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Robust H_∞ Control of Uncertain Switched Delay Systems Using Multiple Lyapunov Functions

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Abstract—**This paper investigates the robust H_∞ control problem for a class of uncertain switched delay systems that involve parameter uncertainties and unknown nonlinear disturbances. Based on the multiple Lyapunov functions method, a sufficient condition for the solvability of the robust H_∞ control problem is derived by employing a hysteresis switching law and variable structure controllers. When the upper bounds of the nonlinear disturbances are unknown, an adaptive variable structure control strategy is developed. The use of adaptive technique is to adapt the unknown upper bounds of the nonlinear disturbances so that the objective of asymptotic stabilization with H_∞ -norm bound is achieved under the hysteresis switching law. A numerical example illustrates the effectiveness of the proposed design method.**

I. Introduction

Switched systems constitute an important class of hybrid systems. A typical switched system consist of a family of subsystems and a switching law specifying which subsystem will be activated at each instant of time. The motivation for studying switched systems is from the fact that many practical systems are inherently multi-models in the sense that several dynamical subsystems are required to describe their behavior depending on various changing environmental factors. In the study of stability analysis for switched systems, the multiple Lyapunov functions method has been considered as an important analysis tool in [1-3]. Robust H_∞ control and stabilization of uncertain switched linear systems were considered by utilizing a state-depend switching strategy with the multiple Lyapunov functions method in [4]. [5] addressed the problem of stability and L_2 -gain analysis for nonlinear switched systems via the multiple Lyapunov functions method. In this paper, the stability result has been generalized by defining more general weak multiple Lyapunov functions. [6] designed a hysteresis switching law to avoid sliding motions that often occur in state-depended switching strategy. The value of the hysteresis switching signal is not determined by the current value of state alone, but depends also on the previous value of switching signal.

Time-delay commonly exists in various industrial systems, and its existence is frequently a source of instability. Switched systems with time-delay are one of the most useful models and have strong engineering background such as power systems [7] and networked control systems [8]. However, very few results on switched delay systems have been reported. Sufficient conditions of asymptotical stability were established for switched linear delay systems under arbitrary and constructed switching signals, respectively in [9]. [10] investigated the problem of delay-dependent common Lyapunov functions for switched linear delay systems, which established the relationship between the delay-dependent common Lyapunov functions and the common Lyapunov functions for corresponding switched systems without delays. The stabilization problem of arbitrary switched linear systems with unknown time varying delays was considered in [11]. For uncertain linear discrete-time switched systems with state delays, sufficient conditions of robust stability and stabilizability in terms of matrix inequalities and Riccati-like inequalities were given in [12]. Stability of a class of switched delay systems was shown in [13] by using a common Lyapunov functional method. [14] studied stability and L_2 -gain of switched delay systems based on the average time technique. However in above results, the value of switching signal only depends on state or time. There are no results for design of hysteresis switching law of uncertain switched delay systems in the current literature, which is indeed our motivation.

Another critical issue for switched systems is to enhance the robustness against system uncertainties and perturbations. Variable structure control with sliding mode or without sliding mode is an effective robust scheme for systems with uncertainties, which employs discontinuous control law to overcome the uncertainties and improve performance and stability.

In this paper, we consider the problem of robust H_∞ control for a class of uncertain switched delay systems with parameter uncertainties and unknown nonlinear disturbances. Based on the multiple Lyapunov functions method, a sufficient condition for robust stability with H_∞ disturbance attenuation level γ is derived, and a hysteresis switching law is designed. For the case of known upper bounds of the nonlinear disturbances, variable structure controllers are developed such that the uncertain switched delay system is asymptotically stabilizable with H_∞ disturbance attenuation

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level γ under the hysteresis switching law. For the case of unknown upper bounds of the nonlinear disturbances, adaptive variable structure controllers are developed so that the objective of asymptotic stabilization with disturbance attenuation level γ is achieved under the hysteresis switching law. A numerical example illustrates the effectiveness of the proposed design methods.

In this paper, $\|\bullet\|$ denotes the Euclidean norm for a vector or the matrix induced norm for a matrix; Z^+ denotes the set of all nonnegative integers.

II. Problem formulation and preliminaries

Consider the uncertain switched delay system of the form

$$\begin{aligned}\dot{x}(t) &= (A_\sigma + \Delta A_\sigma)x(t) + (A_{d\sigma} + \Delta A_{d\sigma})x(t-\tau) \\ &\quad + B_\sigma(u_\sigma(t) + f_\sigma(x,t)) + G_\sigma \omega(t), \qquad (1)\\ x(t) &= \varphi(t),\ t\in[-\bar{\tau},0],\ z(t) = C_\sigma x(t),\end{aligned}$$

where $x(t)\in R^n$ is the system state, $\sigma(t):[0,\infty)\to\Xi=\{1,\ldots,k\}$ is the piecewise constant switching signal, $u_i\in R^m$ is the control input of the $i-th$ subsystem, $\omega(t)\in R^h$ denotes the disturbance input which belongs to $L_2[0,\infty)$, $z(t)\in R^q$ is the controlled output, A_i, A_{di}, B_i G_i C_i are constant matrices with appropriate dimensions, τ denotes unknown constant time-delay which is bounded by the known constant $\bar{\tau}$, $\varphi(t)$ is a differentiable vector-valued initial function on $[-\bar{\tau},0]$, ΔA_i, ΔA_{di} represent the system uncertainties, $f_i(x,t)$ is an unknown nonlinear function. The following assumptions are introduced.

