\ *^T & *^T & *^T & \mu I \end{bmatrix} > \mathbf{0} \tag{14}$$

is feasible.

Proof: (Necessity) Choose $\mathscr{G}\big(x(k)\big) = \mathscr{G}\big(x(k)\big)^T = \mathscr{P}(x(k+1))$.

(Sufficiency) Assume that the inequality (14) are feasible. Hence $\mathscr{G}\big(x(k)\big) + \mathscr{G}\big(x(k)\big)^T > \mathscr{P}\big(x(k+1)\big) > 0$. Note that this implies that $\mathscr{G}\big(x(k)\big)$ is non-singular. Since $\mathscr{P}\big(x(k+1)\big)$ is positive definite the inequality $\Big(\mathscr{P}\big(x(k+1)\big) - \mathscr{G}\big(x(k)\big)\Big)^T \mathscr{P}^{-1}\big(x(k+1)\big)\Big(\mathscr{P}\big(x(k+1)\big) - \mathscr{G}\big(x(k)\big)\Big) > 0$ holds. Therefore establishing $\mathscr{G}^T\big(x(k)\big)\mathscr{P}^{-1}\big(x(k+1)\big)\mathscr{G}\big(x(k)\big) \geq \mathscr{G}\big(x(k)\big) + \mathscr{G}^T\big(x(k)\big) - \mathscr{P}\big(x(k+1)\big)$ which yields

$$\begin{bmatrix} \mathscr{P}\big(x(k)\big) & \mathscr{A}\big(x(k)\big)\mathscr{G}\big(x(k)\big) & \mathscr{B}\big(x(k)\big) & \mathbf{0} \\ *^T & \mathscr{G}\big(x(k)\big)\mathscr{P}^{-1}\big(x(k+1)\big)\mathscr{G}\big(x(k)\big) & \mathbf{0} & \mathscr{G}\big(x(k)\big)\mathscr{C}^T\big(x(k)\big) \\ *^T & *^T & I & \mathscr{D}^T\big(x(k)\big) \\ *^T & *^T & *^T & \mu I \end{bmatrix} > \mathbf{0}$$

which recovers (12) if multiplied on the right by $\mathscr{T}\big(x(k)\big) = diag[\ I,\ \mathscr{G}^{-1}\big(x(k)\big)\mathscr{P}\big(x(k+1)\big),\ I,\ I\ ]$ and on the left by $\mathscr{T}^T\big(x(k)\big)$. ■

Remark 3.5 The introduction of this extra variable provides much advantage in the analysis of nonlinear polynomial systems, because the polynomial feedback control strategy can be chosen as high order to get more excellent performance. The cancelation of nonlinear terms can give convenience for numerical computation in high order polynomial. What's more, for linear system (the system matrix is choose as constant matrix), this new extended stability condition may also provide advantage in the context of synthesis of nonlinear controllers to reduce conservation.

B. Multi-objective Optimization Method for H_∞ Fault Tolerant Control

In this subsection, state-feedback H_∞ control against actuator fault is settled with the aforementioned methodology in part A of subsection II. Throughout this subsection it is the assumed that the state vector $x(k)$ is available for feedback. Moreover, the state information is not corrupted by the input $w(k)$. These assumptions are standard and can be enforced on the measurement equation of system (4) by assigning to the matrices $C_y\big(x(k)\big)$ and $D_{yw}\big(x(k)\big)$ the values $C_y\big(x(k)\big) = I$ and $D_{yw}\big(x(k)\big) = 0$. The following nonlinear static state-feedback control law

$$u(k) = \mathscr{K}\big(x(k)\big)x(k) \tag{15}$$

is sought ($\mathscr{K}\big(x(k)\big)$ is in polynomial form). This feedback structure produces a system in the form (11) where the closed-loop matrices are given by

$$\mathscr{A}\big(x(k)\big) := A\big(x(k)\big) + B_u\big(x(k)\big)\mathscr{K}\big(x(k)\big), \tag{16}$$
$$\mathscr{B}\big(x(k)\big) := B_w\big(x(k)\big), \tag{17}$$
$$\mathscr{C}\big(x(k)\big) := C_z\big(x(k)\big) + D_{zu}\big(x(k)\big)\mathscr{K}\big(x(k)\big), \tag{18}$$
$$\mathscr{D}\big(x(k)\big) := D_{zw}\big(x(k)\big), \tag{19}$$

