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Numerical Methods for Buying-low-and-selling-high Stock Policy

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***Abstract*— This work develops numerical methods using stochastic approximation approach for an optimal stock trading (buy and sell) strategy. Assuming the underlying asset price is governed by a mean-reverting stochastic process, we aim to find buying and selling strategies so as to maximize an overall expected return. One of the advantageous of our approach is that the underlying asset is model free. Only mean reversion is required. Slippage cost is taken into consideration for each transaction. Convergence of the algorithms is provided. Numerical examples are reported to demonstrate the results.**

I. Introduction

This work develops a systematic procedure for an asset trading strategy, where the underlying asset price is subject to random disturbances. There are two main ingredients, namely, buy and sell. A time honored trading strategy is to buy low and sell high. Nevertheless, identifying the lows and highs is a challenging task. In this paper, we develop an easily implementable procedure to identify lows and highs when the underlying asset price is mean reverting. Other than the mean reversion, we deal with model free stock returns by using observed stock prices only.

A mean-reversion formulation is often used in financial and energy markets. It aims to capture price movements that have the tendency to move towards an equilibrium level. Studies of mean-reversion stock returns can be traced back to the 1930's; see Cowles and Jones [6]. The research was carried further in [10], [12] among others. Mean-reversion models were also used to characterize stochastic volatility [14], asset prices in energy markets [1], and option pricing with a mean-reversion asset [2], [11], [16].

Trading rules in financial markets have been studied for many years. Researchers from both academia and industry have devoted their attention to such problems. For example, in [25] a selling rule determined by two threshold levels, a target price and a stop-loss limit, was obtained. One makes a selling decision whenever the price reaches either the target price or the stop-loss limit. The formulation of stock price in mathematical terms is based on a regime-switching geometric Brownian motion models. The objective is to determine the threshold levels to maximize an expected discounted reward function, and the optimal threshold levels are obtained by solving a set of two-point boundary value problems. In [13], the optimal selling rule under a model with switching GBM was considered using a smooth fit technique, and the optimal stopping rule was characterized by a set of algebraic equations under certain smoothness conditions. In [22], aiming at designing a systematic approach, a class of stochastic approximation algorithms is developed for obtaining the optimal selling rule. Further numerical and asymptotic results were obtained in [23]. Along another line, a linear programming approach was developed in [15]; fast Fourier transform was used in [19]; capital gain taxes and transaction costs in connection with selling was found in [3], [5], [7], among others. These papers have been devoted to selling strategies, whereas rigorous mathematical analysis on the buying side of trading has only been developed recently in [24], in which both buy and sell actions were taken. The objective was to buy and sell the underlying asset sequentially to maximize a discounted reward function.

As explained in the aforementioned paper another feature addressed is slippage cost associated with each transaction, where slippage cost usually refers to the spread between expected price and the actual price paid. The study was carried out using a dynamic programming approach and the associated HJB equations or variational inequalities for the value functions. The essence is that the optimal stopping times can be determined by two threshold levels. Suppose that $X(t) \in \mathbb{R}$ is a mean-reverting, Ornstein-Uhlenbeck (OU) process governed by

$$dX(t) = a(b - X(t))dt + \sigma dW(t),\ X(0) = x, \quad (1)$$

where $a > 0$ is the rate of reversion, b is the equilibrium level, $\sigma > 0$ is the volatility, and $W(t)$ is a standard Brownian motion. The asset price is given by

$$S(t) = \exp(X(t)). \quad (2)$$

Then two sequences of stopping times $\{\tau^{\{b_n\}}\}$ and $\{\tau^{\{s_n\}}\}$ with $0 \le \tau^{\{b_1\}} \le \tau^{\{s_1\}} \le \tau^{\{b_2\}} \le \tau^{\{s_2\}} \le \cdots$ are considered. A buying decision is made at $\tau^{\{b_n\}}$ and a selling decision is made at $\tau^{\{s_n\}}$ with $n = 1, 2, \ldots$ We consider the case that the net position at any time can be either flat (no stock holding) or long (with one share of stock holding). Let $i = 0, 1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell the stock before acquiring any shares. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\tau^{\{s_1\}}, \tau^{\{b_2\}}, \tau^{\{s_2\}}, \tau^{\{b_3\}}, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should first buy a stock before selling any shares. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau^{\{b_1\}}, \tau^{\{s_1\}}, \tau^{\{b_2\}}, \tau^{\{s_2\}}, \ldots)$.