Assumption 1. The uncertainties can be represented and emulated as

$$\Delta A_i = D_i\Sigma_i(t)E_i,\ \Delta A_{di} = B_i\Delta M_{di}(t), i\in\Xi,$$

where D_i and E_i are constant matrices with appropriate dimensions, $\Sigma_i(t)$ are unknown matrices with Lebesgue measurable elements and satisfy $\Sigma_i^{\mathrm{T}}(t)\Sigma_i(t)\le I$, $\Delta M_{di}(t)$ are unknown but bounded as $\|\Delta M_{di}(t)\|\le a_i$ with known nonnegative constants a_i.

Assumption 2. There exist known nonnegative scalar-valued functions $\phi_i(x,t)$, $i\in\Xi$ such that $\|f_i(x,t)\|\le\phi_i(x,t)$.

A switching sequence is expressed by

$$\Psi = \{x_0; (i_0,t_0), (i_1,t_1), \ldots, (i_j,t_j), \ldots, |\, i_j\in\Xi, j\in Z^+\}.$$

In which t_0 is the initial time, x_0 is the initial state, (i_k, t_k) means that the $i_k - th$ subsystem is activated for $[t_k, t_{k+1})$.

Lemma 1 ([15]). Given real matrices R_1 and R_2 with appropriate dimensions and an unknown matrix $\Sigma(t)$ with Lebesgue measurable elements such that $\Sigma^{\mathrm{T}}(t)\Sigma(t)\le I$, then we have

$$R_1\Sigma R_2 + R_1^{\mathrm{T}}\Sigma^{\mathrm{T}}R_2^{\mathrm{T}} \le \beta R_1R_1^{\mathrm{T}} + \beta^{-1}R_2^{\mathrm{T}}R_2,$$

where $\beta>0$.

III. Main Results

The objective in this paper is to design a switching law $\sigma(t)$ and an associated controller u_σ such that the system (1) is stabilizable with an H_∞-norm bound. To formulate the problem clearly, we give the following definition.

Definition 1. Given a constant $\gamma>0$, the uncertain switched delay system (1) is said to be robust stabilizable with H_∞ disturbance attenuation level γ if there exists a switching law $\sigma(t)$ and an associated controller u_σ such that

i). the resulting closed-loop system of the system (1) with $\omega(t)=0$ is stable for all admissible uncertainties;

ii). with zero-initial condition $\varphi(\theta)=0,\ \theta\in[-\bar{\tau},0]$, $\|z(t)\|_2 < \gamma\|\omega(t)\|_2$ holds for all admissible uncertainties and all nonzero $\omega\in L_2[0,\infty)$.

For simplicity of presentation only the case for $k=2$ is considered in here. It is easy and straightforward to extend the result to the case for $k>2$.

First, we design a switching law $\sigma(t)$ and an associated controller u_σ such that the system (1) with $\omega(t)=0$ is stabilizable via the multiple Lyapunov functions method.

The following lemma is important to develop our results.

Lemma 2. Suppose Assumptions 1 and 2 hold. The system (1) with $\omega(t)=0$ is stabilizable if there exist matrices $P_i>0$, $Q_i<0$, $Q>0$, scalars $\alpha_i<0$, $\eta_i>0$ $(i=1,2)$, $\varepsilon>0$ such that the following matrix inequalities

$$\begin{bmatrix}\Theta_1 + Q + \alpha_1(P_1 - P_2 + \eta_1 Q_1) & P_1A_{d1}\\ A_{d1}^{\mathrm{T}}P_1 & -Q\end{bmatrix} < 0 \qquad \text{(2-a)}$$

and

$$\begin{bmatrix}\Theta_2 + Q + \alpha_2(P_2 - P_1 + \eta_2 Q_2) & P_2A_{d2}\\ A_{d2}^{\mathrm{T}}P_2 & -Q\end{bmatrix} < 0 \qquad \text{(2-b)}$$

are satisfied with $\Theta_i = A_i^{\mathrm{T}}P_i + P_iA_i - P_iB_iB_i^{\mathrm{T}}P_i + \varepsilon^{-1}P_iD_iD_i^{\mathrm{T}}P_i + \varepsilon E_i^{\mathrm{T}}E_i$ $(i=1,2)$. The stabilizing variable structure controlers for subsystems are given by

$$u_i(t) = -\frac{1}{2}B_i^{\mathrm{T}}P_ix(t) - \hat{u}_i(t), \qquad (3)$$

where $\hat{u}_i(t) = (\lambda a_i\|x(t)\| + \phi_i(x,t) + \mu)sign(s_i(t))$, $s_i(t) = B_i^{\mathrm{T}}P_ix(t)$, μ is positive constant.

