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Robust H_∞ Performance Analysis for Continuous-Time Networked Control Systems

Xun-Lin Zhu and Guang-Hong Yang

***Abstract*— This paper studies the problems of robust H_∞ performance analysis and controller design for continuous-time networked control systems (NCSs). A new type of Lyapunov functionals is exploited to derive sufficient conditions for guaranteeing the robust exponential stability and H_∞ performance of the considered system, and robust H_∞ controller design is presented. It is shown that the newly obtained result is less conservative than the existing corresponding ones. Meanwhile, by using a method of eliminating redundant variables, the computation complexity is reduced. Numerical examples are given to illustrate the effectiveness of the proposed methods.**

I. INTRODUCTION

Networks have received increasing attention in recent years because of the popularization and advantages of using network cables in control systems. The network itself is a dynamic system and induces possible delays via network communication due to limited bandwidth. A realistic networked control system design should take the communication delays into account, since the delays are widely known to degrade the performance of the control system.

In the past decade, the control problem of networked systems with time-delays has received increasing attention. [1] studied the problem of packet dropout and transmission delays induced by communication network of NCSs in both continuous time and discrete time cases. By using the Lyapunov-Razumikhin function techniques, [2] obtained the delay-dependent condition on the stabilization of NCSs in terms of linear matrix inequalities (LMIs). The admissible upper bounds of data packet loss and delays can be computed by using the quasi-convex optimization algorithm. [3] discussed the design of robust H_∞ controllers for uncertain NCSs with both the network-induced delay and data dropout. [4] was concerned with the controller design of NCSs. A new model of NCSs was provided under consideration of both the network-induced delay and the data packet dropout in the transmission, and a controller design method was proposed based on a delay-dependent approach. [5] studied the stabilization problem of NCSs where the main focus was the packet-loss issue. Two types of packet-loss processes were considered. One was the arbitrary packet-loss process, the other was the Markovian packet-loss process.

In this paper, the problem of robust H_∞ performance analysis for continuous-time NCSs is investigated. The objective is to seek for improved LMI-based conditions for ensuring larger delay bounds and better H_∞ performance. A new type of Lyapunov functionals is proposed, and new delay-dependent criteria for the H_∞ performance analysis are derived. Using these criteria, an upper bound of time-delay can be obtained such that the considered system is robustly exponentially stable with a prescribed H_∞ performance bound. It is shown that the new result is less conservative than the existing corresponding ones. Meanwhile, by using a method of eliminating redundant variables, the computational complexity is also reduced.

The organization of this paper is as follows. Section 2 models an NCS with data packet dropout and transmission delays as a linear system with time-varying input delay. Section 3 presents H_∞ performance analysis, controller design, and a method of eliminating redundant variables. It proves that the newly obtained stability condition is less conservative than some latest results. Three numerical examples are given to show the effectiveness of the criteria in Section 4, and finally conclusions are stated in Section 5.

II. SYSTEM DESCRIPTION

Throughout this paper, we assume that the sensor is clock-driven, the controller and actuator are event-driven and hold the latest data, h is the length of sampling period. Single packet transmission is considered throughout this paper. The actuator and the sensor are connected through a communication network with finite bandwidth. Data packet dropout and disordering in an NCS are unavoidable because of limited bandwidth. An NCS with the possibility of dropping data packet and disordering can be described as in Figure 1.
The model presented here is the same as that in [3]:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_\omega \omega(t), \tag{1}$$

$$x(t) = \phi(t), \qquad t \in [t_1 - h_2,\ t_1], \tag{2}$$

$$z(t) = Cx(t) + Du(t), \tag{3}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^p$ is the control input vector, $z(t) \in \Re^q$ is controlled output, $\omega(t) \in L_2[t_1, \infty)$

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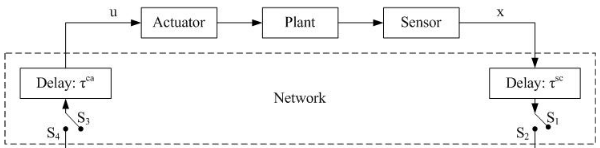

Fig. 1. An NCS with data packet dropout and transmission delays

denotes the external perturbation, and t_1 denotes the instant the actuator receives the 1st control signal. A, B, B_ω, C, D are constant matrices of appropriate dimensions; x_c is the delayed version of x, u is the delayed version of u_c, and $u_c(t) = Kx_c(t)$. Here, K is the state feedback gain matrix. Denote the instant the actuator receives the kth control signal as t_k, and this control signal is based on the state of plant at the instant $i_k h$, thus $\{i_1, i_2, i_3, \cdots\} \subset Z^+$, and

$$u(t^+) = Kx(t-\tau_k), \qquad t \in \{i_k h + \tau_k,\ k = 1,2,\cdots\}, \tag{4}$$

where time-delay τ_k denotes the time from the instant $i_k h$ when the sensor node samples sensor data from a plant to the instant t_k when the actuator receives the control signal, i.e., $\tau_k = t_k - i_k h$, and $\tau_k = \tau_k^{sc} + \tau_k^{ca}$, where τ_k^{sc} is the sensor-to-controller time-delay of $x(i_k h)$, and τ_k^{ca} is the controller-to-actuator time-delay of $\bar{u}(i_k h - \tau_k^{sc})$. Obviously, $\cup_{k=1}^{\infty}[i_k h + \tau_k,\ i_{k+1}h + \tau_{k+1}) = [t_1,\ \infty),\ t_1 \geq 0$.

