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Semi-algebraic Problem Approach for Stability Analysis of a Class of Nonlinear Stochastic Delay System

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Abstract— **In this paper we give a delay dependent stability criterion for stochastic delay systems with *polynomial nonlinearity*. An important benefit of this result is that when an invariant set of the system is available, we can incorporate this information to reduce the conservativeness. The resulting condition is given in terms of a semi-algebraic problem which is known to be efficiently solvable via sums of squares (SOS) relaxations. This work is originally motivated by the design of quantum spin control systems in the face of feedback delays. The effectiveness of the proposed method is evaluated by designing a globally stabilizing control law for the spin-1/2 system.**

Index Terms— **Stochastic delay systems, Sum of squares, Polynomial nonlinearity**

I. Introduction

Stochastic systems have attracted a renewed interest due to their relation to systems biology, financial engineering, quantum mechanical systems, indeterministic behavior of communication networks such as delays and packet loss, systems with Markov switching and so on. In this paper, we derive a new delay-dependent stability criterion for a class of nonlinear stochastic system. In recent years numerous stability criteria for several type of delay systems have been derived. Some of them are applicable to stochastic delay differential systems; see e.g., [7], [9], [22], [24] and references therein.

The original motivation of this work is control of quantum spin systems which is explained in detail in the next section. To deal with this problem, we focus on the delay systems which have the following properties:

1) dynamical equation (more precisely, stochastic delay differential equation) includes polynomial coefficients,
2) systems evolve in a prespecified semi-algebraic set C, i.e., a subset of $\mathbb{R}^n$ defined by

$$\mathsf{C} = \{x \in \mathbb{R}^n : p_i(x) \geq 0,\ i = 1, 2, \cdots, l\} \qquad (1)$$

with some n-variable polynomials p_i.

Firstly, systems with polynomial nonlinearity can cover a wide class of nonlinear system. The main result in [22] is capable of dealing with *norm bounded* nonlinearity, i.e., function $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists a matrix F satisfying

$$\|f(x)\| \leq \|Fx\|$$

where the norm is the standard Euclidean norm. As is seen in the stability analysis of Lur'e systems, it is not so hard to derive sufficient conditions for the stability of this type of nonlinear system. However, if one deals with polynomial nonlinearity as a norm bounded one, this often introduces excessive conservativeness.

Secondly, a good example of such systems are *positive systems*, such as *compartmental systems*, whose state variables do not exit the positive orthant; see e.g., [4]. There also exist many physical systems with signals that take only bounded values. Such a priori knowledge about the region is expected to be useful for less conservative analysis.

Among existing results, delay-dependent stability criteria in terms of LMIs are the most attractive due to their low computational complexity. However, this kind of approach is not suitable for our problem formulation. One reason is that it is impossible to deal with the polynomial nonlinearity. Another is the small freedom of the choice of Lyapunov-Krasovskii functionals; to obtain LMI conditions, Lyapunov-Krasovskii functionals consisting of quadratic forms of state variables and its (double) integrations are necessary. When we need to search for functionals that are globally positive, this restriction may not be too conservative. However, the systems of interest in this paper evolve only in a prespecified region, and consequently we do not need globally positive Lyapunov-Krasovsii functionals. For example, any polynomial with positive coefficients is positive in the positive orthant while it is not globally positive in general.

In view of these facts, we derive a delay-dependent stability criterion in the form of *semi-algebraic problem*. This approach has already been applied for the stability analysis of (deterministic) delay systems by Papachristodoulou et al. [11], [12]. The differences between these existing results and the main result in this paper are

- we deal with stochastic systems, and
- we can easily incorporate the information about the region where the system evolves.

This paper is organized as follows. Section II is devoted to a review of our motivating example, control of quantum spin systems. Section III is the main part of this paper; we provide a delay dependent stability criterion for the class of nonlinear stochastic delay system with the properties stated above. The effectiveness of the result is illustrated by designing a globally stabilizing controller for the spin-$1/2$ particle which takes the feedback delay into explicit account.