Many stock prices during certain period of time can reasonably be assumed to fit into a mean reversion model. In the existing literature, concerning practical applications, some

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empirical studies were conducted in [1], [10], [12], among others. We also use the framework of mean reversion in this paper. However, instead of using a dynamic programming approach, we solve the same problem, namely finding buying and selling strategies using numerical methods. Instead of focusing on analytic solutions, our objective is to establish a systematic approach for numerical solutions. Focusing on threshold type policies, we develop stochastic approximation algorithms to approximate the optimal threshold values leading to optimal buying-selling strategies. We also assume that the asset price is mean reverting. Nevertheless, one of the distinct features in our approach is that in lieu of assuming the explicit form (1) and (2), we deal with more general model free cases. The main ingredient is that we observe the price at any given t, from which we construct a sequence of estimates of the threshold values. Then we update the threshold estimates by stochastic optimization methods.

The rest of the paper is arranged as follows. Section 2 begins with the problem formulation and the algorithm design. Section 3 proceeds with analysis of the recursive algorithms. Section 4 demonstrates the performance of our algorithms by a number of examples. First simulation results are presented; then real data are used for demonstration. Finally, Section 5 concludes the paper with further remarks.

II. Problem Formulation and Algorithm Design

We are working with an asset price that is mean reverting. We can observe the asset price at S_n, and denote the log price by X_n. Different from the model given in (1), we will not assume the stock returns to follow a particular form. Thus the price could be an OU process, or a more general mean-reverting diffusion, or a mean-reverting jump diffusion, or a mean-reverting process with regime switching, or much more general form. Suppose that $0 < K < 1$ is the percentage of slippage or commission per transaction and $\rho > 0$ is the discount factor. Our objective is to maximize the reward function

$$\begin{aligned}\phi(\theta) = \phi(b, s) \ &= E[\exp(-\rho\tau^{\{s\}})s(1-K) \\ &\quad - \exp(-\rho\tau^{\{b\}})b(1+K)],\end{aligned} \tag{3}$$

where $\tau^{\{b\}} = \inf\{t > 0 : S(t) \leq b\}$, $\tau^{\{s\}} = \inf\{t > \tau^{\{b\}} : S(t) \geq s\}$. Note that $\tau^{\{b\}}$ and $\tau^{\{s\}}$ denote the stopping times for buying and selling, respectively, b and s denote the buying and selling threshold values, and $S(t)$ is the stock price at time t. To solve the problem, we use a stochastic approximation (SA) method. The idea can be explained as follows. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and $c_n = n^{-1/6}$. Denote by $\widetilde{\phi}(\theta, \xi)$ the noisy corrupted observations and or measurements of $\phi(\theta)$.

1) Initialization: Choose initial estimates $\theta_0 = (b_0, s_0)$.
2) Iteration: With $n > 0$ and $\theta_n = (b_n, s_n)$ computed, carry out one step stochastic approximation to find the updated threshold $\theta_{n+1} = (b_{n+1}, s_{n\pm 1})$. .
   a) Find $\tau^{\{b_n+c_n\}} < \tau^{\{s_n\}}$, compute $\widetilde{\phi}(\theta_n + c_n e_1, \xi^+_{n,1})$.
   b) Find $\tau^{\{b_n-c_n\}} < \tau^{\{s_n\}}$, compute $\widetilde{\phi}(\theta_n - c_n e_1, \xi^-_{n,1})$.
   c) Find $\tau^{\{b_n\}} < \tau^{\{s_n+c_n\}}$, compute $\widetilde{\phi}(\theta_n + c_n e_2, \xi^+_{n,2})$.
   d) Find $\tau^{\{b_n\}} < \tau^{\{s_n-c_n\}}$, compute $\widetilde{\phi}(\theta_n - c_n e_2, \xi^-_{n,2})$.
   e) Using (a)-(d), find the gradient estimate $\Delta\widetilde{\phi}(\theta_n, \xi_n)$.
   f) Update one step the parameter estimate by using stochastic approximation method.
3) Repeat Step 2 with $n \leftarrow n + 1$, until $|\theta_{n+1} - \theta_n| < \eta$ with a prescribed tolerance level $\eta > 0$ or $n = N$ for some large N.

