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Fast Information Sharing in Networks of Autonomous Agents

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Abstract— **A new nonlinear protocol is proposed for state consensus of multi-agent systems in this paper. It is shown that this protocol can provide faster convergence rate than the typical linear protocol, presented by Olfati-Saber and Murray, and furthermore guarantees the states of agents reach a consensus in finite time, provided that the interaction topology, represented by a directed graph, has a spanning tree.**

Index Terms— **Multi-agent systems, coordination, finite-time consensus, interaction topology.**

I. INTRODUCTION

The object of this paper is to provide an effective fast-convergent consensus protocol with finite settling time for networks of autonomous agents.

In recent years, consensus problem has attracted considerable attention of researchers and become an active area of research in coordinated control of multi-agent systems. Compared to sole agent, cooperation of multiple agents can perform more complex tasks, enhance efficiency, and be more robust against device failure. In order to reach a cooperation, it is often required that individual agents interact with each other and eventually agree on certain qualities of interest or share a common view of information. Those qualities, for instance, may be the aimed positions in rendezvous problem, the anticipated formation information in formation control, or the favorite attitude in attitude alignment, etc. In the study of consensus problem, consensus protocol (*algorithm*) is an interaction rule, which is designed, based on the local information obtained by each agent, to guarantee those critical information to be shared in a distributed manner. "Consensus" means that the states of agents, which can be viewed as the estimations of sharing information, are all the same.

In [1], Vicsek et al. proposed a simple but interesting discrete-time model of finite agents all moving in the plane. Each agent's motion is updated using a local rule based on its own state and the states of its neighbors. By using graph theory and nonnegative matrix theory, Jadbabaie et al. provided a theoretical explanation of the consensus property of the Vicsek model in [2], where each agent's set of neighbors was supposed to change with time as system evolves. The typical continuous-time model was proposed by Olfati-Saber and Murray in [3], where the concepts of solvability of consensus problem and consensus protocol were first introduced. In [3], the authors used a directed graph to model the interaction topology among agents and studied three consensus problems, namely, directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology, where it was assumed that the directed topology is balanced and strongly connected. In [4], Ren and Beard extended the results of [2] and [3] and presented mathematically weaker conditions for state consensus under dynamically changing directed interaction topology. In [5], [6], Z. Lin et al. studied consensus problem in the context of formation control of autonomous vehicles and proved that formation stabilization to a point is feasible if and only if the sensor digraph has a globally reachable node. In the past several years, consensus problem of multi-agent systems has been developing fast and several research topics have been addressed, such as agreement over random networks [7], [8], asynchronous information consensus [9], networks with nonlinear consensus protocols [10], networks with time-delays [3], [11], [12], [13], and high-dimensional consensus problem [14]. For details, see the survey papers [15] and [16] and references therein.

One main research topic in consensus problem is how to design consensus protocol. Consensus protocol is a distributed interaction rule among agents, which is aimed at ensuring that the concerned states of agents converge to a common value. Convergence rate is an importance index to evaluate the proposed protocol. It was shown that the algebraic connectivity of interaction topology quantifies the convergence speed of the typical linear consensus protocol, given in [3]. Kim and Mesbahi considered the problem of finding the best vertex positional configuration to maximize the associated algebraic connectivity [17]. To increase convergence rate, Xiao and Boyd provided a method of how to choose edge weights for discrete-time systems [18]. In [19], it was shown that small-word network topology is of large algebraic connectivity. Note that the above results are on how to find appropriate interaction topology so as to increase the convergence rate of their proposed protocols but not on how to improve the effectiveness of the proposed protocols themselves when the topology is given. Furthermore, the state consensus can never occur in a finite time under the aforementioned protocols. In some practical situations, it is required that the consensus be reached in a finite time. Therefore, finite-time agreement is more appealing and there are a number of settings where finite-time convergence is desirable. The protocol presented in this paper is shown to be

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with better convergence rate than the typical linear protocol when they are acting on the same topology, and moreover it is with finite settling time. The method used in this paper is of interest itself, which is partly motivated by the work of [20], in which continuous finite-time differential equations were introduced as fast accurate controllers for dynamical systems, and partly by the results of finite-time stability of homogeneous systems [21].

This paper is organized as follows. The problem is formulated in Section II. Convergence results and their technical proofs are given in Section III.

II. Problem Formulation

A. Graph theory preliminary

Graph plays a key role in representing the interaction topology among agents. We first give some basic definitions in graph theory [22].