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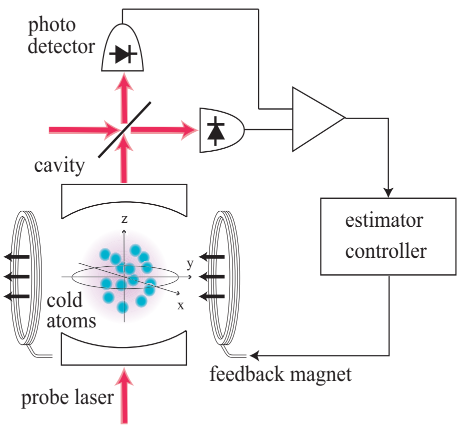

Fig. 1. Control of quantum spin systems by using continuous measurement

Notation

For $z \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$, $\|z\|_M^2 := z^{\mathsf{T}} M z$. The subscript is omitted when M is the identity matrix. For $\mathsf{D} \subset \mathbb{R}^m$, a function $F : \mathsf{D} \to \mathbb{R}$ is said to be *positive* (resp. *negative*) in D if $F(z) \geq 0$ (resp. $F(z) \leq 0$) for any $z \in \mathsf{D}$.

Let C_{C}^h be the set of C-valued uniformly continuous functions on $[-h, 0]$. This is a Banach space equipped with $\|\tilde{x}\|_{C_{\mathsf{C}}^h} := \sup_{\theta\in[-h,0]} \|\tilde{x}(\theta)\|_{\mathsf{C}}$. The 1-dimensional standard Wiener process is denoted by w_t. The set of $\mathcal{W}_0$-measurable C_{C}^h-valued random variables is denoted by $C_{\mathsf{C},\mathcal{W}_0}^h$ where $\mathcal{W}_0$ is the σ-algebra generated by w_0. Expectation is denoted by $\mathbb{E}$. If it exists, the *infinitesimal generator* of a function V along a Markov process $\tilde{x}_t$ is denoted by $\mathscr{A}V$ i.e., $\mathscr{A}V(\tilde{x}) := \lim_{t\to 0} \frac{\mathbb{E}^{\tilde{x}}[\tilde{x}_t] - \tilde{x}}{t}$ where $\mathbb{E}^{\tilde{x}}$ represents the expectation with respect to paths which start at $\tilde{x}_0 = \tilde{x}$; see [9], [7], [22] for a formula.

II. MOTIVATING EXAMPLE: CONTROL OF QUANTUM SPIN

In this section, we consider a cold atomic ensemble trapped in an optical cavity [3], [23], [8] depicted in Figure 1. The total angular momentum of the atom along the i-axis ($i = x, y, z$), denoted by F_i, is a quantum observable. We can partially observe the quantum state by shining a laser (along the z-axis) through the cavity at a homodyne-type photo detector. The system is affected by an external magnetic field along the y-axis where the magnetic field strength u_t is used as a time-varying control input.

The state variable is (conditional) *density matrix* ρ_t that belongs to the following convex set:

$$\mathcal{S} := \{\rho \in \mathbb{C}^{N_s\times N_s} : \rho = \rho^* \geq 0,\ \operatorname{tr}\rho = 1\}. \tag{2}$$

The dynamics of control quantum spin system is described by the following stochastic differential equation called the *Belavkin equation* or *stochastic master equation*:

$$\mathrm{d}\rho_t = \mathrm{i}[u_t F_y, \rho_t]\mathrm{d}t - \frac{1}{2}[F_z, [F_z, \rho_t]]\mathrm{d}t + \sqrt{\eta}\big[F_z\rho_t + \rho_t F_z - 2\operatorname{tr}(F_z\rho_t)\rho_t\big]\mathrm{d}w_t, \tag{3}$$

where $\eta \in (0, 1]$ represents the measurement efficiency and

$$F_y := \frac{\mathrm{i}}{2}\begin{bmatrix} 0 & c_1 & & & \\ -c_1 & 0 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{N_s-2} & 0 & c_{N_s-1} \\ & & & -c_{N_s-1} & 0 \end{bmatrix},$$

$$c_m := \sqrt{(N_s - m)m}, \quad m = 1, 2, \cdots, N_s - 1$$

$$F_z := \frac{1}{2}\operatorname{diag}\{N_s - 1, N_s - 3, \cdots, -(N_s - 3), -(N_s - 1)\}$$

Our goal is to determine a control input u_t such that ρ_t shows a desirable behavior. However, nano-mechanical dynamic systems have very fast dynamics, with time constants orders of magnitude less than the time necessary to compute the control input. From a practical viewpoint, this means that we need to formulate the control problem stated above taking feedback delays into consideration. Actually, such delays are known to degrade the control performance and the dynamical stability. In view of this fact, we should consider the delayed feedback control input $u_t = u(\rho_{t-\tau})$, where $\tau > 0$ means the delay. Note that τ can includes another possible delay that occurs when physically activating the controller as a function of u_t.