As pointed out in [3], under assumption:

$$(i_{k+1} - i_k)h + \tau_{k+1} \leq h_2, \qquad k = 1,2,\cdots, \tag{5}$$

$$0 \leq h_1 \leq \tau_k, \qquad k = 1,2,\cdots, \tag{6}$$

where $h_1,\ h_2$ are constants, then the system (1)-(4) can be rewritten as follows:

$$\dot{x}(t) = Ax(t) + BKx(t - d(t)) + B_\omega \omega(t), \tag{7}$$

$$x(t) = \phi(t), \qquad t \in [t_1 - h_2,\ t_1] \tag{8}$$

$$z(t) = Cx(t) + DKx(t - d(t)), \tag{9}$$

$$0 \leq h_1 \leq d(t) \leq h_2, \tag{10}$$

where $d(t) = t - i_k h,\ t \in [i_k h + \tau_k,\ i_{k+1}h + \tau_{k+1})$, which denotes the time-varying delay in the control signal. Obviously, $d(t)$ is not always differentiable in the interval $[t_1,\ \infty]$.
In this paper, we analyze the robust H_∞ performance of the closed-loop system (7)-(10).

Remark 1. The above mentioned problem was also studied in [3]. In fact, many results of time-delay systems can be applied to deal with this problem, among them the result in [6] is one of the latest and it is listed as follows.

Lemma 1. [6] For given scalars $h_1,\ h_2\ (h_2 > h_1 \geq 0)$ and a matrix K, the linear system (7)-(8) with time-varying delay $d(t)$ satisfying (10) and $B_\omega = 0$ is asymptotically stable if there exist matrices $P_1 > 0$, $S_i > 0$, $Q_i \geq 0$, Y_i, T_i, V_i $(i = 1,2)$, such that the following LMI holds:

$$\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ * & \Lambda_3 \end{bmatrix} < 0, \tag{11}$$

where

$$\Lambda_1 = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & V_1 & -T_1 & h_2 Y_1 & h_{12} T_1 \\ * & \Lambda_{22} & V_2 & -T_2 & h_2 Y_2 & h_{12} T_2 \\ * & * & -Q_1 & 0 & 0 & 0 \\ * & * & * & -Q_2 & 0 & 0 \\ * & * & * & * & -h_2 S_1 & 0 \\ * & * & * & * & * & -h_{12}\sum_{i=1}^{2} S_i \end{bmatrix},$$

$\Lambda_{11} = P_1 A + A^T P_1 + Q_1 + Q_2 + Y_1 + Y_1^T$,
$\Lambda_{12} = P_1(BK) + Y_2^T - Y_1 + T_1 - V_1$,
$\Lambda_{22} = T_2 + T_2^T - Y_2 - Y_2^T - V_2 - V_2^T$,

$$\Lambda_2 = \begin{bmatrix} h_{12} V_1 & A^T(h_2 S_1 + h_{12} S_2) \\ h_{12} V_2 & (BK)^T(h_2 S_1 + h_{12} S_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$\Lambda_3 = diag\{-h_{12} S_2,\ -(h_2 S_1 + h_{12} S_2)\}$,
$h_{12} = h_2 - h_1$.

III. MAIN RESULTS

In this section, a new type of Lyapunov-Krasovskii functionals is proposed to derive new delay-dependent robust exponential stability criteria for the system (7)-(10) with a prescribed H_∞ performance level. By eliminating redundant variables, a method for designing state feedback H_∞ controllers is presented.

A. H_∞ performance analysis

In the following, we will give new sufficient conditions which can ensure the exponential stability of NCS (7)-(10) with a prescribed H_∞ performance bound.

Lemma 2. For given scalars $h_1,\ h_2\ (h_2 > h_1 \geq 0)$, $\gamma > 0$, and a matrix K, the system (7)-(10) is exponentially stable with an H_∞ norm bound γ if there exist $n \times n$ matrices $P_{ij},\ P_{ij} = P_{ji}^T\ (i,j = 1,\ 2,\ 3)$, $P_{11} > 0$, $Q_i \geq 0$, $Z_i \geq 0$, N_i, $S_i \geq 0$ $(i = 1,\ 2)$, and matrices M_i $(i = 1,\ 2,\ \cdots,\ 9)$, such that

$$\Gamma < 0, \tag{12}$$

and

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{bmatrix} \geq 0, \quad H_i = \begin{bmatrix} Z_i & N_i \\ * & S_i \end{bmatrix} \geq 0, \tag{13}$$

where

$$\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ * & \Gamma_3 \end{bmatrix} - M\mathscr{A} - \mathscr{A}^T M^T,$$

$$\Gamma_1 = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & P_{13} & -P_{12} - P_{13} \\ * & \Gamma_{22} & 0 & 0 & 0 \\ * & * & \Gamma_{33} & h_{12}^{-1} S_2 & h_{12}^{-1}(S_1 + S_2) \\ * & * & * & -Q_1 - h_{12}^{-1} S_2 & 0 \\ * & * & * & * & \Gamma_{55} \end{bmatrix},$$