For simplicity, let $\mathcal{I}_n$ denote the set $\{1, 2, \cdots, n\}$. A directed graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ consists a vertex set $\mathcal{V}(\mathcal{G}) = \{v_i, i \in \mathcal{I}_n\}$ and an edge set $\mathcal{E}(\mathcal{G}) \subset \{(v_i, v_j) : i, j \in \mathcal{I}_n\}$. If (v_i, v_j) is an edge of $\mathcal{G}$, v_i is called the parent vertex. The set of neighbors of vertex v_i in $\mathcal{G}$ is defined by $\mathcal{N}(\mathcal{G}, v_i) = \{v_j : (v_j, v_i) \in \mathcal{E}(\mathcal{G}), j \neq i\}$. The associated index set is denoted by $\mathcal{N}(\mathcal{G}, i) = \{j : v_j \in \mathcal{N}(\mathcal{G}, v_i)\}$. A *path* in directed graph $\mathcal{G}$ is a sequence $v_{i_1}, \cdots, v_{i_k}$ of vertices such that $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}(\mathcal{G})$ for $j = 1, \cdots, k-1$. A *directed tree* is a directed graph, where every vertex, except one special vertex without any parent, which is called the *root vertex*, has exactly one parent, and the root vertex can be connected to any other vertices through paths. Directed graph $\mathcal{G}$ is *strongly connected* if between every pair of distinct vertices v_i, v_j, there exists a path that begins at v_i and ends at v_j.

A *subgraph* $\mathcal{G}_s$ of $\mathcal{G}$ is a directed graph such that the vertex set $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$ and the edge set $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$. If $\mathcal{V}(\mathcal{G}_s) = \mathcal{V}(\mathcal{G})$, $\mathcal{G}_s$ is called a *spanning subgraph*. For any $v_i, v_j \in \mathcal{V}(\mathcal{G}_s)$, if $(v_i, v_j) \in \mathcal{E}(\mathcal{G}_s) \iff (v_i, v_j) \in \mathcal{E}(\mathcal{G})$, $\mathcal{G}_s$ is called an *induced subgraph*. We also say that $\mathcal{G}_s$ is induced by $\mathcal{V}(\mathcal{G}_s)$. A *spanning tree* of $\mathcal{G}$ is a directed tree that is a spanning subgraph of $\mathcal{G}$. A directed graph is said to have a spanning tree if a subset of the edges forms a spanning tree. A *strongly connected component* of directed graph $\mathcal{G}$ is an induced subgraph that is maximal, subject to being strongly connected. Since any subgraph consisting of only one vertex is strongly connected, it follows that each vertex lies in a strongly connected component, and therefore the strongly connected components of $\mathcal{G}$ partition its vertices. Moreover, the relation determined by lying in the same strongly connected component is an equivalence relation. We introduce another directed graph, denoted by $\mathcal{G}^c$, consisting of all strongly connected components $u_1, u_2, \cdots, u_k$ of $\mathcal{G}$, such that $(u_i, u_j) \in \mathcal{E}(\mathcal{G}^c)$ if and only if there exist $v_{i'} \in \mathcal{V}(u_i)$ and $v_{j'} \in \mathcal{V}(u_j)$ satisfying $(v_{i'}, v_{j'}) \in \mathcal{E}(\mathcal{G})$. A *weighted directed graph* $\mathcal{G}(A)$ is a directed graph $\mathcal{G}$ plus a nonnegative (element-wise) matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that $(v_i, v_j) \in \mathcal{G} \iff a_{ji} > 0$. A is called the weight matrix and a_{ij} is called the weight of edge (v_j, v_i). In what follows, matrix A is said nonnegative or positive, denoted by $A \geq 0$ or $A > 0$ separately, if entries of matrix A are not less or larger than zero. We say $\mathcal{G}$ is undirected if $(v_i, v_j) \in \mathcal{V}(\mathcal{G})$ implies that $(v_j, v_i) \in \mathcal{V}(\mathcal{G})$, and we say $\mathcal{G}(A)$ is undirected if $A^T = A$. An undirected graph is connected if the graph, as a special directed graph, is strongly connected.

B. Model

The system studied in this paper consists of n autonomous agents, labeled 1 through n. All these agents share a common state space $\mathbb{R}$. Let x_i denote the state of agent i, let x denote $[x_1, x_2, \cdots, x_n]^T$, and suppose that agent i takes the following continuous-time dynamics

$$\dot{x}_i(t) = u_i(t), i \in \mathcal{I}_n, \tag{1}$$

where $u_i(t)$ is the state feedback, called *protocol*, to be designed based on the local state information obtained by agent i from its neighbors.

We use a weighted directed graph $\mathcal{G}(A)$ to represent the interaction topology among agents, where the diagonal entries of A are supposed to be zeros. Vertex v_i represents agent i; edge (v_i, v_j) represents information channel from agent i to agent j, which means that agent j can receive the state information of agent i; the neighbors of agent i are defined as those agents, whose information can be received by agent i, namely, they correspond to $\mathcal{N}(\mathcal{G}(A), v_i)$. The local interaction topology of a group of agents is the subgraph induced by their associated vertices.

Given protocol u_i, $i \in \mathcal{I}_n$, u_i or this multi-agent system is said to solve a consensus problem if for any initial states and any $i, j \in \mathcal{I}_n$, $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ (cf. [3]), and it is said to solve a finite-time consensus problem if for any initial states, there exists a finite-time t^* such that $x_i(t) = x_j(t)$ for any i, j and any $t \geq t^*$. If protocol u_i solves a consensus problem under proper conditions, then the protocol is said to be *consensus* protocol. Next section will give a nonlinear protocol, which will be proved to be effective in consensus building.