Note that a control system which takes into account these delays can be described by using stochastic delay differential systems. Moreover, we can restrict our analysis to the class of system which have polynomial coefficients and evolve in a bounded set. Let us see this point in more detail. System Eq. (3) has the complex matrix-valued state variable ρ_t. By concatenating the real and imaginary part of all elements into a column vector, we can rewrite (3) as a real vector-valued nonlinear stochastic system with polynomial coefficients. In other words, any Hermitian matrix $\rho \in \mathbb{C}^{N_s\times N_s}$ such that $\operatorname{tr}\rho = 1$ can be represented by using $(N_s^2 - 1)$ real variables. It is known ([23, Proposition 1]) that the positive semi-definiteness of ρ can be represented in the form of a semi-algebraic set with respect to newly introduced real variables; see also Section IV. Therefore, we obatain a polynomial type nonlinear stochastic delay system which evolves only in a bounded semi-algebraic set determined by $\mathcal{S}$.

It should be mentioned that the diffusion term (the coefficient of stochastic noise) is free from the control input which possibly suffers from delays.

III. MAIN RESULT

A. Delay-dependent stability criterion

We investigate the class of stochastic delay systems with delay free diffusion term. The mathematical description of the problem in this paper is given as follows.

Problem 1: Let $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ be polynomials satisfying $f(0, 0) = g(0) = 0$ and C a bounded semi-algebraic set in $\mathbb{R}^n$ including the origin. Suppose that for any initial condition $\tilde{x}_{\mathrm{i}} \in C_{\mathsf{C},\mathcal{W}_0}^\tau$ the solution to the delay differential stochastic equation

$$\mathrm{d}x_t = f(x_t, x_{t-\tau})\mathrm{d}t + g(x_t)\mathrm{d}w_t \quad (4)$$
$$x_\theta = \tilde{x}_{\mathrm{i}}(\theta) \in \mathsf{C}, \quad \theta \in [-\tau, 0] \quad (5)$$

does not exit C almost surely. Then, determine whether the solution converges to the origin almost surely for any $\tilde{x}_{\mathrm{i}} \in C^{\tau}_{\mathsf{C},\mathcal{W}_0}$.

A delay-dependent stability criterion for Problem 1 is given in the following:

Theorem 1: For Problem 1, suppose there exist n-variable polynomials V_i $(i = 0, 1)$, $N \in \mathbb{R}^{2n\times n}$, and positive-definite matrices R, $T \in \mathbb{R}^{n\times n}$ such that $\Upsilon : \mathsf{C} \times \mathsf{C} \times \mathbb{R}^{2n} \to \mathbb{R}$ defined below is negative:

$$\begin{aligned}
F(x, x_{\mathrm{d}}) &:= \left(\tfrac{\partial V_0(x)}{\partial x}\right)^{\mathsf{T}} f(x, x_{\mathrm{d}}) \\
&+\tfrac{1}{2}\, g(x)^{\mathsf{T}} \tfrac{\partial}{\partial x}\left(\tfrac{\partial V_0(x)}{\partial x}\right)^{\mathsf{T}} g(x) \\
&+V_1(x) - V_1(x_{\mathrm{d}}) + \|x\|^2 + \tau\|g(x)\|_T^2 \\
&+2\begin{bmatrix} x^{\mathsf{T}} & x_{\mathrm{d}}^{\mathsf{T}} \end{bmatrix} N(x - x_{\mathrm{d}}) + \tau\|f(x, x_{\mathrm{d}})\|_R^2 \qquad (6)\\
\Upsilon(x, x_{\mathrm{d}}, y) &:= F(x, x_{\mathrm{d}}) \\
&+ \begin{bmatrix} x \\ x_{\mathrm{d}} \\ y \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & N & \tau N \\ N^{\mathsf{T}} & -T & 0 \\ \tau N^{\mathsf{T}} & 0 & -\tau R \end{bmatrix} \begin{bmatrix} x \\ x_{\mathrm{d}} \\ y \end{bmatrix},
\end{aligned}$$

Then, the unique continuous solution of Eqs. (4), (5) converges to the origin almost surely for any initial condition $\tilde{x}_{\mathrm{i}} \in C^{\tau}_{\mathsf{C},\mathcal{W}_0}$.