$\Gamma_{11} = Q_1 + Q_2 + h_2 Z_1 + h_{12} Z_2 + P_{12} + P_{12}^T + C^T C - h_2^{-1} S_1$,
$\Gamma_{12} = h_2 N_1 + h_{12} N_2 + P_{11}$,
$\Gamma_{13} = C^T DK + h_2^{-1} S_1$,
$\Gamma_{22} = h_2 S_1 + h_{12} S_2$,
$\Gamma_{33} = (DK)^T(DK) - h_2^{-1} S_1 - h_{12}^{-1}(S_1 + S_2) - h_{12}^{-1} S_2$,
$\Gamma_{55} = -Q_2 - h_{12}^{-1}(S_1 + S_2)$,

$$\Gamma_2 = \begin{bmatrix} P_{22} - h_2^{-1}N_1^T & P_{22}+P_{23} & P_{23} & 0 \\ P_{12} & P_{12}+P_{13} & P_{13} & 0 \\ h_2^{-1}N_1^T & \Gamma_{73} & h_{12}^{-1}N_2^T & 0 \\ P_{23}^T & P_{23}^T+P_{33} & P_{33}-h_{12}^{-1}N_2^T & 0 \\ P_{22}-P_{23}^T & \Gamma_{75} & -P_{23}-P_{33} & 0 \end{bmatrix},$$
$$\Gamma_{73} = -h_{12}^{-1}(N_1^T+N_2^T),$$
$$\Gamma_{75} = -P_{22}-P_{23}^T-P_{23}-P_{33}+h_{12}^{-1}(N_1^T+N_2^T),$$
$$\Gamma_3 = diag\{-h_2^{-1}Z_1,\ -h_{12}^{-1}(Z_1+Z_2),\ -h_{12}^{-1}Z_2,\ -\gamma^2 I\},$$
$$M = \begin{bmatrix} M_1^T & \cdots & M_9^T \end{bmatrix}^T,$$
$$\mathscr{A} = \begin{bmatrix} A & -I & BK & 0 & 0 & 0 & 0 & 0 & B_\omega \end{bmatrix},$$
$$h_{12} = h_2 - h_1.$$

Proof: Construct a Lyapunov-Krasovskii functional as

$$\begin{aligned} V(t) = &\begin{bmatrix} x(t) \\ \int_{t-h_2}^{t} x(s)ds \\ \int_{t-h_2}^{t-h_1} x(s)ds \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \int_{t-h_2}^{t} x(s)ds \\ \int_{t-h_2}^{t-h_1} x(s)ds \end{bmatrix} \\ &+ \sum_{i=1}^{2} \int_{t-h_i}^{t} x(s)^T Q_i x(s) ds \\ &+ \int_{-h_2}^{0}\int_{t+\beta}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T H_1 \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds d\beta \\ &+ \int_{-h_2}^{-h_1}\int_{t+\beta}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T H_2 \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds d\beta, \end{aligned} \tag{14}$$

where $P \geq 0$, $H_i \geq 0$ $(i = 1,\ 2)$ are defined in (13) and $Q_i \geq 0$ $(i = 1,\ 2)$.

Using the Cauchy-Schwarz inequality [7], and denoting $\zeta = [x^T(t)\ \dot{x}^T(t)\ x^T(t-d(t))\ x^T(t-h_1)\ x^T(t-h_2)$ $(\int_{t-d(t)}^{t} x(s)ds)^T\ (\int_{t-h_2}^{t-d(t)} x(s)ds)^T\ (\int_{t-d(t)}^{t-h_1} x(s)ds)^T\ \omega^T(t)]^T$, we can get

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) \leq \zeta^T\Gamma\zeta, \tag{15}$$

which implies that $||z(t)||_2 \leq \gamma||\omega(t)||_2$ under zero initial condition.

Similar to [3], we can prove the exponential stability of system (7)-(10).

Thus, the proof is completed. ■

Remark 2. In Lemma 2, a sufficient condition of exponential stability for the system (7)-(10) with an H_∞ norm bound γ is given in terms of solutions to a set of LMIs. Different sufficient conditions are also given in [3] and [4]. The numerical comparison between Lemma 2 and the results in [3] and [4] will be given in Section 4. Note that the Lyapunov-Krasovskii functional (14) is more general, and lead to a less conservative stability condition than that in [6]. The details will be discussed in the sequel.

By the elimination Lemma ([14], p.22), it is readily seen that if there exist matrices M_i $(i = 1,\ 2,\ \cdots,\ 9)$ that solve $\Gamma < 0$, if and only if

$$\mathscr{N}_{\mathscr{A}}{}^T \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ * & \Gamma_3 \end{bmatrix} \mathscr{N}_{\mathscr{A}} < 0 \tag{16}$$

holds, where $\mathscr{N}_{\mathscr{A}}$ denotes the full-rank matrix representations of the right annihilator of $\mathscr{A}$. By the Schur complement, it yields that