The SA algorithm is:

$$\begin{aligned}\theta_{n+1} \ &= \theta_n + \varepsilon_n \Delta\widetilde{\phi}(\theta_n, \xi_n) \\ &= \theta_n + \varepsilon_n \text{ gradient estimate of } \phi(\theta_n),\end{aligned} \tag{4}$$

where $\{\varepsilon_n\}$, satisfying $\varepsilon_n \geq 0$, $\varepsilon_n \to 0$ as $n \to \infty$, and $\sum_n \varepsilon_n = \infty$ is known as the step size sequence. For simplicity, in what follows, we choose $\varepsilon_n = 1/n$. Thus the algorithm becomes

$$\theta_{n+1} \ = \theta_n + \frac{1}{n}\Delta\widetilde{\phi}(\theta_n, \xi_n), \tag{5}$$

where $\theta_n = (b_n, s_n)$, and the gradient estimate $\Delta\widetilde{\phi}(\theta_n, \xi_n) = (\Delta^i\widetilde{\phi}(\theta_n, \xi_n))$ of $\phi(\theta) = \phi(b, s)$ can be obtained by

$$\Delta^i\widetilde{\phi}(\theta_n, \xi_n) = \frac{\left[\widetilde{\phi}(\theta_n + c_n e_i, \xi^+_{n,i}) - \widetilde{\phi}(\theta_n - c_n e_i, \xi^-_{n,i})\right]}{2c_n},$$

where $\xi^{\pm}_{n,i}$ are noise sequences. The algorithm proposed above is of stochastic approximation type. To proceed, we use the techniques developed in [18] to analyze the algorithm.

III. Asymptotic Properties of Algorithm

In the convergence analysis, we use the idea that on each "small" interval, the noise ξ varies much faster than the 'state' θ. Thus with θ 'fixed', the noise will be eventually averaged out resulting in an averaged system that can be characterized by a system of ordinary differential equations. Define $t_n = \sum_{j=1}^{n} \frac{1}{j}$; $\quad m(t) = \max\{n : \ t_n \leq t\}$. Let $\theta^0(t)$ be the piecewise constant interpolation of θ_n on the interval $[t_n, t_{n+1})$ and let $\theta^n(t) = \theta^0(t + t_n)$.

To proceed with the analysis of algorithm (5), we assume the following conditions.

For convenience, we first rewrite (5) by separating the bias and noise terms. We use E_n to denote conditional expectation with respect to the σ-algebra $\mathcal{F}_n = \{\xi^{\pm}_j : j < n\}$. Define for $i = 1, 2$,

$$\begin{aligned}b^i_n &= \frac{\phi(\theta_n + c_n e_i) - \phi(\theta_n - c_n e_i)}{2c_n} - \frac{\partial\phi(\theta_n)}{\partial\theta^i}, \\ \rho^i_n &= [\widetilde{\phi}(\theta_n + c_n e_i, \xi^+_{n,i}) - \widetilde{\phi}(\theta_n - c_n e_i, \xi^-_{n,i})] \\ &\quad - E_n[\widetilde{\phi}(\theta_n + c_n e_i, \xi^+_{n,i}) - \widetilde{\phi}(\theta_n - c_n e_i, \xi^-_{n,i})], \\ \psi^i_n &= [E_n\widetilde{\phi}(\theta_n + c_n e_i, \xi^+_{n,i}) - \phi(\theta_n + c_n e_i)] \\ &\quad - [E_n\widetilde{\phi}(\theta_n - c_n e_i, \xi^-_{n,i}) - \phi(\theta_n - c_n e_i)],\end{aligned}$$

and define $b_n = (b^1_n, b^2_n)'$, $\psi_n = (\psi^1_n, \psi^2_n)'$, $\rho_n = (\rho^1_n, \rho^2_n)'$. Then algorithm (5) can be rewritten as

$$\theta_{n+1} = \theta_n + \frac{1}{n}\nabla\phi(\theta_n) + \frac{1}{n}\frac{\psi_n}{2c_n} + \frac{1}{n}\frac{\rho_n}{2c_n} + \frac{1}{n}b_n. \tag{6}$$

Note that in fact, both ψ_n and b_n are θ dependent. In what follows, when it is needed, we write $\psi_n = \psi(\theta_n, \xi_n)$, where ξ_n includes $\xi_n^{\pm}$.

(A1) For each ξ, $\widetilde{\phi}(\cdot, \xi)$ is a continuous function.