This theorem states that the origin of the stochastic system considered in Problem 1 is stable if a semi-algebraic problem is feasible. While semi-algebraic problems are in general NP-hard, SOS relaxation enables us to systematically search for polynomials V_0 and V_1 which satisfy the required conditions [13], [14]. In the numerical example in the next section, we utilized MATLAB SOSTOOLS [15], [16].

B. Proof of Theorem 1

In this section, we prove Theorem 1 by using a standard Lyapunov-Krasovskii type argument: Let x_t be the solution of the stochastic delay differential equations (4) and (5). Define

$$\tilde{x}_t(\theta) := x_{t+\theta},\ \theta \in [-2\tau, 0]$$

for $t \geq \tau$.

Consider the following Lyapunov-Krasovskii functional candidate $V : C^{2\tau}_{\mathsf{C}} \to \mathbb{R}_+$

$$\begin{aligned}
V(\tilde{x}) &:= V_0(\tilde{x}(0)) + \int_{-\tau}^{0} V_1(\tilde{x}(\theta))\mathrm{d}\theta \\
&+\int_{-\tau}^{0}\int_{v}^{0} \left\{\|f(\tilde{x}(\theta), \tilde{x}(-\tau+\theta))\|_R^2 + \|g(\tilde{x}(\theta))\|_T^2\right\} \mathrm{d}\theta\mathrm{d}v.
\end{aligned} \qquad (7)$$

Direct computation yields

$$\begin{aligned}
&0 \leq \tau e^{\mathsf{T}} X e - \int_{-\tau}^{0} e^{\mathsf{T}} X e \mathrm{d}s \\
&0 = (2-2)\cdot e^{\mathsf{T}} N \left\{\tilde{x}(0) - \tilde{x}(-\tau) - \int_{-\tau}^{0} \underline{f}(s)\mathrm{d}s\right\} \\
&\quad \leq 2e^{\mathsf{T}} N(\tilde{x}(0) - \tilde{x}(-\tau)) - \int_{-\tau}^{0} 2e^{\mathsf{T}} N \underline{f}(s)\mathrm{d}s \\
&\qquad + e^{\mathsf{T}} N T^{-1} N^{\mathsf{T}} e + \left\|\tilde{x}(0) - \tilde{x}(-\tau) - \int_{-\tau}^{0} \underline{f}(s)\mathrm{d}s\right\|_T^2
\end{aligned}$$

where $e := \begin{bmatrix} \tilde{x}(0)^{\mathsf{T}} & \tilde{x}(-\tau)^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$, $\underline{f}(s) := f(\tilde{x}(s), \tilde{x}(-\tau + s))$ and $X := NR^{-1}N^{\mathsf{T}} \geq 0$. We used Lemma 1 in Appendix. Combining these inequalities and

$$\begin{aligned}
\mathscr{A}V(\tilde{x}) &= \left(\left.\frac{\partial V_0(x)}{\partial x}\right|_{\tilde{x}(0)}\right)^{\mathsf{T}} f(\tilde{x}(0), \tilde{x}(-\tau)) \\
&+\tfrac{1}{2}\, g(\tilde{x}(0))^{\mathsf{T}} \left.\tfrac{\partial}{\partial x}\left(\tfrac{\partial V_0(x)}{\partial x}\right)^{\mathsf{T}}\right|_{\tilde{x}(0)} g(\tilde{x}(0)) \\
&+V_1(\tilde{x}(0)) - V_1(\tilde{x}(-\tau)) + \tau(\|\underline{f}(0)\|_R^2 + \|g(\tilde{x}(0))\|_T^2) \\
&-\int_{-\tau}^{0} \left\{\|\underline{f}(s)\|_R^2 + \|g(\tilde{x}(s))\|_T^2\right\} ds,
\end{aligned}$$

we obtain

$$\mathscr{A}V(\tilde{x}) + \|\tilde{x}(0)\|^2 \leq \tilde{\Upsilon}(\tilde{x}(0), \tilde{x}(-\tau)) - G_1(\tilde{x}) - G_2(\tilde{x})$$