$$\Pi < 0, \tag{17}$$

where

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix},$$
$$\Pi_1 = \begin{bmatrix} \Pi_{11} & \Pi_{12} & P_{13} & -P_{12}-P_{13} & \Pi_{15} \\ * & \Pi_{22} & h_{12}^{-1}S_2 & h_{12}^{-1}(S_1+S_2) & \Pi_{25} \\ * & * & \Pi_{33} & 0 & P_{23}^T \\ * & * & * & \Pi_{44} & -P_{22}-P_{23}^T \\ * & * & * & * & -h_2^{-1}Z_1 \end{bmatrix},$$
$$\begin{aligned} \Pi_{11} = &\ Q_1+Q_2+h_2Z_1+h_{12}Z_2+P_{12}+P_{12}^T+(P_{11} \\ &+h_2N_1+h_{12}N_2)A+A^T(P_{11}+h_2N_1+h_{12}N_2)^T \\ &+C^TC-h_2^{-1}S_1, \end{aligned}$$
$$\Pi_{12} = (P_{11}+h_2N_1+h_{12}N_2)BK+C^T(DK)+h_2^{-1}S_1,$$
$$\Pi_{15} = P_{22}+A^TP_{12}-h_2^{-1}N_1^T,$$
$$\Pi_{22} = (DK)^T(DK)-h_2^{-1}S_1-h_{12}^{-1}(S_1+S_2)-h_{12}^{-1}S_2,$$
$$\Pi_{25} = (BK)^TP_{12}+h_2^{-1}N_1^T,$$
$$\Pi_{33} = -Q_1-h_{12}^{-1}S_2,$$
$$\Pi_{44} = -Q_2-h_{12}^{-1}(S_1+S_2),$$
$$\Pi_2 = \begin{bmatrix} \Pi_{16} & P_{23}+A^TP_{13} & \Pi_{18} & \Pi_{19} \\ \Pi_{26} & (BK)^TP_{13}+h_{12}^{-1}N_2^T & 0 & \Pi_{29} \\ P_{23}^T+P_{33} & P_{33}-h_{12}^{-1}N_2^T & 0 & 0 \\ \Pi_{46} & -P_{23}-P_{33} & 0 & 0 \\ 0 & 0 & P_{12}^TB_\omega & 0 \end{bmatrix},$$
$$\Pi_{16} = P_{22}+P_{23}+A^T(P_{12}+P_{13}),$$
$$\Pi_{18} = (h_2N_1+h_{12}N_2+P_{11})B_\omega,$$
$$\Pi_{19} = A^T(h_2S_1+h_{12}S_2),$$
$$\Pi_{26} = (BK)^T(P_{12}+P_{13})-h_{12}^{-1}(N_1^T+N_2^T),$$
$$\Pi_{29} = (BK)^T(h_2S_1+h_{12}S_2),$$
$$\Pi_{46} = -P_{22}-P_{23}^T-P_{23}-P_{33}+h_{12}^{-1}(N_1^T+N_2^T),$$
$$\Pi_3 = \begin{bmatrix} -h_{12}^{-1}(Z_1+Z_2) & 0 & (P_{12}^T+P_{13}^T)B_\omega & 0 \\ * & -h_{12}^{-1}Z_2 & P_{13}^TB_\omega & 0 \\ * & * & -\gamma^2 I & \Pi_{89} \\ * & * & * & \Pi_{99} \end{bmatrix},$$
$$\Pi_{89} = B_\omega^T(h_2S_1+h_{12}S_2),$$
$$\Pi_{99} = -(h_2S_1+h_{12}S_2),$$
$$h_{12} = h_2-h_1.$$

On the other hand, if $\Pi < 0$ holds, then it is very easy to see that $\Gamma < 0$ holds by taking

$$\begin{aligned} &M_1 = -(P_{11}+h_2N_1+h_{12}N_2), \quad M_2 = -(h_2S_1+h_{12}S_2), \\ &M_6 = -P_{12}^T, \quad M_7 = -(P_{12}+P_{13})^T, \quad M_8 = -P_{13}^T, \\ &M_3 = M_4 = M_5 = M_9 = 0. \end{aligned}$$

This implies that $M_3,\ M_4,\ M_5,\ M_9$ are all redundant in Γ.

Remark 3. From above analysis, it is shown that M_6 is redundant when $P_{12} = 0$, M_8 is redundant when $P_{13} = 0$, and $M_6,\ M_7,\ M_8$ are redundant when $P_{12} = P_{13} = 0$ in Lemma 2.

Thus, we have the following result for the H_∞ performance analysis, which is equivalent to Lemma 2 and has fewer decision variables.

Theorem 1. For given scalars $h_1,\ h_2$ $(h_2 > h_1 \geq 0)$, $\gamma > 0$, and a matrix K, the system (7)-(10) is exponentially stable with an H_∞ norm bound γ if there exist $n \times n$ matrices P_{ij}, $P_{ij} = P_{ji}^T$ $(i, j = 1,\ 2,\ 3)$, $P_{11} > 0$, $Q_i \geq 0$, $Z_i \geq 0$, N_i, $S_i \geq 0$ $(i = 1,\ 2)$, and matrices M_i $(i = 1,\ 2,\ \cdots,\ 5)$, such that

$$\Theta < 0, \tag{18}$$

and

$$P=\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{bmatrix} \geq 0, \quad H_i=\begin{bmatrix} Z_i & N_i \\ * & S_i \end{bmatrix} \geq 0 \ (i=1,\ 2), \tag{19}$$

where

$$\Theta=\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ * & \Gamma_3 \end{bmatrix}+\Theta_1+\Theta_1^T,$$
$$\Theta_1=-\begin{bmatrix} M_1^T & M_2^T & 0 & 0 & 0 & M_3^T & M_4^T & M_5^T & 0 \end{bmatrix}^T \mathscr{A},$$
$$\mathscr{A}=\begin{bmatrix} A & -I & BK & 0 & 0 & 0 & 0 & 0 & B_\omega \end{bmatrix},$$

and $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ are defined in Lemma 2 .