Then the nonlinear transformation (change-of-variables)

$$X\big(x(k)\big) := \mathscr{G}(x(k)), \tag{20}$$
$$L\big(x(k)\big) := \mathscr{K}(x(k))\mathscr{G}(x(k)), \tag{21}$$
$$P\big(x(k)\big) := \mathscr{P}(x(k)), \tag{22}$$

is able to reduce partial nonlinear conditions after replacing (16)—(22) into the inequalities of Theorems 3.4, then polynomial matrix inequality (PMI) on the synthesis variables X, L and P is given as follow.
Denote

$$\begin{aligned}
&\Gamma_k(A,X,B_u,L) = A(k)X(k) + B_u(k)L(k),\\
&\Gamma_k(A,X,B_u\rho,L) = A(k)X(k) + B_u(k)L(k),\\
&\Gamma_k(X,C,L,D_{zu}) = X^T(k) + C_z(k) + L^T(k)D_{zu}^T(k).
\end{aligned}$$

Theorem 3.6 (H_∞ state-feedback FTC): Suppose that for the system (4) there exist polynomial matrices $X(k)$ and $L(k)$ and the symmetric polynomial matrices $P(k),\ P(k+1)$, a constant $\varepsilon_1 > 0$, and a sum of squares $\varepsilon_2(x(k))$ with $\varepsilon_2(x(k)) > 0$ for $x \neq 0$, such that the following SOS optimization problem has zero optimum of μ

$$\begin{aligned}
&\text{minimize} \quad \mu\\
&\text{subject to}\\
&\quad v^T[P\big(x(k)\big) - \varepsilon_1 I]v \quad \in \Phi_{sos},\\
&\quad v^T[P\big(x(k+1)\big) - \varepsilon_1 I]v \quad \in \Phi_{sos},
\end{aligned}$$

$$\begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}^T \begin{bmatrix} P(k) & \Gamma_k(A,X,B_u,L) & B_w(k) & \mathbf{0}\\ *^T & X(k)+X^T(k)-P(k+1) & \mathbf{0} & \Gamma_k(X,C,L,D_{zu})\\ *^T & *^T & I & D_{zw}^T(k)\\ *^T & *^T & *^T & \mu I\end{bmatrix} \begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\end{bmatrix} \in \Phi_{sos} \tag{23}$$

$$\begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\end{bmatrix}^T \begin{bmatrix} P(k) & \Gamma_k(A,X,B_u\rho,L) & B_w(k) & \mathbf{0}\\ *^T & X(k)+X^T(k)-P(k+1) & \mathbf{0} & \Gamma_k(X,C,L,D_{zu})\\ *^T & *^T & I & D_{zw}^T(k)\\ *^T & *^T & *^T & \mu I\end{bmatrix} \begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\end{bmatrix} \in \Phi_{sos} \tag{24}$$

where $v,\ v_1,\ v_2,\ v_3,\ v_4$ are mutually independent scalar vectors. Then for the state feedback law $u(k) = \mathscr{K}\big(x(k)\big)x(k)$, the zero equilibrium of the closed-loop system is asymptotically stable, and the closed-loop system has $\|H_{wx}(\zeta)\|_\infty^2 < \mu$ in the normal or faulty situation.

Proof: This theorem can be proved with the introduction of change-of-variables by theorem 3.4. The key point is that the inequality (14) is affine in the set of extreme matrices $N_{\rho j}$. ■

Remark 3.7 For the existence of nonlinear term in $\mathscr{P}(k+1)$, the set of $V(k)$ and $\mathscr{K}(k)$ satisfying these conditions is not jointly convex, hence a simultaneous search for such $V(k)$ and $\mathscr{K}(k)$ is hard. The following theorem converts this problem into a semi-definite programming.
Denote

$$\overrightarrow{p}_{i,j} = [p_{i,j}^{(1)}, \cdots, p_{i,j}^{(n)}]^T \tag{25}$$

$$\Xi_{(v,p)} = \Sigma_{i,j=1,\cdots,n}\ v_i v_j \overrightarrow{P}_{i,j} \tag{26}$$

$$\tilde{P}\big(x(k+1)\big) = (\tilde{p}_{i,j})_{n\times n} \tag{27}$$

$$= \Big(p_{i,j}^{(0)} + [p_{i,j}^{(1)}, \cdots, p_{i,j}^{(n)}]^T A\big(x(k)\big)x(k)\Big)_{n\times n} \tag{28}$$

Theorem 3.8 (Optimization for H_∞ state-feedback FTC): Suppose that for the system (4), there exist polynomial matrices $X(k)$ and $L(k)$ and the symmetric polynomial matrices $P(k),\ P(k+1)$, a constant $\varepsilon_1 > 0$, and a sum of squares $\varepsilon_2(x(k))$ with $\varepsilon_2(x(k)) > 0$ for $x \neq 0$, such that the following