Proof. We define regions

$$\Phi_1 = \{x(t)\,|\,x^{\mathrm{T}}(t)(P_1 - P_2 + \eta_1Q_1)x(t)\le 0\},$$
$$\Phi_2 = \{x(t)\,|\,x^{\mathrm{T}}(t)(P_2 - P_1 + \eta_2Q_2)x(t)\le 0\}. \qquad (4)$$

Obviously $\Phi_1\cup\Phi_2 = R^n$.

Then, we design a hysteresis switching law by

$$\sigma = \begin{cases}1, if\,(x(0)\in\Phi_1)\ or\ (x(t)\in\Phi_1\ and\ \sigma(t^-)=1)\\ \quad or(x(t)\notin\Phi_2\ and\ \sigma(t^-)=2),\\ 2, if\,(x(0)\notin\Phi_1)\ or\ (x(t)\in\Phi_2\ and\ \sigma(t^-)=2)\\ \quad or(x(t)\notin\Phi_1\ and\ \sigma(t^-)=1).\end{cases} \qquad (5)$$

Choose the Lyapunov functional candidate

$$V_i(t) = x^{\mathrm{T}}(t)P_i x(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(\theta)Qx(\theta)d\theta \,, \tag{6}$$

where P_i $(i=1,2)$, Q satisfy (2-a) and (2-b). Differentiating (6) with respect to t, we obtain

$$\dot{V}_i(t) = 2x^{\mathrm{T}}(t)P_i\dot{x}(t) + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau) \,. \tag{7}$$

Using (3) in the systems (1) with $\omega(t)=0$, we have

$$\begin{aligned}\dot{x}(t) &= (A_i - \frac{1}{2}B_iB_i^{\mathrm{T}}P_i + \Delta A_i)x(t) + (A_{di} + \Delta A_{di})x(t-\tau) \\ &\quad - B_i\hat{u}_i(t) + B_i f_i(x,t).\end{aligned} \tag{8}$$

Let $\xi(t) = \left[x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau)\right]^{\mathrm{T}}$. Substituting (8) into (7) and rearranging terms, we have

$$\begin{aligned}\dot{V}_i(t) &= \xi^{\mathrm{T}}(t)\Pi_i\xi(t) + 2x^{\mathrm{T}}(t)P_iB_i\Delta M_{di}x(t-\tau) \\ &\quad - 2x^{\mathrm{T}}(t)P_iB_i\hat{u}_i(t) + 2x^{\mathrm{T}}(t)P_iB_if_i(x,t),\end{aligned} \tag{9}$$

where

$$\Pi_i = \begin{bmatrix} A_i^{\mathrm{T}}P_i + P_iA_i - P_iB_iB_i^{\mathrm{T}}P_i + Q + \Delta A_i^{\mathrm{T}}P_i + P_i\Delta A_i & P_iA_{di} \\ A_{di}^{\mathrm{T}}P_i & -Q \end{bmatrix}.$$

It follows from Lemma 1 that

$$\Delta A_i^{\mathrm{T}}P_i + P_i\Delta A_i \le \varepsilon^{-1}P_iD_iD_i^{\mathrm{T}}P_i + \varepsilon E_i^{\mathrm{T}}E_i \,.$$

Applying the Razumikin theorem [16] we know that for any solution $x(t-\tau)$ of (1) there exists a constant $\lambda > 1$ such that

$$\|x(t-\tau)\| < \lambda\|x(t)\|, 0 \le \tau < \bar{\tau} \,. \tag{10}$$

Thus, it holds that

$$\dot{V}_{i_j}(t) < 0 \,,\; [t_j, t_{j+1}) \,.$$

Let

$$\tilde{\Phi}_1 = \{x(t) \,|\, x^{\mathrm{T}}(t)(P_1 - P_2 + \eta_1 Q_1)x(t) = 0\} \,,$$

$$\tilde{\Phi}_2 = \{x(t) \,|\, x^{\mathrm{T}}(t)(P_2 - P_1 + \eta_2 Q_2)x(t) = 0\} \,. \tag{11}$$

Obviously, $\tilde{\Phi}_i$ is the boundary of $\Phi_i (i=1,2)$.