with

$$\begin{aligned}
\tilde{\Upsilon}(x, x_{\mathrm{d}}) &:= F(x, x_{\mathrm{d}}) + \left\|\begin{bmatrix} x^{\mathsf{T}} & x_{\mathrm{d}}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}\right\|_{\tau X + NT^{-1}N^{\mathsf{T}}}^2 \\
G_1(\tilde{x}) &:= \int_{-\tau}^{0} \left\|\begin{bmatrix} e^{\mathsf{T}} & \underline{f}(s)^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}\right\|_{\Xi}^2 \mathrm{d}s \geq 0 \\
G_2(\tilde{x}) &:= \int_{-\tau}^{0} \|g(\tilde{x}(s))\|_T^2 \mathrm{d}s \\
&\quad - \left\|\tilde{x}(0) - \tilde{x}(-\tau) - \int_{-\tau}^{0} \underline{f}(s)\mathrm{d}s\right\|_T^2, \\
\Xi &:= \begin{bmatrix} X & N \\ N^{\mathsf{T}} & R \end{bmatrix} \geq 0.
\end{aligned}$$

Moreover, we can show

$$\begin{aligned}
&\mathbb{E}\left[G_2(\tilde{x}_t)\right] \\
=\ &\mathbb{E}\left[\int_{-\tau}^{0} \|g(\tilde{x}(s))\|_T^2 \mathrm{d}s - \left\|\int_{-\tau}^{0} g(\tilde{x}(s))\mathrm{d}w_s\right\|_T^2\right] \\
=\ &0
\end{aligned}$$

by using Itô isometry. We thus have

$$\mathbb{E}\left[\mathscr{A}V(\tilde{x}_t) + \|\tilde{x}_t(0)\|^2\right] \leq \mathbb{E}\left[\tilde{\Upsilon}(\tilde{x}_t(0), \tilde{x}_t(-\tau))\right].$$

Finally, by using Lemma 2 in Appendix II and the assumption on Υ, we can show that $\tilde{\Upsilon}$ is negative, and consequently

$$\mathbb{E}\left[\mathscr{A}V(\tilde{x}_t)\right] \leq -\mathbb{E}\left[\|\tilde{x}_t(0)\|^2\right] \leq 0 \qquad (8)$$

follows.

Recall that x_t evolves only in bounded domain C. Hence Fubini's theorem yields

$$\mathbb{E}\left[\int_0^t \mathscr{A}V(\tilde{x}_s)ds\right] = \int_0^t \mathbb{E}\left[\mathscr{A}V(\tilde{x}_s)\right] ds.$$

By combining this equality, (8) and Dynkin's formula, we obtain

$$\begin{aligned} V(\tilde{x}_t) - V(\tilde{x}_0) &= \mathbb{E}\left[\int_0^t \mathscr{A}V(\tilde{x}_s)ds\right] \\ &= -\int_0^t \mathbb{E}\left[\|x_t\|^2\right] ds \le 0 \end{aligned}$$

Without loss of generality, we can assume that polynomials V_i $(i = 0, 1)$ are non-negative on C. This is because both of these are bounded from the below due to the continuity of polynomials and also the boundedness of C. Therefore we conclude that $V(\tilde{x}_t)$ is a nonnegative super-martingale. The remaining proof is the same as the standard Lyapunov-Krasovsii argument; see e.g., Theorem 6.1 and 6.2 in [6] and their proofs. This completes the proof.

Remark 1: The assumption on the boundedness of C is not essential. When we consider an unbounded domain C we need to add some assumptions including the non-negativity of V_i. See [6] for the detail.

IV. APPLICATION: 2-DIMENSIONAL QUANTUM SPIN

This section focuses on a *spin-1/2* model of the atom such that the system is composed of only a single particle. This system is very important, since it is the most basic component in quantum information processing [10]. In (3), the angular momentum operators are

$$F_y = \frac{1}{2}\begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \quad F_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is well known as *quantum state reduction* that the equation (3) without the input $u_t = 0$ shows the following probabilistic convergence:

$$\rho_t \to \rho_\uparrow := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{or} \quad \rho_t \to \rho_\downarrow := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The density matrices $\rho_\uparrow$ and $\rho_\downarrow$ represent that the monitored spin state of the atom is up and down, respectively. Note that only these two matrices are equilibrium points of Eq. (3) when $u_t = 0$. Our goal is to design a feedback control law $u_t = u(\rho_{t-\tau})$ that achieves the deterministic convergence to the prescribed equilibrium state ρ_f which is $\rho_\uparrow$ or $\rho_\downarrow$. This problem has been solved in [23] when there is no delay. We should remark that, even in such a simple setting, the controlled equation (3) shows a significant dependence on the delay, and eventually the control problem becomes much harder than the previous one.