III. Convergence Results

A. Consensus protocol

We first introduce the function $\mathrm{sig}(r)^\alpha \triangleq \mathrm{sign}(r)|r|^\alpha$, where $r \in \mathbb{R}$, $\alpha > 0$ and $\mathrm{sign}(\cdot)$ is the sign function, defined by

$$\mathrm{sign}(r) = \begin{cases} 1, & r > 0 \\ 0, & r = 0 \\ -1, & r < 0 \end{cases}.$$

Function $\mathrm{sig}(r)^\alpha$ can be easily proved to be continuous with respect to r. And $\frac{d|r|^{\alpha+1}}{dr} = (\alpha+1)\,\mathrm{sig}(r)^\alpha$. For simplicity, if r is a vector, then $\mathrm{sig}(r)^\alpha$ is also a vector with the same dimensions as r, obtained by operating $\mathrm{sig}(\cdot)^\alpha$ on each entry of r.

With the above preparation, we present the following protocol

$$u_i = \beta \,\mathrm{sig}\left(\sum_{j\in\mathcal{N}(\mathcal{G}(A),i)} a_{ij}(x_j - x_i)\right)^{\alpha} + \gamma \sum_{j\in\mathcal{N}(\mathcal{G}(A),i)} a_{ij}(x_j - x_i), \tag{2}$$

where $0 < \alpha < 1$, $\beta > 0$, $\gamma \geq 0$.

By the property of function $\mathrm{sig}(r)^{\alpha}$, u_i is continuous with respect to state variables x_i, $i \in \mathcal{I}_n$. If $\beta = 0$, $\gamma = 1$, then the above protocol becomes the typical linear protocol[3], which was presented by Olfati-Saber and Murray and was proved to solve a consensus problem under appropriate conditions. If $\alpha = 0$, $\beta \neq 0$, then the above protocol becomes discontinuous with respect to state variables. This case is beyond the scope of our research. However, it is worthy of mentioning that the case when $\alpha = 0$, $\beta = 1$, $\gamma = 0$, and A is a $0-1$ symmetric matrix was studied by Cortés. Interested reader may refer to [23].

Next, we list some basic properties of system (1) under protocol (2).

Property 1: Protocol (2) is continuous with respect to state variables $x_1, x_2, \ldots, x_n$. Moreover, under this protocol and given any initial state $x(0)$, there exists at least one solution of differential equations (1) on $[0, \infty)$. Furthermore, $\max_i x_i(t)$ is non-increasing and $\min_i x_i(t)$ is non-decreasing. Hence, $\|x(t)\|_{\infty}$ is also non-increasing and $\|x(t)\|_{\infty} \leq \|x(0)\|_{\infty}$ for all $t \geq 0$.

B. Finite-time convergence

Now, we present one of the main results.

Theorem 1: If the interaction topology $\mathcal{G}(A)$ has a spanning tree, the proposed protocol (2) solves a finite-time consensus problem.

To prove Theorem 1, we need the following lemmas. Proofs of them are omitted due to the page limitation.

Lemma 1 ([3], [4]): Let $L(A) = [l_{ij}] \in \mathbb{R}^{n\times n}$ denote the *graph Laplacian* of $\mathcal{G}(A)$, which is defined by

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases}.$$

Then

(i) 0 is an eigenvalue of $L(A)$ and $\mathbf{1}$ is the associated eigenvector, where $\mathbf{1} = [1, 1, \cdots, 1]^T$ with compatible dimensions;

(ii) if $\mathcal{G}(A)$ has a spanning tree, then eigenvalue 0 is algebraically simple and all other eigenvalues are with positive real parts;

(iii) if $\mathcal{G}(A)$ is strongly connected, then there exists a positive column vector $\omega \in \mathbb{R}^n$ such that $\omega^T L(A) = 0$;

if $\mathcal{G}(A)$ is undirected, namely, $A^T = A$, and connected, then $L(A)$ has the following properties:

(iv) $\xi^T L(A)\xi = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\xi_j - \xi_i)^2$ for any $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{R}^n$, and therefore $L(A)$ is semi-positive definite, which implies that all eigenvalues of $L(A)$ are nonnegative real numbers;

(v) denote the eigenvalues of $L(A)$ by $0, \lambda_2(L(A)), \cdots, \lambda_n(L(A))$ in the increasing order. The second smallest eigenvalue of $L(A)$, $\lambda_2(L(A))$, called the *algebraic connectivity* of $\mathcal{G}(A)$, is larger than zero;

(vi) the algebraic connectivity of $\mathcal{G}(A)$ is equal to $\min_{\xi\neq 0, \mathbf{1}^T\xi=0} \frac{\xi^T L(A)\xi}{\xi^T\xi}$, and therefore, if $\mathbf{1}^T\xi = 0$, then

$$\xi^T L(A)\xi \geq \lambda_2(L(A))\xi^T\xi.$$

Corollary 1: Suppose $\mathcal{G}(A)$ is strongly connected, and let $\omega > 0$ such that $\omega^T L(A) = 0$. Then $\mathrm{diag}(\omega)L(A) + L(A)^T\mathrm{diag}(\omega)$ is the graph Laplacian of the undirected weighted graph $\mathcal{G}(\mathrm{diag}(\omega)A + A^T\mathrm{diag}(\omega))$. And therefore it is semi-positive definite, 0 is its algebraically simple eigenvalue and $\mathbf{1}$ is the associated eigenvector.