From Theorem 1 and the equivalence of the inequality (17) and (18), we can derive the following stability conditions for the system (7)-(8).

Corollary 1. For given scalar $h_1,\ h_2\ (h_2 > h_1 \geq 0)$ and matrix K, the linear system (7)-(8) with time-varying delay $d(t)$ satisfying (10) and $B_\omega = 0$ is asymptotically stable if there exist matrices $P_{ij},\ P_{ij} = P_{ji}^T\ (i,j = 1,\ 2,\ 3),\ P_{11} > 0,\ Q_i \geq 0,\ Z_i \geq 0,\ N_i,\ S_i \geq 0\ (i = 1,\ 2)$, such that the following LMIs hold:

$$\Omega < 0, \tag{20}$$

and

$$P=\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{bmatrix} \geq 0, \quad H_i=\begin{bmatrix} Z_i & N_i \\ * & S_i \end{bmatrix} \geq 0 \ (i=1,\ 2), \tag{21}$$

where

$$\Omega=\begin{bmatrix} \Omega_1 & \Omega_2 \\ * & \Omega_3 \end{bmatrix},$$
$$\Omega_1=\begin{bmatrix} \Omega_{11} & \Omega_{12} & P_{13} & -P_{12}-P_{13} & \Omega_{15} \\ * & \Omega_{22} & h_{12}^{-1}S_2 & h_{12}^{-1}(S_1+S_2) & \Omega_{25} \\ * & * & \Omega_{33} & 0 & P_{23}^T \\ * & * & * & \Omega_{44} & -P_{22}-P_{23}^T \\ * & * & * & * & -h_2^{-1}Z_1 \end{bmatrix},$$
$$\Omega_{11}=Q_1+Q_2+h_2Z_1+h_{12}Z_2+P_{12}+P_{12}^T-h_2^{-1}S_1 + (P_{11}+h_2N_1+h_{12}N_2)A+A^T(P_{11}+h_2N_1+h_{12}N_2)^T,$$
$$\Omega_{12}=(P_{11}+h_2N_1+h_{12}N_2)BK+h_2^{-1}S_1,$$
$$\Omega_{15}=P_{22}+A^TP_{12}-h_2^{-1}N_1^T,$$
$$\Omega_{22}=-h_2^{-1}S_1-h_{12}^{-1}(S_1+S_2)-h_{12}^{-1}S_2,$$
$$\Omega_{25}=(BK)^TP_{12}+h_2^{-1}N_1^T,$$
$$\Omega_{33}=-Q_1-h_{12}^{-1}S_2,$$
$$\Omega_{44}=-Q_2-h_{12}^{-1}(S_1+S_2),$$
$$\Omega_2=\begin{bmatrix} \Omega_{16} & P_{23}+A^TP_{13} & \Omega_{18} \\ \Omega_{26} & (BK)^TP_{13}+h_{12}^{-1}N_2^T & \Omega_{28} \\ P_{23}^T+P_{33} & P_{33}-h_{12}^{-1}N_2^T & 0 \\ \Omega_{46} & -P_{23}-P_{33} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\Omega_{16}=P_{22}+P_{23}+A^T(P_{12}+P_{13}),$$
$$\Omega_{18}=A^T(h_2S_1+h_{12}S_2),$$
$$\Omega_{26}=(BK)^T(P_{12}+P_{13})-h_{12}^{-1}(N_1^T+N_2^T),$$
$$\Omega_{28}=(BK)^T(h_2S_1+h_{12}S_2),$$
$$\Omega_{46}=-P_{22}-P_{23}^T-P_{23}-P_{33}+h_{12}^{-1}(N_1^T+N_2^T),$$
$$\Omega_3=diag\{-h_{12}^{-1}(Z_1+Z_2),\ -h_{12}^{-1}Z_2,\ -(h_2S_1+h_{12}S_2)\},$$
$$h_{12}=h_2-h_1.$$

The comparison between Corollary 1 and Lemma 1 is given as follows.

Theorem 2. If the LMI in Lemma 1 is feasible, the LMIs in Corollary 1 are also feasible.

Proof: Denoting

$$\Delta_4=\begin{bmatrix} -h_2^{-1}I & 0 & 0 & 0 \\ h_2^{-1}I & -h_{12}^{-1}I & h_{12}^{-1}I & 0 \\ 0 & 0 & -h_{12}^{-1}I & 0 \\ 0 & h_{12}^{-1}I & 0 & 0 \end{bmatrix} \tag{22}$$

pre- and post-multiplying both sides of Λ by $\begin{bmatrix} I & \Delta_4 \\ 0 & I \end{bmatrix}$ and its transpose, respectively, then it is easy to see that $\Omega < 0$ from $\Lambda < 0$ by taking $P_{11} = P_1,\ P_{12} = P_{13} = P_{22} = P_{23} = P_{33} = 0$ and $Z_i = \varepsilon I,\ N_i = 0\ (i = 1,\ 2)$ with $\varepsilon > 0$ being sufficient small scalar. ■

Remark 4. By Theorem 2, it proves theoretically that Corollary 1 is less conservative than Lemma 1.