According to the switching law (5), if $\sigma(t^-)=1$ and $x(t)\in\Phi_1$, then the trajectory will remains in Φ_1 until it hits the boundary $\tilde{\Phi}_1$. This means that switching only takes place on $\tilde{\Phi}_1$ or $\tilde{\Phi}_2$. In fact, $x(t_k)\in\tilde{\Phi}_1$ means that $x^{\mathrm{T}}(t_k)(P_1 - P_2 + \eta_1 Q_1)x(t_k) = 0$ and $x(t_k)\in\Phi_2$. Thus,

$$x^{\mathrm{T}}(t_k)P_1x(t_k) = x^{\mathrm{T}}(t_k)(P_2 - \eta_1 Q_1)x(t_k) > x^{\mathrm{T}}(t_k)P_2x(t_k) \,.$$

Then according to the switching law (5), at each switching time t_j,

$$V_{i_{j+1}}(t_j) < V_{i_j}(t_j)$$

is true. In view of the multiple Lyapunov functions method, the system (1) with $\omega(t)=0$ is asymptotically stabilizable. This completes the proof.

Remark 1. The **c**onventional state-depended switching rules appeared in many references [4, 17], which may result in sliding motions in switching surface. From the proof of Lemma 2, it is obviously that regions Φ_1 and Φ_2 overlap, thus we refer [6] to introduce hysteresis to avoid sliding motions by design a hysteresis switching law.

Next, based on the previous arguments, we consider stabilization with H_∞ disturbance attenuation level γ of the uncertain switched delay system (1).

Theorem 1. Suppose Assumptions 1 and 2 hold. Given a constant $\gamma > 0$, the uncertain switched delay system (1) is stabilizable with H_∞ disturbance attenuation level γ, if there exist matrices $P_i > 0$, $Q_i < 0$, $Q > 0$ and scalars $\alpha_i < 0$, $\eta_i > 0$ $(i=1,2)$, $\varepsilon > 0$ such that the following matrix inequalities

$$\begin{bmatrix} W_1 + Q + \alpha_1(P_1 - P_2 + \eta_1 Q_1) & P_1A_{d1} \\ A_{d1}^{\mathrm{T}}P_1 & -Q \end{bmatrix} < 0 \tag{12-a}$$

and

$$\begin{bmatrix} W_2 + Q + \alpha_2(P_2 - P_1 + \eta_2 Q_2) & P_2A_{d2} \\ A_{d2}^{\mathrm{T}}P_2 & -Q \end{bmatrix} < 0 \tag{12-b}$$

are satisfied with $W_i = A_i^{\mathrm{T}}P_i + P_iA_i - P_iB_iB_i^{\mathrm{T}}P_i + C_i^{\mathrm{T}}C_i + \varepsilon^{-1}P_iD_iD_i^{\mathrm{T}}P_i + \varepsilon E_i^{\mathrm{T}}E_i + \gamma^{-2}P_iG_iG_i^{\mathrm{T}}P_i$ $(i=1,2)$. In this case, the switching law $\sigma(t)$ and the controllers u_i are taken as (5) and (3) respectively where P_i $(i=1,2)$, Q satisfy (12-a) and (12-b).

Proof. First, take the multiple Lyapunov functions $V_i(t)$ $(i=1,2)$ as (6), where P_i, Q satisfy (12-a) and (12-b). By Lemma 2, the system (1) is asymptotically stabilizable with $\omega(t)=0$ for all admissible uncertainties.

Secondly, under zero initial condition, introduce the performance

$$J = \int_0^\infty (z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega)dt \,. \tag{13}$$

Without loss of generality, we assume that the subsystem 1 is first activated, and the switching sequence is

$$\Sigma \,\big|\, \{t_0, t_1, t_2, t_3, \ldots\} \,. \tag{14}$$

The subsystem 1 is activated on $[t_{2m}, t_{2m+1})$, the subsystem 2 is activated on $[t_{2m+1}, t_{2m+2})$, where $m \in Z^+$, $t_0 = 0$. Then for $\forall\omega \in L_2[0,\infty)$, we have

$$\begin{aligned} J &= \int_{t_0}^{t_1}(z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega + \frac{d}{dt}V_1)dt - V_1(x(t_1)) + V_1(x(t_0)) \\ &\quad + \int_{t_1}^{t_2}(z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega + \frac{d}{dt}V_2)dt - V_2(x(t_2)) + V_2(x(t_1)) \\ &\quad + \int_{t_2}^{t_3}(z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega + \frac{d}{dt}V_1)dt - V_1(x(t_3)) + V_1(x(t_2)) + \ldots \\ &\le \int_{t_0}^{t_1}(z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega + \frac{d}{dt}V_1)dt + \int_{t_1}^{t_2}(z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega + \frac{d}{dt}V_2)dt \\ &\quad + \int_{t_2}^{t_3}(z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega + \frac{d}{dt}V_1)dt + \ldots, \end{aligned} \tag{15}$$

where $V_i(t)$ $(i=1,2)$, as (6) P_i, Q satisfy (12-a) and (12-b). Then the right hand side of (15) can be written as

$$\sum_{0}^{\infty}\int_{t_{2m}}^{t_{2m+1}}(z^{\mathrm T}z-\gamma^2\omega^{\mathrm T}\omega+\frac{d}{dt}V_1)dt \\ +\sum_{0}^{\infty}\int_{t_{2m+1}}^{t_{2m+2}}(z^{\mathrm T}z-\gamma^2\omega^{\mathrm T}\omega+\frac{d}{dt}V_2)dt. \tag{16}$$