Corollary 2: Let $b = [b_1, b_2, \ldots, b_n]^T \geq 0$, $b \neq 0$, and let $\mathcal{G}(A)$ be undirected and connected. Then $L(A) + \mathrm{diag}(b)$ is positive definite, where $\mathrm{diag}(b)$ is the diagonal matrix with the ith diagonal entry being b_i.

Lemma 2: Let $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and let $0 < p \leq 1$. Then

$$\sum_{i=1}^{n}\xi_i^p \geq \left(\sum_{i=1}^{n}\xi_i\right)^p.$$

Proof of Theorem 1:

Proof: This theorem is proved through the following three steps.

Step 1: Suppose that $\mathcal{G}(A)$ is strongly connected.

Then by Lemma 1, there exists vector $\omega = [\omega_1, \omega_2, \cdots, \omega_n]^T \in \mathbb{R}^n$ such that $\omega > 0$ and $\omega^T L(A) = 0$. Let $y_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i)$ and $y = [y_1, y_2, \cdots, y_n]^T$. Then $\omega \perp y$ and

$$\dot{y}_i = \sum_{j=1}^{n} a_{ij}\big(\beta\,\mathrm{sig}(y_j)^{\alpha} + \gamma y_j - \beta\,\mathrm{sig}(y_i)^{\alpha} - \gamma y_i\big), i \in \mathcal{I}_n,$$

which is equivalent to

$$\dot{y} = -\beta L(A)\,\mathrm{sig}(y)^{\alpha} - \gamma L(A)y.$$

Here, $\mathrm{sig}(y)^{\alpha}$ represents the column vector $[\mathrm{sig}(y_1)^{\alpha}, \mathrm{sig}(y_2)^{\alpha}, \cdots, \mathrm{sig}(y_n)^{\alpha}]^T$.

Consider Lyapunov candidate

$$V_1(t) = \sum_{i=1}^{n}\omega_i\left(\frac{\beta}{1+\alpha}|y_i|^{1+\alpha} + \frac{\gamma}{2}y_i^2\right).$$

Obviously, $V_1(t) \geq 0$. And

$$\begin{aligned}\frac{dV_1(t)}{dt} &= \sum_{i=1}^{n}\omega_i\big(\beta\,\mathrm{sig}(y_i)^{\alpha} + \gamma y_i\big)\dot{y}_i \\ &= -\big(\beta\,\mathrm{sig}(y)^{\alpha} + \gamma y\big)^T\mathrm{diag}(\omega)L(A)\big(\beta\,\mathrm{sig}(y)^{\alpha} + \gamma y\big).\end{aligned}$$

Claim 1: Given initial state $x(0)$, there exists $K_1 > 0$ such that

$$\big(\beta\,\mathrm{sig}(y)^{\alpha} + \gamma y\big)^T\mathrm{diag}(\omega)L(A)\big(\beta\,\mathrm{sig}(y)^{\alpha} + \gamma y\big) \geq K_1 V_1(t)^{\frac{2\alpha}{1+\alpha}}.$$

Therefore,

$$\frac{dV_1(t)}{dt} \leq -K_1 V_1(t)^{\frac{2\alpha}{1+\alpha}}.$$

By Comparison Principle of differential equations, $V_1(t) \le v_1(t)$, where

$$v_1(t) = \begin{cases} \Big(- K_1 \frac{1-\alpha}{1+\alpha} t + V_1(0)^{\frac{1-\alpha}{1+\alpha}} \Big)^{\frac{1+\alpha}{1-\alpha}}, & t < t^* \\ 0, & t \ge t^* \end{cases},$$

where $t^* = \frac{(1+\alpha)V(0)^{\frac{1-\alpha}{1+\alpha}}}{K_1(1-\alpha)}$.

Thus, $V_1(t)$ will reach zero in finite time t^*, which implies that y will be zero. Since $y = -L(A)x$ and $\text{rank}(L(A)) = n-1$ (by Lemma (1)), $y = 0$ implies that $x \in \text{span}\{\mathbf{1}\} \triangleq \{r\mathbf{1} : r \in \mathbb{R}\}$ and $\dot{x}(t) = 0$. Therefore, this system solves a finite-time consensus problem.

It is left to us to prove that Claim 1 holds.