B. Robust performance analysis

Next, consider the following system with parameter uncertainties given by

$$\dot{x}(t)=[A+\Delta A(t)]x(t)+[B+\Delta B(t)]u(t)+B_\omega\omega(t), \tag{23}$$

where $\Delta A(t)$ and $\Delta B(t)$ denote the parameter uncertainties satisfying the following condition:

$$\begin{bmatrix} \Delta A(t) & \Delta B(t) \end{bmatrix}=GF(t)\begin{bmatrix} E_a & E_b \end{bmatrix}, \tag{24}$$

where G, E_a and E_b are constant matrices of appropriate dimensions and $F(t)$ is an unknown time-varying matrix, which is Lebesque measurable in t and satisfies

$$F^T(t)F(t) \leq I,\ \forall\ t \geq 0. \tag{25}$$

In this case, from Theorem 1, (18) is substituted by

$$\Theta+(-\tilde{M})F(t)\begin{bmatrix} E_a & 0 & E_bK & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} E_a & 0 & E_bK & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T F^T(t)(-\tilde{M})^T < 0, \tag{26}$$

where

$$\tilde{M}=\begin{bmatrix} M_1^T & M_2^T & 0 & 0 & 0 & M_3^T & M_4^T & M_5^T & 0 \end{bmatrix}^T G,$$

thus, according to the definition of robust exponential stability in [3], it is easy to get the following result.

Theorem 3. For given scalars $h_1,\ h_2\ (h_2 > h_1 \geq 0),\ \gamma > 0$, and a matrix K, the system described by (23)-(25) and (8)-(10) is robustly exponentially stable with an H_∞ norm bound γ if there exist $n \times n$ matrices $P_{ij},\ P_{ij} = P_{ji}^T,\ P_{11} > 0\ (i,j = 1,\ 2,\ 3),\ Q_i \geq 0,\ Z_i \geq 0,\ N_i,\ S_i \geq 0\ (i = 1,\ 2)$, and matrices $M_i\ (i = 1,\ 2,\ \cdots,\ 5)$, and scalar $\varepsilon > 0$, such that

$$\Phi=\begin{bmatrix} \Phi_1 & \Phi_2 \\ * & -\varepsilon I \end{bmatrix} < 0, \tag{27}$$

and

$$P=\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{bmatrix} \geq 0, \quad H_i=\begin{bmatrix} Z_i & N_i \\ * & S_i \end{bmatrix} \geq 0,\ (i=1,\ 2) \tag{28}$$

where

$$\Phi_1 = \Theta + \varepsilon \left[\begin{array}{cccccccccc} E_a & 0 & E_b K & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]^T \times \left[\begin{array}{cccccccccc} E_a & 0 & E_b K & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right],$$
$$\Phi_2 = -\left[\begin{array}{ccccccccc} M_1^T & M_2^T & 0 & 0 & 0 & M_3^T & M_4^T & M_5^T & 0 \end{array} \right]^T G,$$

and Θ is defined in Theorem 1.

C. Robust H_∞ controller design

Based on Theorem 3, we are now in a position to design the feedback gain K, which can ensure the robustly exponential stability of the uncertain system described by (23)-(25) and (8)-(10) with H_∞ norm bound γ.

Obviously, (18) implies M_2 is nonsingular, so there exist matrices U_1, U_3, U_4, U_5, such that $M_1 = M_2U_1$, $M_3 = M_2U_3$, $M_4 = M_2U_4$ and $M_5 = M_2U_5$.

Pre- and post-multiplying both sides of (27) by $diag\{M_2^{-1}, \ \cdots, \ M_2^{-1}, \ I, \ I\}$ and its transpose, pre- and post-multiplying both sides of H_i $(i = 1, \ 2)$ in (28) by $diag\{M_2^{-1}, \ M_2^{-1}\}$ and its transpose, pre- and post-multiplying both sides of P in (28) by $diag\{M_2^{-1}, \ M_2^{-1}, \ M_2^{-1}\}$ and its transpose, respectively, and denoting

$$\bar{M}_2 = M_2^{-1}, \quad \bar{P}_{ij} = \bar{M}_2 P_{ij} \bar{M}_2^T \ (i,j = 1, \ 2, \ 3), \quad Q = K\bar{M}_2^T,$$
$$\mu = \varepsilon^{-1}, \quad \bar{Q}_i = \bar{M}_2 Q_i \bar{M}_2^T, \quad \bar{Z}_i = \bar{M}_2 Z_i \bar{M}_2^T,$$
$$\bar{N}_i = \bar{M}_2 N_i \bar{M}_2^T, \quad \bar{S}_i = \bar{M}_2 S_i \bar{M}_2^T \ (i = 1, \ 2).$$

and using the Schur complement, we can obtain the following theorem.