It is shown ([23]) that the control input $u_t = u(\rho_t)$ with

$$u(\rho) := k_1(1 - \mathrm{tr}(\rho\rho_\mathrm{f})) + k_2\,\mathrm{tr}(\mathrm{i}[F_y, \rho]\rho_\mathrm{f}) \qquad (9)$$

achieves the control objective $\rho_t \to \rho_\mathrm{f}$ when k_1 and k_2 are chosen appropriately[1].

In this section, we derive a sufficient condition for this form of control input to globally stabilize a quantum spin control system in the face of feedback delay. Let us rewrite Eq. (3) in terms of the regulation error

$$\begin{bmatrix} x_t^{(1)} & x_t^{(2)} + \mathrm{i}x_t^{(3)} \\ x_t^{(2)} - \mathrm{i}x_t^{(3)} & -x_t^{(1)} \end{bmatrix} := \begin{cases} \rho_\mathrm{f} - \rho_t, & \text{if } \rho_f = \rho_\uparrow \\ \rho_t - \rho_\mathrm{f}, & \text{if } \rho_f = \rho_\downarrow. \end{cases}$$

We obtain a stochastic delay differential systems with respect to real variables $x_t^{(1)}$, $x_t^{(2)}$ and $x_t^{(3)}$ The followings can easily be verified:

- $x_t^{(3)}$ affects neigher $x_t^{(1)}$ nor $x_t^{(2)}$,
- if both $x_t^{(1)}$ and $x_t^{(2)}$ converege to 0, then so does $x^{(3)}$.

Hence the dynamics of $x^{(3)}$ can be ignored. An explicit expression of the dynamics of

$$x := \begin{bmatrix} x^{(1)} & x^{(2)} \end{bmatrix}^\mathsf{T}$$

is given by (4) with

$$\begin{aligned} f(x, x_\mathrm{d}) &:= \begin{bmatrix} -kx_\mathrm{d}x^{(2)} \\ kx_\mathrm{d}\left(x^{(1)} - \frac{1}{2}\right) - \frac{1}{2}x^{(2)} \end{bmatrix}, \\ g(x) &:= \sqrt{\eta}\begin{bmatrix} 2x^{(1)}(x^{(1)} - 1) \\ (2x^{(1)} - 1)x^{(2)} \end{bmatrix}, \\ k &:= \begin{bmatrix} k_1 & k_2 \end{bmatrix}. \end{aligned}$$

Let us consider the invariant subset of this dynamics. Recall that original state variable ρ_t is positive semi-definite. By this property and the definition of x_t, an invariant subset is given by the following circular domain C:

$$\mathsf{C} := \left\{x \in \mathbb{R}^2 : \Psi(x) \le 0\right\}, \qquad (10)$$

$$\Psi(x) := x^{(1)}(x^{(1)} - 1) + x^{(2)2}. \qquad (11)$$

Actually, we can show that for any initial condition $\tilde{x}_\mathrm{i} \in C^\tau_{\mathsf{C},\mathcal{W}_0}$ the solution to delay differential stochastic equation (4) does not exit C almost surely.

In summary, the following statements are equivalent:

- In Problem 1 with the definitions above, x_t converges to the origin almost surely.
- With $u(\cdot)$ given by (9), $u_t = u(\rho_{t-\tau})$ regulates the state ρ_t to the target state ρ_f almost surely.

Therefore, the required global stabilization of the target state is achieved if the following SOS problem has a solution:

Problem 2 (SOS programming): Under the above definitions, find polynomials V_i $(i = 0, 1)$, h, h_d and $N \in \mathbb{R}^{2n\times n}$ and positive-definite matrices R, $T \in \mathbb{R}^n$ such that

$$\begin{aligned} &-\Upsilon(x, x_\mathrm{d}, y) - h(x, x_\mathrm{d}, y)\Psi(x) - h_\mathrm{d}(x, x_\mathrm{d}, y)\Psi(x_d), \\ &h(x, x_\mathrm{d}, y), \\ &h_\mathrm{d}(x, x_\mathrm{d}, y) \end{aligned}$$

are the sum of squares with respect to x, x_d and y.

[1]The interpretation of this control law is as follows: the second term with $k_2 > 0$ *locally* stabilizes ρ_f. However, unfortunately, any eigenstate is an equilibrium of the closed-loop system. Hence, when ρ_t is close to an eigenstate that is not the regulation point, we need to prevent ρ_t from converging to it. This is done by the first term.