Suppose that $V_1(t) \neq 0$, namely, $y \neq 0$. Let $f(y)$ denote $\frac{\big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)^T \text{diag}(\omega) L(A) \big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)}{V_1(t)^{\frac{2\alpha}{1+\alpha}}}$, let M denote the matrix $\frac{1}{2}\big(\text{diag}(\omega)L(A) + L(A)^T \text{diag}(\omega)\big)$, and let $\mathcal{U} = \{\xi \in \mathbb{R}^n : \xi^T\xi = 1, \text{ and}, \ \xi = \beta\,\text{sig}(\zeta)^\alpha + \gamma\zeta \text{ for some } \zeta \perp \omega \}$. Then $\mathcal{U}$ is a bounded closed set. Since function $\xi^T M \xi$ is continuous and for any $\xi \in \mathcal{U}$, $\xi^T M \xi \neq 0$ (by Corollary 1), we have that $\min_{\xi \in \mathcal{U}} \xi^T M \xi$, denoted by k_1, exists and larger that zero. Since $\omega \perp y$, we have that $\frac{\beta\,\text{sig}(y)^\alpha + \gamma y}{\sqrt{(\beta\,\text{sig}(y)^\alpha + \gamma y)^T(\beta\,\text{sig}(y)^\alpha + \gamma y)}} \in \mathcal{U}$ and

$$\frac{\big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)^T M \big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)}{\big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)^T \big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)} \ge k_1.$$

Therefore,

$$\begin{aligned} f(y) \ge & \frac{k_1 \big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)^T \big(\beta\,\text{sig}(y)^\alpha + \gamma y\big)}{V(t)^{\frac{2\alpha}{1+\alpha}}} \\ \ge & \frac{k_1 \sum_{i=1}^n \big(\beta\,\text{sig}(y_i)^\alpha + \gamma y_i\big)^2}{\sum_{i=1}^n \left(\frac{\beta\omega_i}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{2\alpha} + \sum_{i=1}^n \left(\frac{\gamma\omega_i}{2}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{\frac{4\alpha}{1+\alpha}}} \\ \ge & \frac{k_1 \beta^2 \sum_{i=1}^n |y_i|^{2\alpha}}{\sum_{i=1}^n \left(\frac{\beta\omega_i}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{2\alpha} + \sum_{i=1}^n \left(\frac{\gamma\omega_i}{2}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{\frac{4\alpha}{1+\alpha}}} \\ \triangleq & \ g(y). \end{aligned}$$

The second inequality follows from Lemma 2 and the third inequality follows from the fact that y_i and $\text{sig}(y_i)^\alpha$ are with the same sign.

Let $\mathcal{W} = \{\xi \in \mathbb{R}^n : 1 \le \|\xi\|_\infty \le \|L(A)\|_{i\infty}\|x(0)\|_\infty\}$, where $\|\cdot\|_{i\infty}$ is the induced matrix norm of maximum norm $\|\cdot\|_\infty$. Suppose $\mathcal{W} \neq \emptyset$. Then it is compact, $g(\xi) \neq 0$ when $\xi \in \mathcal{W}$, and thus $\min_{\xi \in \mathcal{W}} g(\xi)$, denoted by k_2, exists, and is also larger than zero. Therefore, if $y \in \mathcal{W}$, then $f(y) \ge k_2$.

Because $|y_i(t)| \le \|y(t)\|_\infty = \|-L(A)x(t)\|_\infty \le \|L(A)\|_{i\infty}\|x(t)\|_\infty \le \|L(A)\|_{i\infty}\|x(0)\|_\infty$, if $y \notin \mathcal{W}$, then $0 < \|y\|_\infty < 1$, and thus

$$g(y) > \frac{k_1\beta^2 \sum_{i=1}^n |y_i|^{2\alpha}}{2\omega_0 \sum_{i=1}^n |y_i|^{2\alpha}} = \frac{k_1\beta^2}{2\omega_0},$$

where $\omega_0 = \max\Big\{ \left(\frac{\beta\omega_i}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}}, i \in \mathcal{I}_n; \left(\frac{\gamma\omega_i}{2}\right)^{\frac{2\alpha}{1+\alpha}}, i \in \mathcal{I}_n \Big\}$.

Let $K_1 = \min\big\{k_2, \frac{k_1\beta^2}{2\omega_0}\big\}$. Then Claim 1 holds.

Step 2: Next, we suppose that $\mathcal{G}(A)$ has a spanning tree, the associated root vertex is v_i, and the subgraph induced by the remaining vertices is strongly connected. Furthermore, we suppose that there exists not any path connecting those vertices to vertex v_i.

Without loss of generality, assume that the root vertex v_i is v_n. Therefore, we get that $a_{n1} = a_{n2} = \cdots = a_{nn} = 0$ and $a_{1n}, a_{2n}, \cdots, a_{n-1,n}$ are not all zeros. For convenience, let $m = n-1$, $b = [b_1, b_2, \cdots, b_m]^T = [a_{1n}, a_{2n}, \cdots, a_{mn}]^T$ and denote $z_i = x_i - x_n$, $i = 1, 2\cdots, m$, $A^0 = [a_{ij}]_{1 \le i,j \le m}$. Then

$$\begin{aligned} \dot{z}_i = \dot{x}_i = & \beta\,\text{sig}\left(\sum_{j=1}^m a_{ij}(z_j - z_i) - b_i z_i\right)^\alpha \\ & + \gamma\left(\sum_{j=1}^m a_{ij}(z_j - z_i) - b_i z_i\right). \end{aligned}$$