Theorem 4. For prescribed scalars h_1, h_2 $(h_2 > h_1 \geq 0)$, $\gamma > 0$, and some tuning matrix parameters U_i, $(i = 1, \ 3, \ 4, \ 5)$, the system described by (23)-(25) and (8)-(10) is robustly exponentially stable with an H_∞ norm bound γ if there exist $n \times n$ matrices $\bar{P}_{ij}$, $\bar{P}_{ij} = \bar{P}_{ji}^T$ $(i,j = 1, \ 2, \ 3)$, $\bar{P}_{11} > 0$, $\bar{Q}_i \geq 0$, $\bar{Z}_i \geq 0$, $\bar{N}_i$, $\bar{S}_i \geq 0$ $(i = 1, \ 2)$, and matrices Q, $\bar{M}_2$ and scalar $\mu > 0$, such that

$$\Psi < 0, \tag{29}$$

and

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} & \bar{P}_{13} \\ * & \bar{P}_{22} & \bar{P}_{23} \\ * & * & \bar{P}_{33} \end{bmatrix} \geq 0, \ \bar{H}_i = \begin{bmatrix} \bar{Z}_i & \bar{N}_i \\ * & \bar{S}_i \end{bmatrix} \geq 0, \ (i = 1, \ 2) \tag{30}$$

where

$$\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 & \Psi_3 \\ * & \Psi_4 & \Psi_5 \\ * & * & \Psi_6 \end{bmatrix} + \Psi_7 + \Psi_7^T + \mu \Psi_8 \Psi_8^T,$$

$$\Psi_1 = \begin{bmatrix} \Psi_{11} & \Psi_{12} & h_2^{-1}\bar{S}_1 & \bar{P}_{13} & -\bar{P}_{12} - \bar{P}_{13} \\ * & \Psi_{22} & 0 & 0 & 0 \\ * & * & \Psi_{33} & h_{12}^{-1}\bar{S}_2 & h_{12}^{-1}(\bar{S}_1 + \bar{S}_2) \\ * & * & * & \Psi_{44} & 0 \\ * & * & * & * & \Psi_{55} \end{bmatrix},$$

$$\Psi_{11} = \bar{Q}_1 + \bar{Q}_2 + h_2\bar{Z}_1 + h_{12}\bar{Z}_2 + \bar{P}_{12} + \bar{P}_{12}^T - h_2^{-1}\bar{S}_1,$$
$$\Psi_{12} = h_2\bar{N}_1 + h_{12}\bar{N}_2 + \bar{P}_{11},$$
$$\Psi_{22} = h_2\bar{S}_1 + h_{12}\bar{S}_2,$$
$$\Psi_{33} = -h_2^{-1}\bar{S}_1 - h_{12}^{-1}(\bar{S}_1 + \bar{S}_2) - h_{12}^{-1}\bar{S}_2,$$

TABLE I

ALLOWABLE UPPER BOUND OF h_2 WITH GIVEN h_1

Methods	h_1	0	0.5	0.8	1	2
He et al.[6]	h_2	1.2817	1.4407	1.5719	1.6626	2.1071
Corollary 1	h_2	1.3423	1.5405	1.7065	1.8147	2.2458

$$\Psi_{44} = -\bar{Q}_1 - h_{12}^{-1}\bar{S}_2,$$
$$\Psi_{55} = -\bar{Q}_2 - h_{12}^{-1}(\bar{S}_1 + \bar{S}_2),$$

$$\Psi_2 = \begin{bmatrix} \bar{P}_{22} - h_2^{-1}\bar{N}_1^T & \bar{P}_{22} + \bar{P}_{23} & \bar{P}_{23} \\ \bar{P}_{12} & \bar{P}_{12} + \bar{P}_{13} & \bar{P}_{13} \\ h_2^{-1}\bar{N}_1^T & -h_{12}^{-1}(\bar{N}_1^T + \bar{N}_2^T) & h_{12}^{-1}\bar{N}_2^T \\ \bar{P}_{23}^T & \bar{P}_{23}^T + \bar{P}_{33} & \bar{P}_{33} - h_{12}^{-1}\bar{N}_2^T \\ -\bar{P}_{22} - \bar{P}_{23}^T & \Psi_{57} & -\bar{P}_{23} - \bar{P}_{33} \end{bmatrix},$$

$$\Psi_{57} = -\bar{P}_{22} - \bar{P}_{23} - \bar{P}_{23}^T - \bar{P}_{33} + h_{12}^{-1}(\bar{N}_1^T + \bar{N}_2^T),$$

$$\Psi_3 = \begin{bmatrix} 0 & \bar{M}_2 E_a^T & \bar{M}_2 C^T \\ 0 & 0 & 0 \\ 0 & Q^T E_b^T & Q^T D^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_4 = diag\{-h_2^{-1}\bar{Z}_1, \ -h_{12}^{-1}(\bar{Z}_1 + \bar{Z}_2), \ -h_{12}^{-1}\bar{Z}_2\},$$
$$\Psi_6 = diag\{-\gamma^2 I, \ -\mu I, \ -I\},$$
$$\Psi_7 = -U \left[\begin{array}{ccccccccc} A M_2^T & -M_2^T & BQ & 0 & \cdots & 0 & B_\omega & 0 & 0 \end{array} \right],$$
$$\Psi_8 = UG,$$
$$U = \left[\begin{array}{ccccccccccc} U_1^T & I & 0 & 0 & 0 & U_3^T & U_4^T & U_5^T & 0 & 0 & 0 \end{array} \right]^T,$$
$$h_{12} = h_2 - h_1.$$

The state feedback gain matrix is then given by:

$$K = Q\bar{M}_2^{-T}. \tag{31}$$

Remark 5. To design robust H_∞ controllers, we take $M_i = M_2U_i$, where M_2 is nonsingular and U_i $(i = 1, \ 3, \ 4, \ 5)$ are tuning matrix parameters. By applying a numerical optimization algorithm [11], such as *fminunc* in the Optimization Toolbox, the tuning matrix parameters can be obtained.