SOS optimization problem has zero optimum of μ

$$\text{minimize} \quad \gamma+\mu$$

subject to

$$v^T[P(k)) - \varepsilon_1 I]v \quad \in \Phi_{sos}, \tag{29}$$

$$v^T[P(k+1) - \varepsilon_1 I]v \quad \in \Phi_{sos}, \tag{30}$$

$$v_5^T \begin{bmatrix} \gamma & \Xi_{(v,p)}B_u(k) \\ *^T & I \end{bmatrix} v_5 \quad \in \Phi_{sos}, \tag{31}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T \begin{bmatrix} P(k) & \Gamma_k(A,X,B_u,L) & B_w(k) & \mathbf{0} \\ *^T & X(k)+X^T(k)-\tilde{P}(k+1) & \mathbf{0} & \Gamma_k(X,C,L,D_{zu}) \\ *^T & *^T & I & D_{zw}^T(k) \\ *^T & *^T & *^T & (\mu-\varepsilon_2(x(k))I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Phi_{sos} \tag{32}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}^T \begin{bmatrix} P(k) & \Gamma_k(A,X,B_u\rho,L) & B_w(k) & \mathbf{0} \\ *^T & X(k)+X^T(k)-\tilde{P}(k+1) & \mathbf{0} & \Gamma_k(X,C,L,D_{zu}) \\ *^T & *^T & I & D_{zw}^T(k) \\ *^T & *^T & *^T & (\mu-\varepsilon_2(x(k))I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Phi_{sos} \tag{33}$$

for all $\rho \in \{\rho_1, \cdots, \rho_L\}$

where v, v_1, v_2, v_3, v_4, v_5 are mutually independent scalar vectors. Then for the state feedback law $u(k) = \mathscr{K}\left(x(k)\right)x(k)$, the zero equilibrium of the closed-loop system is asymptotically stable, and the closed-loop system has $\|H_{wx}(\zeta)\|_\infty^2 < \mu$ in the normal or faulty situation.

Proof: Note that

$$\mathscr{P}\left(x(k+1)\right) = \left(p_{ij}(k+1)\right)_{n\times n} = \left(p_{ij}^{(0)} + \sum_{l=1}^{n} p_{ij}^{(l)} x_l(k+1)\right)_{n\times n}$$
$$= \left(p_{i,j}^{(0)} + [p_{i,j}^{(1)}, \cdots, p_{i,j}^{(n)}]^T \left(A(k) + B_u(k)\mathscr{K}(k)\right)x(k)\right)_{n\times n}$$

$$v_2^T \mathscr{P}\left(x(k+1)\right)v_2 = v_2^T \tilde{P}\left(x(k+1)\right)v_2 + \Xi_{(v,p)}B_u\left(x(k)\right)\mathscr{K}\left(x(k)\right)x(k)$$

The nonlinear terms in (23)-(24) equal to $\Xi_{(v,p)}B_u(k)\mathscr{K}(k)x(k)$ and $\Xi_{(v,p)}B_u(k)\rho\mathscr{K}(k)x(k)$. By the Schur complement,

$$\begin{bmatrix} \gamma & \Xi_{(v,p)}B_u(k) \\ *^T & I \end{bmatrix} \geq 0 \tag{34}$$

implies $\left(\Xi_{(v,p)}B_u(k)\right)\left(\Xi_{(v,p)}B_u(k)\right)^T \leq \gamma$. It can be seen that γ is nonnegative. If the minimum of γ is zero, then $\Xi_{(v,p)}B_u(k) = 0$, which makes two nonlinear terms disappear. By Proposition 2.3, it follows that (31) is the sum of squares relaxation of (34). ■

Remark 3.3 If the optimum value of μ is not zero, from (29)-(30), by Hölder's inequality

$$-v_4^T S(k)v_4 + \Xi_{(v,p)}B_u(k)u(k) \geq \varepsilon_2(x(k))v_4^T v_4 + \Xi_{(v,p)}B_u(k)u(k)$$
$$\geq \varepsilon_2(x(k))v_4^T v_4 - \sqrt{(\Xi_{(v,p)}B_u(k))(\Xi_{(v,p)}B_u(k))^T u(k)^T u(k)}$$

$$-v_4^T S_\rho(k)v_4 + \Xi_{(v,p)}B_u(k)\rho u(k)$$
$$\geq \varepsilon_2(x(k))v_4^T v_4 + \Xi_{(v,p)}B_u(k)\rho u(k)$$
$$\geq \varepsilon_2(x(k))v_4^T v_4 - \sqrt{(\Xi_{(v,p)}B_u(k))(\Xi_{(v,p)}B_u(k))^T u(k)^T \rho^T \rho u(k)}$$