Let $y_i = \sum_{j=1}^m a_{ij}(z_j - z_i) - b_i z_i$, $i = 1, 2, \cdots, m$, and let $y^0 = [y_1, y_2, \cdots, y_m]^T$. Then

$$\begin{aligned} \dot{y}_i = & \sum_{j=1}^m a_{ij}\big(\beta\,\text{sig}(y_j)^\alpha + \gamma y_j - \beta\,\text{sig}(y_i)^\alpha - \gamma y_i\big) \\ & - b_i\big(\beta\,\text{sig}(y_i)^\alpha + \gamma y_i\big). \end{aligned}$$

Because the subgraph induced by $v_1, v_2, \cdots, v_m$ is strongly connected, i.e., $\mathcal{G}(A^0)$ is strongly connected, by lemma 1, there exists a positive m-column vector $\omega^0 = [w_1, w_2, \cdots, w_m]^T$ such that $\omega^0 L(A^0) = 0$. Consider Lyapunov candidate

$$V_2(t) = \sum_{i=1}^m \omega_i \left(\frac{\beta}{1+\alpha}|y_i|^{1+\alpha} + \frac{\gamma}{2}y_i^2\right).$$

$$\begin{aligned} \frac{dV_2(t)}{dt} = & \sum_{i=1}^m w_i\big(\beta\,\text{sig}(y_i)^\alpha + \gamma y_i\big)\dot{y}_i \\ = & \big(\beta\,\text{sig}(y^0)^\alpha + \gamma y^0\big)^T \big(-\text{diag}(w^0)L(A^0) - \text{diag}(b)\big) \\ & \times \big(\beta\,\text{sig}(y^0)^\alpha + \gamma y^0\big). \end{aligned}$$

Claim 2: Given initial state $x(0)$, there exists $K_2 > 0$ such that

$$\begin{aligned} & \big(\beta\,\text{sig}(y^0)^\alpha + \gamma y^0\big)^T \big(\text{diag}(w^0)L(A^0) + \text{diag}(b)\big) \\ & \qquad \times \big(\beta\,\text{sig}(y^0)^\alpha + \gamma y^0\big) \ge K_2 V_2(t)^{\frac{2\alpha}{1+\alpha}}. \end{aligned}$$

If Claim 2 holds, by the same arguments as in the first step, we have that $V_2(t)$ will reach zero in finite time. $V_2(t) = 0$ implies that $y^0 = 0$. Since $L(A) = \begin{bmatrix} L(A^0) + \text{diag}(b) & -b \\ 0 & 0 \end{bmatrix}$, by Lemma 1, $L(A^0) + \text{diag}(b)$ is of full rank. And from $y^0 = \big(-L(A^0) - \text{diag}(b)\big)z$, we have $z = 0$, where $z = [z_1, z_2, \cdots, z_m]^T$. Therefore, in this case, the system also solves a finite-time consensus problem and the final state is x_n.

Next, we prove Claim 2. By Corollaries 1 and 2, $\frac{1}{2}\big(\text{diag}(w^0)L(A^0) + L(A^0)^T \text{diag}(w^0)\big) + \text{diag}(b)$ is positive definite. Denote the smallest eigenvalue of it by λ. Then if $V_2(t) \neq 0$,

$$\begin{aligned}\frac{-dV_2(t)/dt}{V_2(t)^{\frac{2\alpha}{1+\alpha}}} &\geq \frac{\lambda \sum_{i=1}^m \left(\beta \operatorname{sig}(y_i)^\alpha + \gamma y_i\right)^2}{V_2(t)^{\frac{2\alpha}{1+\alpha}}} \\ &\geq \frac{\lambda \sum_{i=1}^m \left(\beta \operatorname{sig}(y_i)^\alpha + \gamma y_i\right)^2}{\sum_{i=1}^m \left(\frac{\omega_i \beta}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{2\alpha} + \sum_{i=1}^m \left(\frac{\omega_i \gamma}{2}\right)^{\frac{2\alpha}{1+\alpha}} |y_i|^{\frac{4\alpha}{1+\alpha}}} \\ &\triangleq g^0(y^0)\end{aligned}$$

It can be observed that $\begin{bmatrix} y^0 \\ 0 \end{bmatrix} = -L(A)x$ and thus for any $i \in \{1, 2, \cdots, m\}$, $|y_i| \leq \|-L(A)\|_{i\infty} \|x(0)\|_\infty$. With the similar arguments as in the first step, there exists $K_2 > 0$, such that $g^0(y^0) \geq K_2$. Therefore, Claim 2 holds.

Step 3: Finally, we prove the general case, namely, the case when $\mathcal{G}(A)$ has a spanning tree, by induction.

Consider the directed graph $\mathcal{G}^c(A)$ consisting of the strongly components in $\mathcal{G}(A)$. Obviously, $\mathcal{G}^c(A)$ is a directed tree.

(i) The dynamics of agents corresponding to the vertex set of the root of $\mathcal{G}^c(A)$ is not affected by others and the local interaction topology among them is strongly connected. By the conclusion of the first step, the states of them will reach consensus in a finite time. The final state is denoted by x_0.