Thus, during any $[t_k, t_{k+1})$, $k \in Z^+$, it holds that

$$z^{\mathrm T}z-\gamma^2\omega^{\mathrm T}\omega+\frac{d}{dt}V_i \\ \le \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^{\mathrm T}\begin{bmatrix} W_i+Q & P_iA_{di} \\ A_{di}^{\mathrm T}P_i & -Q\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix} \\ -(\gamma^{-1}G_i^{\mathrm T}P_ix(t)-\gamma\omega)^{\mathrm T}(\gamma^{-1}G_i^{\mathrm T}P_ix(t)-\gamma\omega)<0.$$

In view of $V_{\sigma(t_{k+1})}(t_k) < V_{\sigma(t_k)}(t_k)$, we have

$$J<0$$

for $\forall\omega \in L_2[0,\infty)$ and all admissible uncertainties. That is $\|z(t)\|_2 < \gamma\|\omega(t)\|_2$. This completes the proof.

From practical point of view, the bounds $\phi_i(x,t)$ of the disturbances $f_i(x,t)$ are not easy to known. In order to handle this problem, we introduce the following assumption.

Assumption 3. There exist unknown positive constants g_{i1} and g_{i2} $(i=1,2)$ such that

$$\|f_i(x,t)\| \le g_{i1}+g_{i2}\|x(t)\|. \tag{17}$$

We now have to deal with g_{i1} and g_{i2}. To this end, we employ parameter estimates $\hat{g}_{i1}$ and $\hat{g}_{i2}$ to adapt the unknown constant parameters g_{i1} and g_{i2}, respectively. The adaptation error of each parameter estimated is defined as $\tilde{g}_{i1}=\hat{g}_{i1}-g_{i1}$ and $\tilde{g}_{i2}=\hat{g}_{i2}-g_{i2}$.

Theorem 2. Suppose Assumptions 1 and 3 hold. Given a constant $\gamma>0$, the uncertain switched delay system (1) is stabilizable with H_∞ disturbance attenuation level γ, if the matrix inequalities (12-a) and (12-b) are feasible. The switching law is taken as (5) and the stabilizing adaptive variable structure controllers for subsystems are given by

$$u_i(t)=-\frac{1}{2}B_i^{\mathrm T}P_ix(t)-\tilde{u}_i(t), \tag{18}$$

where $\tilde{u}_i(t)=(\lambda a_i\|x(t)\|+\hat{g}_{i1}+\hat{g}_{i2}\|x(t)\|+\mu)sign(s_i(t))$, $s_i(t) = B_i^{\mathrm T}P_ix(t)$ and λ, μ are positive constants, P_i, $(i=1,2)$ Q satisfy (12-a) and (12-b). The parameter update laws of the $i-th$ subsystem are chosen as

$$\dot{\hat{g}}_{i1}=\begin{cases}0, \sigma\ne i, \\ \beta_1\|s_i(t)\|, \sigma=i,\end{cases} \quad \dot{\hat{g}}_{i2}=\begin{cases}0, \sigma\ne i, \\ \beta_2\|s_i(t)\|\|x(t)\|, \sigma=i,\end{cases} \tag{19}$$

where β_1 and β_2 are positive constants specified by the designer.

Proof. We choose the Lyapunov functional candidate

$$V_i(t)=x^{\mathrm T}(t)P_ix(t)+\int_{t-\tau}^{t}x^{\mathrm T}(\theta)Qx(\theta)d\theta \\ +\beta_1^{-1}\sum_{j\in\Xi}\tilde{g}_{j1}^{\,2}+\beta_2^{-1}\sum_{j\in\Xi}\tilde{g}_{j2}^{\,2}, (i=1,2), \tag{20}$$

where P_i, Q satisfy (12-a) and (12-b).

In subsequent arguments, we shall first verify the stabilizability of the system (1) with $\omega(t)=0$.

Note that

$$\dot{\tilde{g}}_{i1}=\dot{\hat{g}}_{i1},\ \dot{\tilde{g}}_{i2}=\dot{\hat{g}}_{i2}. \tag{21}$$

Differentiating (20) with respect to t, we obtain

$$\dot{V}_i(t)=2x^{\mathrm T}(t)P_i\dot{x}(t)+x^{\mathrm T}(t)Qx(t)-x^{\mathrm T}(t-\tau)Qx(t-\tau) \\ +2\beta_1^{-1}\sum_{j\in\Xi}\tilde{g}_{j1}\dot{\hat{g}}_{j1}+2\beta_2^{-1}\sum_{j\in\Xi}\tilde{g}_{j2}\dot{\hat{g}}_{j2}. \tag{22}$$