(A2) For each $0 < N < \infty$ and each $0 < T < \infty$, the set $\{\sup_{|\theta|\le N} |\widetilde{\phi}(\theta, \xi_n)|,\ n \le m(T)\}$ is uniformly integrable.

(A3) The sequences $\{\xi_n^{\pm}\}$ are bounded. For each θ in a bounded set and for each $T < \infty$,

$$\sup_n \sum_{j=n}^{m(T+t_n)-1} \frac{1}{j} E^{1/2} |E_j \frac{\psi_j(\theta, \xi_j)}{2c_j}| < \infty,$$
$$\lim_n \sup_{0 \le i \le m(T+t_n)} E|\psi_i^n| = 0,$$

where ψ_i^n is defined by

$$\psi_i^n = (n+i) \sum_{j=n+i}^{m(T+t_n)+i-1} \frac{1}{j2c_j} E_{n+i}[\psi_j(\theta_{n+i+1}, \xi_j) - \psi_j(\theta_{n+i}, \xi_j)],\ i < m(T+t_n).$$

(A4) The second derivative of $\phi(\cdot)$ is continuous.

Remark 1: Note that unbounded noise can also be added. For example, we may require that in addition to the non-additive correlated noise, there are unbounded and additive noise sequences $\{\xi_{2,n}^{\pm}\}$ that are martingale differences with the condition $\sup_n |\xi_{2,n}^{\pm}|^2 < \infty$. All the subsequent analysis go through without modification. Condition (A3) that seems to be technical is motivated by mixing type of noise.

To present the main ideas without undue technical complication, we have chosen to use simplified notation. Weaker conditions and more general setup can be found in [18]; see also [22], [23] for the associated problem in applications to asset liquidation.

Theorem 2: Under conditions (A1)–(A4), $\theta^n(\cdot)$ converges weakly to $\theta(\cdot)$ that is a solution of

$$\dot{\theta} = \nabla \phi(\theta), \tag{7}$$

provided that the ordinary differential equation above has a unique solution for each initial condition.

Proof. The proof of the weak convergence requires that we first verify that the sequence $\{\theta^n(\cdot)\}$ is tight. By virtue of the Prohorov's theorem, we can then extract a convergent subsequence. We next characterize the limit by showing it is the solution of a martingale problem with a desired operator. Recall the definition of weak convergence. Let X_n and X be $\mathbb{R}^r$-valued random variables. We say that X_n converges weakly to X iff for any bounded and continuous function $g(\cdot)$, $Eg(X_n) \to Eg(X)$ as $n \to \infty$. $\{X_n\}$ is said to be tight iff for each $\eta > 0$, there is a compact set K_η such that $P(X_n \in K_\eta) \ge 1 - \eta$ for all n. The definitions of weak convergence and tightness extend to random variables in a metric space. The notion of weak convergence is a substantial generalization of convergence in distribution. It implies much more than just convergence in distribution since $g(\cdot)$ can be chosen in many interesting ways. On a complete separable metric space, tightness is equivalent to sequential compactness. This is known as the Prohorov's Theorem. Due to this theorem, we are able to extract convergent subsequences once tightness is verified. The above discussion also extends to function spaces. Let $D^r[0, \infty)$ denote the space of $\mathbb{R}^r$-valued functions that are right continuous and have left-hand limits, endowed with the Skorohod topology. For various notations and terms in weak convergence theory such as Skorohod topology, Skorohod representation etc. and many others, we refer to [9], [18] and the references therein.

To carry out the analysis, a convenient device is a truncation method. Let $\nu > 0$ be fixed but otherwise arbitrary. Let S_ν be the sphere with radius ν, i.e., $S_\nu = \{x \in \mathbb{R}^r, |x| \le \nu\}$. We say that $\theta^{n,\nu}(\cdot)$ is an ν-truncation of $\theta^n(\cdot)$ if, $\theta^{n,\nu}(t) = \theta(t)$ up until the first exit from S_ν, and

$$\lim_{K_0 \to \infty} \limsup_n P(\sup_{t \le T} |\theta^{n,\nu}(t)| \ge K_0) = 0 \ \text{ for each } \ T < \infty.$$

Let $q^\nu(\cdot)$ be a smooth function such that $q^\nu(\theta) = \begin{cases} 1 & \text{if } \theta \in S_\nu, \\ 0 & \text{if } \theta \in \mathbb{R}^r - S_{\nu+1}. \end{cases}$ Consider the truncated sequence

$$\theta_{n+1}^\nu = \theta_n^\nu + \Big[\frac{1}{n} \nabla \phi(\theta_n) + \frac{1}{n} \frac{\psi_n}{2c_n} + \frac{1}{n} \frac{\rho_n}{2c_n} + \frac{1}{n} b_n\Big] q^\nu(\theta_n^\nu). \tag{8}$$