(ii) Consider the dynamics of agents, denoted by $v_{i_1}, v_{i_2}, \cdots, v_{i_{k_i}}$, corresponding to the vertex set of some vertex (not the root vertex) of $\mathcal{G}^c(A)$. It is only affected by those agents, such that there exist paths connecting them to $v_{i_l}, l = 1, \cdots, k_i$. Suppose those agents excluding $v_{i_l}, l = 1, 2, \cdots, k_i$, are $v_{j_1}, v_{j_2}, \cdots, v_{i_{k_j}}$, the states of them have already reached consensus, and the consensus state is x_0. Then for any $l \in \{1, 2, \cdots, k_i\}$,

$$\begin{aligned}&\sum_{s=1}^{k_i} a_{i_l, i_s}(x_{i_s} - x_{i_l}) + \sum_{s=1}^{k_j} a_{i_l, j_s}(x_{j_s} - x_{i_l}) \\ =&\sum_{s=1}^{k_i} a_{i_l, i_s}(x_{i_s} - x_{i_l}) + \left(\sum_{s=1}^{k_j} a_{i_l, j_s}\right)(x_0 - x_{i_l})\end{aligned}$$

Therefore, $v_{j_1}, v_{j_2}, \cdots, v_{i_{k_j}}$, as a whole, can be seen as one (virtual) agent and as the leader of $v_{i_l}, l = 1, \cdots, k_i$. Specifically, if we relabel the vertices $v_{i_l}, l = 1, \cdots, k_i$ by $v_l, l = 1, \cdots, k_i$ and label the virtual agent by $k_i + 1$, then the dynamics of $v_l, l = 1, \cdots, k_i$, is the same as when the local interaction topology among $v_l, l = 1, 2, \cdots, k_i + 1$ is $\mathcal{G}(A')$ and protocol (2) is applied, where

$$A' = \begin{bmatrix} a_{i_1, i_1} & \cdots & a_{i_1, i_{k_i}} & \sum_{s=1}^{k_j} a_{i_1, j_s} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i_{k_i}, k_1} & \cdots & a_{i_{k_i}, i_{k_i}} & \sum_{s=1}^{k_j} a_{i_{k_i}, j_s} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is not difficult to get that $\mathcal{G}(A')$ satisfies the assumption of the second step. Thus, the states of agents $i_l, l = 1, 2, \cdots, k_i$, will reach consensus in finite time, and the final state of them is x_0.

(iii) With the above conclusion and by induction, the system solves a finite-time consensus problem, and the final state is also x_0. ■

Remark: V_1 and V_2 can be viewed as the measurement of the length of y and y^0 respectively. Since $\omega \perp y$, for any $x \in \mathbb{R}^n$, there exists $r \in \mathbb{R}$ such that $x = r\mathbf{1} + y$. Therefore, the length of y reflects the difference of x_i, $i \in \mathcal{I}_n$, namely, $V_1(t)$ measures the disagreement of agents' states at time t. On the other hand, because $y^0 = -(L(A^0) + \operatorname{diag}(b))z$ and $(L(A^0) + \operatorname{diag}(b))$ is invertible, the length of y_0 also can be viewed as the length of z in the sense of norm equivalence. And because $z_i = x_i - x_n$, $i = 1, 2, \cdots, n-1$, $V_2(t)$ measures the difference between agents' current states and their final agreement state.

C. Comparison with the typical linear protocol[3]

In fact, protocol (2) represents a class of protocols with free parameters α, β, and γ. If $\beta = 0$ and $\gamma \neq 0$, protocol (2) becomes the typical linear consensus protocol, presented by Olfati-Saber and Murray, under interaction topology $\mathcal{G}(\gamma A)$. In this subsection, parameter γ in (2) is supposed to be equal to one, that is, we compare the convergence speeds of the following two protocols acting on the same topology $\mathcal{G}(A)$.

(i) the linear protocol, presented by Olfati-Saber and Murray,

$$u_i = \sum_{j \in \mathcal{N}(\mathcal{G}(A(t)), i)} a_{ij}(x_j - x_i). \tag{3}$$

(ii) protocol (2) with $\gamma = 1$,

$$\begin{aligned}u_i =& \beta \operatorname{sig}\left(\sum_{j \in \mathcal{N}(\mathcal{G}(A), i)} a_{ij}(x_j - x_i)\right)^\alpha \\ &+ \sum_{j \in \mathcal{N}(\mathcal{G}(A), i)} a_{ij}(x_j - x_i);\end{aligned} \tag{4}$$

To show our presented protocol (4) converge faster than protocol (3), we first study the case when $\mathcal{G}(A)$ is undirected and connected. In this case, the two systems have a common Lyapunov candidate $V_3(t) = \frac{1}{4} \sum_{i,j=1}^n a_{ij}\left(x_j(t) - x_i(t)\right)^2$, with the property that $\frac{\partial V_3}{\partial x_i} = -\sum_{j=1}^n a_{ij}(x_j - x_i)$. If $V_3(t) \neq 0$,

$$\begin{aligned}\left.\frac{dV_3(t)}{dt}\right|_{(3)} &= \sum_{i=1}^n \frac{\partial V_3(t)}{\partial x_i} \dot{x}_i \\ &= -\sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x_j - x_i)\right)^2 \\ &= -\frac{x^T L(A)^T L(A) x}{\frac{1}{2} x^T L(A) x} V_3(t) \\ &\leq -2\lambda_2(L(A)) V_3(t),\end{aligned} \tag{5}$$

and

$$\left.\frac{dV_3(t)}{dt}\right|_{(4)} = -\beta\sum_{i=1}^{n}\left|\sum_{j=1}^{n}a_{ij}(x_j-x_i)\right|^{1+\alpha} \tag{6}$$

$$-\sum_{i=1}^{n}\left(\sum_{j=1}^{n}a_{ij}(x_j-x_i)\right)^{2}$$

$$\leq -\beta\big(2\lambda_2(L(A))V_3(t)\big)^{\frac{1+\alpha}{2}} - 2\lambda_2(L(A))V_3(t). \tag{7}$$

Therefore, the system under protocol (4) has a better convergence rate than the system under protocol (3).