Substituting (18) into the system (1), gives

$$\dot{x}(t)=(A_i-\frac{1}{2}B_iB_i^{\mathrm T}P_i+\Delta A_i)x(t)+(A_{di}+\Delta A_{di})x(t-\tau) \\ -B_i\tilde{u}_i(t)+B_if_i(x,t). \tag{23}$$

Substituting (23) into (22) and rearranging terms, we have

$$\dot{V}_i(t)=\xi^{\mathrm T}(t)\Pi_i\xi(t)+2x^{\mathrm T}(t)P_iB_i\Delta M_{di}x(t-\tau) \\ -2x^{\mathrm T}(t)P_iB_i\tilde{u}_i(t)+2x^{\mathrm T}(t)P_iB_if_i(x,t) \\ +2\beta_1^{-1}(\hat{g}_{i1}-g_{i1})\dot{\hat{g}}_{i1}+2\beta_2^{-1}(\hat{g}_{i2}-g_{i2})\dot{\hat{g}}_{i2}, \tag{24}$$

where $\xi(t)=[x^{\mathrm T}(t)\quad x^{\mathrm T}(t-\tau)]^{\mathrm T}$, Π_i are defined as (9).

Thus, it holds that

$$\dot{V}_{i_j}(t)<0,\ [t_j, t_{j+1}). \tag{25}$$

Note that $\sum_{j\in\Xi}\tilde{g}_{j1}^{\,2}$ and $\sum_{j\in\Xi}\tilde{g}_{j2}^{\,2}$ are continuous all time. Similar to the proof of Lemma 1, at each switching time t_j, we have

$$V_{i_{j+1}}(t_j)<V_{i_j}(t_j).$$

In view of multiple Lyapunov functions method, the system (1) with $\omega(t)=0$ is asymptotically stabilizable.

Secondly, let

$$J=\int_0^\infty(z^{\mathrm T}z-\gamma^2\omega^{\mathrm T}\omega)dt. \tag{26}$$

Without loss of generality, we assume that the subsystem 1 is first activated and the switching sequence is expressed as (14). Then, similar to the proof of Theorem 1, we have

$$J\le\sum_{0}^{\infty}\int_{t_{2m}}^{t_{2m+1}}(z^{\mathrm T}z-\gamma^2\omega^{\mathrm T}\omega+\frac{d}{dt}V_1)dt \\ +\sum_{0}^{\infty}\int_{t_{2m+1}}^{t_{2m+2}}(z^{\mathrm T}z-\gamma^2\omega^{\mathrm T}\omega+\frac{d}{dt}V_2)dt, \tag{27}$$

where $V_i(t)$ $(i=1,2)$ as (19).

Similar to the proof of Theorem 1, in view of $V_{\sigma(t_{k+1})}(t_k) < V_{\sigma(t_k)}(t_k)$, we have

$$J<0$$

for $\forall\omega \in L_2[0,\infty)$ and all admissible uncertainties. That is $\|z(t)\|_2 < \gamma\|\omega(t)\|_2$. This completes the proof.

Remark 2. From practical of point, chattering phenomenon may occur around $\|x(t)\|=0$. Therefore, $\|x(t)\|$ can not be precisely guaranteed equal to zero. However, according to (19), $\hat{g}_{i1}$ and $\hat{g}_{i2}$ will be increasing all the time as long as $\|x(t)\|\ne 0$. Motivated by [18], to overcome this problem, we may apply the following modified parameter update laws

$$\dot{\hat{g}}_{i1}=\begin{cases}0, if\begin{cases}\sigma\neq i,\\ \|x(t)\|<d,\sigma=i,\end{cases}\\ \beta_1\|s_i(t)\|, if\sigma=i,\end{cases}\quad \dot{\hat{g}}_{i2}=\begin{cases}0, if\begin{cases}\sigma\neq i,\\ \|x(t)\|<d,\sigma=i,\end{cases}\\ \beta_2\|s_i(t)\|\cdot\|x(t)\|, if\sigma=i.\end{cases}\tag{28}$$

Remark 3. The use of sign function $sign(x)$ may causes chattering effect. This may be undesirable in practical engineering systems. To overcome this drawback, the function $\tanh(x)$ can be used to replace the function $sign(x)$ [19].

IV. EXAMPLES

In this section, we present a numerical example to demonstrate the effectiveness of proposed design method.