We provide a numerical example to illustrate the effectiveness of Theorem 1. If we fix the degrees of free decision polynomials, Problem 2 can be solved via reduction to the semi definite programming. Polynomials V_0 and V_1 are restricted to quadratic functions

$$V_0(x) := x^{(1)} + x^{\mathsf{T}} Q_0 x,$$
$$V_1(x) := q x^{(1)} + x^{\mathsf{T}} Q_1 x.$$

It should be stressed that $x^{(1)}$ in these functions is equivalent to $\mathrm{dist}(\cdot)$ which is defined below and represents a distance from the target state ρ_{f}. Note that $Q_i \in \mathbb{R}^{2\times 2}$ does not need to be positive definite because $x^{(1)}$ is positive in C.

We take $\eta = 0.9$ and $\rho_t \equiv \frac{1}{2}(\rho_{\downarrow} + \rho_{\uparrow})$ for $-\tau \le t \le 0$. In what follows, time responses of

$$\mathrm{dist}(\rho) := 1 - \mathrm{tr}(\rho\rho_{\mathrm{f}}) : \mathcal{S} \to [0, 1]$$

are shown (10 sample paths (blue) and their average (red)). This function gives the distance from the target state, i.e., $\mathrm{dist}(\rho) = 0$ (resp. $\mathrm{dist}(\rho) = 1$) if and only if $\rho = \rho_{\mathrm{f}}$ (resp. ρ is another eigenstate, different from the target state). The initial state satisfies $\mathrm{dist}(\rho_0) = 1/2$.

Figure 2 is the time responses of dynamics without control input, i.e., $u_t \equiv 0$. In this case, $\mathrm{dist}(\rho_t)$ converges to 0 or 1. This is a result of the quantum state reduction. We design control law such that $\rho_t \to \rho_{\mathrm{f}} := \rho_{\uparrow}$ as $t \to \infty$.

Let $k_1 = k_2 = 1.0$ and $\rho_{\mathrm{f}} := \rho_{\uparrow}$. Figure 3 is the time responses of ρ_t controlled without delay. We can see that global stability is achieved. Then, we solved Problem 2 for this control input and $\tau = 1.0$. This means that this control input stabilized the target state despite the feedback delay. Figure 4 shows the time response of this case.

Let $k_1 = k_2 = 10.0$. In this case, Problem 2 had no solution. The time response in Figure 5 does not converge to the target state.

Roughly speaking, Problem 1 covers not only the spin-1/2 model but also other higher dimensional spin systems. Using the numerical approach introduced in the next section, we can show that the general multi-spin system is also globally stabilized by a feedback controller despite time-delays. See also [5] for the effect of delays in switching control law.

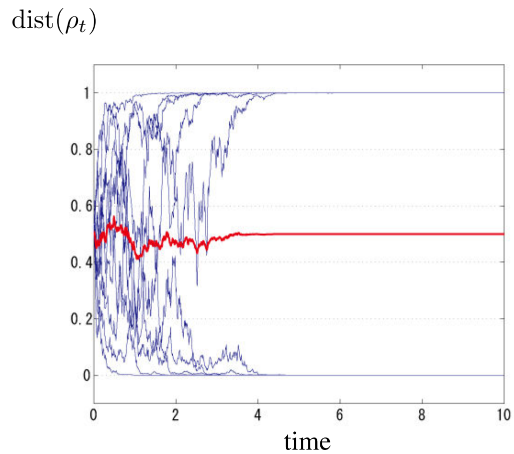

Fig. 2. Time response: without control input, i.e., $u_t \equiv 0$

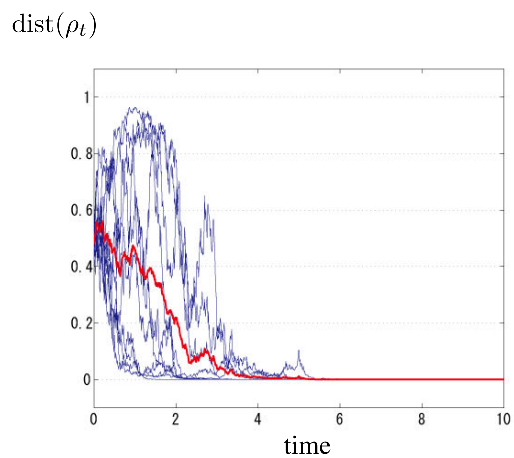

Fig. 3. Time response: delay free case with $k_1 = k_2 = 1.0$

V. CONCLUSION

In this paper, we gave a delay dependent stability criterion for a class of stochastic delay systems with polynomial nonlinearity. An important advantage of this result is that when an invariant set of the system is available, we can incorporate this information to reduce the conservativeness. The resulting condition was given in terms of semi-algebraic problem which is known to be effectively solvable via SOS relaxations. However, we have not so far discussed the computational complexity.