The following theorem also illustrates this fact, although it is of a little conservation.

Theorem 2: Assume that $\mathcal{G}(A)$ is undirected and connected, and $\lambda_n(L(A)) \neq \lambda_2(L(A))$. Given initial state $x(0)$, denote $V_3(t)$ under protocol (3) and protocol (4) by $V_L(t)$ and $V_N(t)$ separately. If $V_L(0) = V_N(0) \leq \left(\frac{\beta\left(2\lambda_2(L(A))\right)^{\frac{2}{1-\alpha}}}{2\left(\lambda_n(L(A))-\lambda_2(L(A))\right)}\right)^{\frac{2}{1-\alpha}}$, then $V_N(t) \leq V_L(t)$ for all $t \geq 0$.

Proof: Since $\dot{V}_N(t) \leq 0$, $V_N(t) \leq V_N(0)$, and thus $V_N(t) \leq \left(\frac{\beta\left(2\lambda_2(L(A))\right)^{\frac{1+\alpha}{2}}}{2\left(\lambda_n(L(A))-\lambda_2(L(A))\right)}\right)^{\frac{2}{1-\alpha}}$.

$$V_N(t) \leq \left(\frac{\beta\big(2\lambda_2(L(A))\big)^{\frac{1+\alpha}{2}}}{2\big(\lambda_n(L(A))-\lambda_2(L(A))\big)}\right)^{\frac{2}{1-\alpha}}$$

$$\Longleftrightarrow V_N(t)^{\frac{1-\alpha}{2}} \leq \frac{\beta\big(2\lambda_2(L(A))\big)^{\frac{1+\alpha}{2}}}{2\big(\lambda_n(L(A))-\lambda_2(L(A))\big)}$$

$$\Longleftrightarrow 2\big(\lambda_n(L(A))-\lambda_2(L(A))\big)V_N(t)^{\frac{1-\alpha}{2}} \leq \beta\big(2\lambda_2(L(A))\big)^{\frac{1+\alpha}{2}}$$

$$\Longleftrightarrow 2\big(\lambda_n(L(A))-\lambda_2(L(A))\big)V_N(t) \leq \beta\big(2\lambda_2(L(A))V_N(t)\big)^{\frac{1+\alpha}{2}}$$

$$\Longleftrightarrow -\beta\big(2\lambda_2(L(A))V_N(t)\big)^{\frac{1+\alpha}{2}} - 2\lambda_2(L(A))V_N(t) \leq -2\lambda_n(L(A))V_N(t),$$

which further implies by (7) that $\frac{dV_N(t)}{dt} \leq -2\lambda_n(L(A))V_N(t)$. On the other hand, by (5), $\frac{dV_L(t)}{dt} \geq -2\lambda_n(L(A))V_L(t)$. By Comparison Principle of differential equations, $V_N(t) \leq V_L(t)$ for all $t \geq 0$. ■

In the general case, that is, we only know that $\mathcal{G}(A)$ has a spanning tree, we take the Lyapunov function $V_4(t) = \max_{i\in\mathcal{I}_n} x_i - \min_{i\in\mathcal{I}_n} x_i$ for consensus state, which was presented in[24]. Clearly, $V_4(t) = 0$ if and only if $x \in \text{span}\{\mathbf{1}\}$.

Since for any i, x_i is differentiable, $V_4(t)$ is piecewise differentiable and its right-derivative $\dot{V}_{4+}(t)$ exists. Given $x(t)$, let $\mathcal{I}_1 = \{j : x_j = \max_{i\in\mathcal{I}_n} x_i\}$ and $\mathcal{I}_2 = \{j : x_j = \min_{i\in\mathcal{I}_n} x_i\}$. It can be observed from the equations (3) and (4) that $\max_{i\in\mathcal{I}_1} u_i|_{(4)} \leq \max_{i\in\mathcal{I}_1} u_i|_{(3)} \leq 0$ and $\min_{i\in\mathcal{I}_2} u_i|_{(4)} \geq \min_{i\in\mathcal{I}_2} u_i|_{(3)} \geq 0$. Since $\dot{V}_{4+}(t) = \max_{i\in\mathcal{I}_1} u_i - \min_{i\in\mathcal{I}_2} u_i$, we have that

$$\dot{V}_{4+}(t)|_{(4)} \leq \dot{V}_{4+}(t)|_{(3)},$$

which also shows that the system under protocol (4) converges faster than that under protocol (3).

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