Consider the following uncertain switched delay system

$$\begin{aligned}\dot{x}(t)&=(A_\sigma+\Delta A_\sigma)x(t)+(A_{d\sigma}+\Delta A_{d\sigma})x(t-\tau)\\&+B_\sigma(u_\sigma+f_\sigma(x,t))+G_\sigma\omega(t),\\ x(t)&=\varphi(t),\ t\in[-\tau,0], z(t)=C_\sigma x(t),\end{aligned}\tag{29}$$

where $\sigma(t)\in\Xi=\{1,2\}$, $\tau\leq 1$,

$A_1=\begin{bmatrix}-5 & 1\\ 0.1 & -2\end{bmatrix}$, $A_2=\begin{bmatrix}-2 & 1\\ -5 & -4\end{bmatrix}$, $A_{d1}=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$, $A_{d2}=\begin{bmatrix}2 & 3\\ 2 & 2\end{bmatrix}$, $B_1=\begin{bmatrix}1\\ 0.5\end{bmatrix}$, $B_2=\begin{bmatrix}1\\ 0\end{bmatrix}$, $C_1=\begin{bmatrix}0.1\\ 0\end{bmatrix}^{\mathrm{T}}$, $C_2=\begin{bmatrix}0\\ 0.1\end{bmatrix}^{\mathrm{T}}$, $G_1=\begin{bmatrix}1\\ 0.5\end{bmatrix}$, $G_2=\begin{bmatrix}1\\ 0.5\end{bmatrix}$, the state uncertainties $\Delta A_1=D_1\Sigma_1(t)E_1$, $\Delta A_2=D_2\Sigma_2(t)E_2$, where $E_1=\begin{bmatrix}0.1 & 0\end{bmatrix}$, $D_1=\begin{bmatrix}-0.1 & 0.2\\ 0.1 & 0.1\end{bmatrix}$, $\Sigma_1=\begin{bmatrix}0.5\nu_{11} & 0.5\nu_{12}\end{bmatrix}$, $\nu_{11},\nu_{12}\in[-1,1]$, $D_2=\begin{bmatrix}0.2 & 0.1\\ -0.1 & 0.1\end{bmatrix}$, $E_2=\begin{bmatrix}0\\ 0.1\end{bmatrix}^{\mathrm{T}}$, $\Sigma_2=\begin{bmatrix}0.5\nu_{21} & 0.5\nu_{22}\end{bmatrix}$, ν_{21}, $\nu_{22}\in[-1,1]$, the delay state uncertainties $\Delta A_{d1}=\begin{bmatrix}1\\ 0.5\end{bmatrix}2\nu_1\times\begin{bmatrix}1 & 2\end{bmatrix}$, $\nu_1\in[-1,1]$, $\Delta A_{d2}=\begin{bmatrix}1\\ 0\end{bmatrix}2\nu_2\begin{bmatrix}1 & 2\end{bmatrix}$, $\nu_2\in[-1,1]$ the unknown nonlinear functions $f_1=0.5x_1\cos(x_2)-0.5\sin(x_1+x_2)$, $f_2=1.5x_1x_2+1.5\sin(x_1+x_2)$.

We adopt adaptive control to estimate the upper bounds of the disturbances. It is easy to verify that the conditions of Theorem 2 are satisfied. Let $\varepsilon=0.1$, $\alpha_1=\alpha_2=-5$, $\eta_1=\eta_2=0.1$. The disturbance attenuation level is given by $\gamma=1/\sqrt{2}$. Then solving (12-a), (12-b), leads to

$$P_1=\begin{bmatrix}0.1183 & 0.1126\\ 0.1126 & 0.3687\end{bmatrix},\ P_2=\begin{bmatrix}0.1415 & 0.0313\\ 0.0313 & 0.263\end{bmatrix},$$

$$Q=\begin{bmatrix}0.5483 & 0.4009\\ 0.4009 & 0.6249\end{bmatrix},\ Q_1=\begin{bmatrix}-0.0424 & 0.0363\\ 0.0363 & -0.1016\end{bmatrix},$$

$$Q_2=\begin{bmatrix}-0.0266 & -0.0057\\ -0.0057 & -0.0873\end{bmatrix}.$$

Let $\mu=2$. According to (18), the adaptive variable structure controllers are given by

$$\begin{aligned}u_1&=-0.0873x_1-0.1485x_2-(6\|x(t)\|\\&+\hat{g}_{11}+\hat{g}_{12}\|x(t)\|+2)sign(s_1),\\ u_2&=-0.0707x_1-0.0157x_2-(8\|x(t)\|\\&+\hat{g}_{21}+\hat{g}_{22}\|x(t)\|+2)sign(s_2).\end{aligned}\tag{30}$$

The parameter update laws are designed as (28), with $d=0.04$, $\beta_1=\beta_2=0.1$.

$$\begin{aligned}&\dot{\hat{g}}_{11}=\begin{cases}0, if\begin{cases}\sigma\neq 1,\\ \|x(t)\|<d,\sigma=1,\end{cases}\\ \beta_1\|s_1(t)\|, if\sigma=1,\end{cases}\ \dot{\hat{g}}_{12}=\begin{cases}0, if\begin{cases}\sigma\neq 1,\\ \|x(t)\|<d,\sigma=1,\end{cases}\\ \beta_2\|s_1(t)\|\cdot\|x(t)\|, if\sigma=1,\end{cases}\\ &\dot{\hat{g}}_{21}=\begin{cases}0, if\begin{cases}\sigma\neq 2,\\ \|x(t)\|<d,\sigma=2,\end{cases}\\ \beta_1\|s_2(t)\|, if\sigma=2,\end{cases}\ \dot{\hat{g}}_{22}=\begin{cases}0, if\begin{cases}\sigma\neq 2,\\ \|x(t)\|<d,\sigma=2,\end{cases}\\ \beta_2\|s_2(t)\|\|x(t)\|, if\sigma=2,\end{cases}\end{aligned}\tag{31}$$

where $s_1=[0.1747,0.297]x(t)$, $s_2=[0.1415,0.0313]x(t)$.