This work was originally motivated by the design of quantum spin control systems in the face of feedback delays. The effectiveness of the proposed method is evaluated by designing a globally stabilizing control law for the spin-1/2 system. To the best of the author's knowledge, there have been no theoretical result which proves that quantum spin systems in the face of feedback delays can be stabilized by using continuous (i.e., without switching) control law. In principle, higher dimensional spin systems can be dealt with similarly to the case of spin-1/2 system. However, due to the computational complexity issue noted above, it is not clear whether resulting semi-algebraic problems can be solved in a realistic time.

APPENDIX

Lemma 1: For any $x,\ y \in \mathbb{R}^n$ and positive definite matrix $T \in \mathbb{R}^{n\times n}$,

$$2|x^{\mathsf{T}} y| \le \|x\|_T^2 + \|y\|_{T^{-1}}^2.$$

Proof: This result readily follows from

$$(T^{1/2}x \pm T^{-1/2}y)^{\mathsf{T}}(T^{1/2}x \pm T^{-1/2}y) \ge 0.$$

■

Lemma 2 (Schur complement): Let D be a subset of $\mathbb{R}^n$, $N \in \mathbb{R}^{n\times m}$ and negative-definite $W \in \mathbb{R}^{m\times m}$. If $F : \mathsf{D} \to$

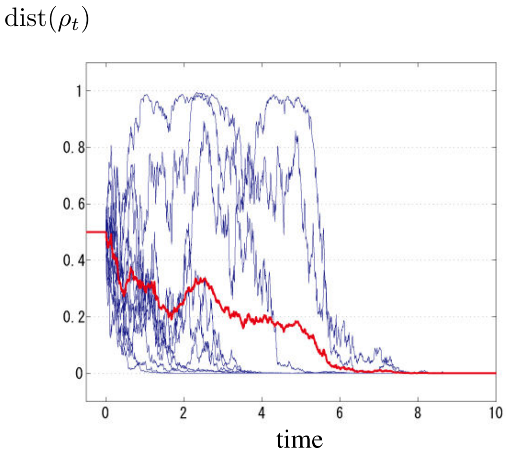

Fig. 4. Time response: the case of $\tau = 1.0$ with $k_1 = k_2 = 1.0$

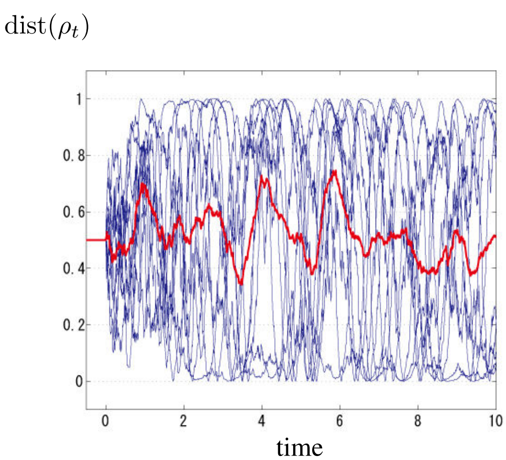

Fig. 5. Time response: the case of $\tau = 1.0$ with $k_1 = k_2 = 10.0$

$\mathbb{R}$ makes

$$\Upsilon(z, y) := F(z) + \begin{bmatrix} z \\ y \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & N \\ N^{\mathsf{T}} & W \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}$$

negative in $\mathsf{D} \times \mathbb{R}^m$, then

$$\tilde{\Upsilon}(z) := F(z) - z^{\mathsf{T}} N W^{-1} N^{\mathsf{T}} z$$

is also negative in D.

Proof: The result readily follows from $\Upsilon(z, \bar{y}) = \tilde{\Upsilon}(z)$ with $\bar{y} := -W^{-1} N^{\mathsf{T}} z \in \mathbb{R}^m$. ■

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