IV. NUMERICAL EXAMPLES

Example 1. Consider the following system

$$\dot{x}(t) = \begin{bmatrix} -1.8 & -2.3 \\ -0.8 & -1.2 \end{bmatrix} x(t) + \begin{bmatrix} -0.9 & 0.6 \\ 0.2 & 0.1 \end{bmatrix} x(t - d(t)), \tag{32}$$

and

$$h_1 \leq d(t) \leq h_2, \tag{33}$$

where $h_2 > h_1 \geq 0$. The maximum upper bound of h_2 is 0.9917 with $h_1 = 0$ by the method in [13]. But when $h_1 > 0$, the result in [13] is not applicable. The computed upper bounds, h_2, which guarantee the stability of the system (32) for given lower bounds, h_1, are listed in Table 1. It is clear that the method given in Corollary 1 is less conservative than those given in [13] and [6].

Example 2. Consider the following system [3]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t). \tag{34}$$

TABLE II

ALLOWABLE UPPER BOUND OF h_2 WITH $h_1 = 0$

Methods	Allowable upper bound of h_2
Yue et al. [3]	0.8871
Yue et al. [4]	0.8695
Naghshtabrizi et al. [15]	0.8695
Theorem 1	1.0081

For this example, we employ the same feedback controller as in [3], that is, $K = \begin{bmatrix} -3.75 & -11.5 \end{bmatrix}$. It is found that the maximum allowable value of h_2 with $h_1 = 0$ can be 1.0081 by Theorem 1. For convenience of comparison, the allowable upper bounds of h_2 obtained by various methods are listed in Table 2. For the case of $h_2 - h_1 = 0.3$, we find that the maximum allowable value of h_1 is 0.7501 by Corollary 1, and corresponding maximum allowable value of h_1 was 0.6916 in [3].

Next, we consider the effect of the external perturbation on the system. Just as shown in [3], (34) can be expressed as

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \omega(t), \\ z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0.1u(t). \end{aligned} \tag{35}$$

When $h_2 = 0.8695$, we find that the minimum allowable H_∞ norm bound $\gamma_{\min}$ is 1.00 with $h_1 = 0$, and the H_∞ norm bound $\gamma_{\min}$ is 0.85 with $h_1 = 0.5695$ by Theorem 1, while corresponding values of $\gamma_{\min}$ were 6.82 and 1.26 in [3], respectively.

Example 3. Consider the following uncertain system controlled over a network:

$$\begin{aligned} \dot{x}(t) &= \left(\begin{bmatrix} -1 & 0 & -0.5 \\ 1 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} + \Delta A(t) \right) x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \omega(t), \\ z(t) &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t) + 0.1u(t), \end{aligned} \tag{36}$$

where $||\Delta A(t)|| \leq 0.01$, $u(t) = Kx(t - d(t))$.

This example was also used in [3], where $h_1 = 0.1$, and the H_∞ norm bound $\gamma_{\min}$ was 1.9 when $h_2 = 0.5$. Using Theorem 4 with $U_1 = 1.2I$, $U_3 = U_4 = U_5 = 0$, it is found that, the H_∞ norm bound $\gamma_{\min}$ is 1.7 for $h_2 = 0.5$. For convenience, supposing $U_3 = U_4 = U_5 = \lambda$, where λ is a scalar, by applying a numerical optimization algorithm which is similar to the one in [11], it yields that

$$U_1 = \begin{bmatrix} 1.3323 & -0.1090 & -0.0455 \\ 0.0152 & 1.2057 & -0.0118 \\ 0.2177 & -0.2262 & 1.0809 \end{bmatrix}, \ \lambda = -0.0166,$$

and $\gamma_{\min}$ is 1.6, $\bar{M}_2$ and Q are given by

$$\bar{M}_2 = \begin{bmatrix} -4.7194 & -3.3291 & 4.1586 \\ 3.7084 & -3.3387 & -2.6421 \\ 1.5100 & 0.9831 & -3.2380 \end{bmatrix},$$

$$Q = \begin{bmatrix} -3.0629 & 1.5339 & 3.7297 \end{bmatrix},$$

respectively. Thus, the state feedback gain is given by

$$K = \begin{bmatrix} -0.6177 & -0.0048 & -1.4414 \end{bmatrix}.$$

V. CONCLUSIONS

In this paper, a new type of Lyapunov functionals is exploited to derive sufficient conditions for guaranteeing the robust exponential stability and H_∞ performance of the continuous-time networked control systems (NCSs). A method of eliminating redundant variables to reduce computation complexity is given, and it is shown that the new result is less conservative than the existing corresponding ones. A robust H_∞ controller design method is also presented. Numerical examples are given to illustrate the effectiveness of the proposed methods.

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