By (5), we design the following switching law with initial state $x_0=[2,-2]^{\mathrm{T}}$

$\sigma(0)=1$,

$$\sigma=\begin{cases}1, if(x(t)\in\Phi_1\ and\ \sigma(t^-)=1)or(x(t)\notin\Phi_2\ and\ \sigma(t^-)\neq 1),\\ 2, if(x(t)\in\Phi_2\ and\ \sigma(t^-)=2)or(x(t)\notin\Phi_1\ and\ \sigma(t^-)\neq 2).\end{cases}\tag{32}$$

where $\Phi_1=\left\{x(t)\middle|x^{\mathrm{T}}(t)\begin{bmatrix}-0.0274 & 0.085\\ 0.085 & 0.0955\end{bmatrix}x(t)\right\}<0$,

$$\Phi_2=\left\{x(t)\middle|x^{\mathrm{T}}(t)\begin{bmatrix}0.0205 & -0.0819\\ -0.0819 & -0.1144\end{bmatrix}x(t)\right\}<0.$$

The simulation results are depicted in Fig. 1-Fig. 4.

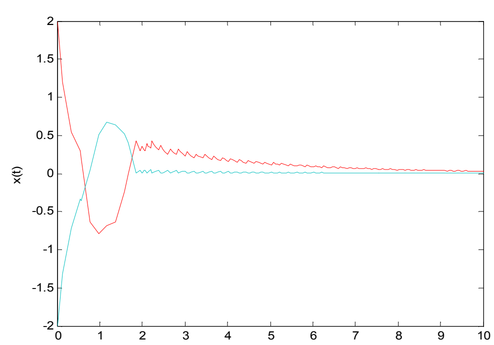

Fig. 1. The state responses of the switched system (29)

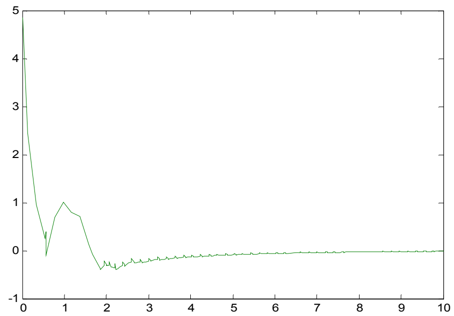

Fig. 2. The input signal of the switched system (29)

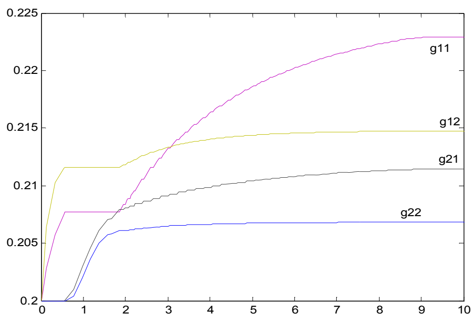

Fig. 3. The trajectories of the parameter update laws (30)

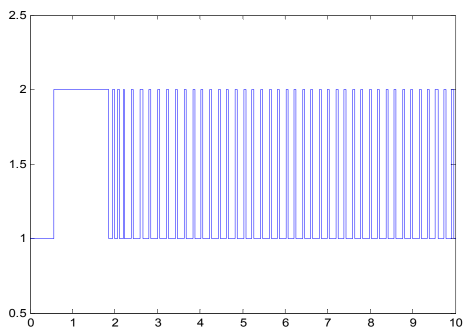

Fig. 4. The switching signal (32)

The system states in the closed-loop are shown in Fig. 1. It is clearly seen that the closed-loop system of the switched system (29) with the designed controllers (30), (31) and the switching law (32) is asymptotically stable. Fig. 2 is the input signal of the switched system (29). Fig. 3 gives the estimations of the unknown nonlinear disturbances. The switching signal is given by Fig. 4.

V. Conclusion

In this paper, the problem of robust H_∞ control has been studied for a class of uncertain switched delay systems with uncertainties and unknown nonlinear disturbances. Based on the multiple Lyapunov functions method, the sufficient conditions are derived for robust stability with prescribed disturbance attenuation level γ. The hysteresis switching law has been designed. The variable structure control strategy and the adaptive variable structure control strategy have been developed to stabilize the uncertain switched delay system with H_∞ disturbance attenuation level γ under the hysteresis switching law for the cases of known and unknown upper bounds of the nonlinear disturbances respectively.

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