where S and S_ρ represents (21) and (22). It can be seen that if

$$u(k)^T u(k) \leq \frac{\varepsilon_2^2(x(k))(v_4^T v_4)^2}{\max\{\overline{\rho_i}^k\}^2\gamma} \tag{35}$$

holds, we can also get the same result, thus, (35) can be used as a discriminant condition of γ. When the input u and γ satisfy (35), then the objective of this fault tolerant control strategy is attained. So $\varepsilon_2(x(k))$ should be enlarged to get a feasible control law. It can be seen that if γ equals zero, (35) is obviously satisfied.

IV. Example

Consider a nonlinear system with an approximate polynomial form given by

$$A(k) = \begin{bmatrix} -1+x_2(k) & \frac{2}{3}-x_1(k)^2 \\ x_1(k) & x_2(k) \end{bmatrix}, B_w(k) = \begin{bmatrix} x_2(k) & 1 \\ 1 & 3 \end{bmatrix},$$
$$B_u(k) = \begin{bmatrix} 1 & 0 \\ x_1(k) & 2 \end{bmatrix}, C_z(k) = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix},$$
$$D_{zw}(k) = \begin{bmatrix} 0 & 0 \\ 6 & 7 \end{bmatrix}, D_{zu}(k) = \begin{bmatrix} 0 & 0 \\ 1 & 7 \end{bmatrix},$$

Choose the four faulty modes as follows:

Normal mode 1: Both of the two actuators are normal, that is, $\rho_1^1 = \rho_1^1 = 1$.

Faulty mode 2: The first actuator is outrage and the second actuator may be normal or loss of effectiveness, described by $\rho_1^2 = 0$, $0.5 \leq \rho_2^2 \leq 1$.

Faulty mode 3: The second actuator is outrage and the first actuator may be normal or loss of effectiveness, described by $\rho_2^3 = 0$, $0.4 \leq \rho_1^3 \leq 1$.

Faulty mode 4: Both actuators may be normal or loss of effectiveness, described by $0.4 \leq \rho_1^4 \leq 1$, $0.6 \leq \rho_2^4 \leq 1$.

In the simulation, the disturbance that used is

$$w_1(k) = w_2(k) = \begin{cases} 1, & 10 \leq t \leq 11, \\ 0, & \text{otherwise} \end{cases}$$

and the fault case is at 2 second , the first actuator becomes loss of effectiveness of 50%.

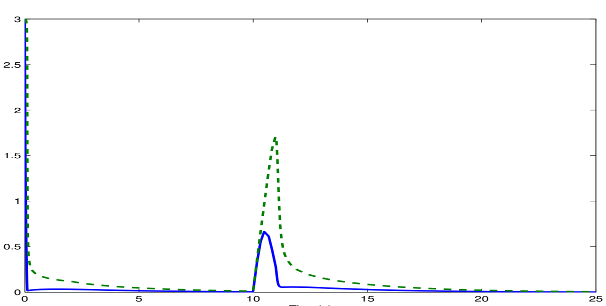

Fig. 1. Response curve of the first state of system in the faulty situation : controller without FTC consideration (dash), controller with FTC consideration (solid).

A state feedback controller is designed using the result stated in Theorem 3.8, augmented with the minimization of γ. The values of ε_1 and $\varepsilon_2(x(k))$ are chosen as 0.1 and $2(x_1^2(k)+x_2^2(k))$.The optimization returns 1.26 as the optimal value of μ. Hence, we conclude that the H_∞ gain from w to z of the closed-loop system is no greater than 1.26. For comparison, a state feedback controller without fault tolerant consideration designed for the original system in faulty condition returns 2.13 as the optimal H_∞ gain of the closed-loop system. In the normal situation, this controller returns 0.97 as H_∞ gain. This value is a lower bound on the best achievable nonlinear H_∞ performance, and thus we see that the FTC nonlinear design is not overly conservative. Although μ obtained from the FTC based design is higher than that of the original one, the performance of the controller designed for the original nonlinear system is guaranteed in faulty situation.

V. CONCLUDING REMARKS

In this paper, we have addressed the state feedback FTC H_∞ problems for a class of nonlinear discrete-time systems. Our approach is built upon representing the nonlinear systems in a state dependent linear-like form, and the solution is stated in terms of state dependent linear polynomial inequalities that incorporate index optimization. It is then converted into sum of squares optimization problem, which can be solved using semidefinite programming. A numerical example is presented to illustrate the availability and efficiency of the method.

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