Next define the interpolation of θ_n^ν as

$$\theta^{0,\nu}(t) = \theta_n^\nu \ \text{ for } \ t \in [t_n, t_{n+1}), \ \text{ and } \ \theta^{n,\nu}(t) = \theta^{0,\nu}(t).$$

We shall first obtain the convergence of $\theta^{n,\nu}(\cdot)$ and then let $\nu \to \infty$ to conclude the proof. The rest of the proof is divided into three steps.

(i) Tightness of $\{\theta^{n,\nu}(\cdot)\}$. First

$$\theta^{n,\nu}(t) = \widetilde{\theta}^{n,\nu}(t) + \sum_{k=1}^{m(t+t_n)-1} \frac{1}{k} \Big[\frac{\psi_k}{2c_k} + \frac{\rho_k}{2c_k}\Big] q^\nu(\theta_k^\nu),$$

where $\widetilde{\theta}^{n,\nu}(t) = \sum_{k=1}^{m(t+t_n)-1} \frac{1}{k} [\nabla \phi(\theta_k^\nu) + b_k] q^\nu(\theta_k^\nu)$. For any $\eta > 0$, let $t, s \ge 0$ with $0 \le s \le \eta$.

We have

$$E|\widetilde{\theta}^{n,\nu}(t+s) - \widetilde{\theta}^{n,\nu}(t)|^2 \le KE\Big| \sum_{k=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{k} \Big[\nabla \phi(\theta_k^\nu) + b_k\Big] q^\nu(\theta_k^\nu)\Big|^2.$$

By means of the boundedness of $\{\theta_k^\nu\}$, we have

$$E\Big| \sum_{k=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{k} \nabla \phi(\theta_k^\nu) q^\nu(\theta_k^\nu)\Big|^2 \le K[(t+s+t_n) - (t+t_n)]^2 \le Ks^2 \le K\eta^2. \tag{9}$$

Likewise,

$$E\Big| \sum_{k=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{k} b_k q^\nu(\theta_k^\nu)\Big|^2 \le Ks^2 \le K\eta^2. \tag{10}$$

Using (9)–(13), taking $\limsup_n$ followed by letting $\eta \to 0$, we obtain

$$\lim_{\eta \to 0} \limsup_{n \to \infty} E|\theta^{n,\nu}(t+s) - \theta^{n,\nu}(t)|^2 = 0. \tag{11}$$

To proceed, we claim that the following lemma holds.

Lemma 3: Under the conditions of Theorem 2,

$$\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1} \frac{1}{k}\Big[\frac{\psi_k}{2c_k}+\frac{\rho_k}{2c_k}\Big]q^\nu(\theta_k^\nu)\to 0$$

in probability as $n\to\infty$ and convergence is uniform in t.

Proof of Lemma 3. First note that $\{\rho_n\}$ is a martingale difference sequence. Thus the orthogonality implies that

$$\begin{aligned} E\Big|\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1}\frac{1}{k}\frac{\rho_k}{2c_k}q^\nu(\theta_k^\nu)\Big|^2 \\ \leq K\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1}\frac{1}{k^{10/6}}E|\rho_k|^2\to 0 \quad\text{as}\quad n\to\infty. \end{aligned} \tag{12}$$

Using the same technique as in the proof of [22, Lemma 3.1], we can show

$$E\Big|\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1}\frac{1}{k}\frac{\psi_k}{2c_k}q^\nu(\theta_k^\nu)\Big|^2\to 0 \quad\text{as}\quad n\to\infty. \tag{13}$$

Thus the lemma is proved. □

Combining (11) with the result of Lemma 3, the tightness criterion in [17, p. 47] yields the tightness of $\{\theta^{n,\nu}(\cdot)\}$.

(ii) Characterization of the limit process. Since $\{\theta^{n,\nu}(\cdot)\}$ is tight, we can extract a weakly convergent subsequence by Prohorov's theorem. Do so and still index the sequence by n and write it as $\{\theta^{n,\nu}(\cdot)\}$ with limit denoted by $\theta^\nu(\cdot)$. We characterize the limit process. For each $t,s\geq 0$, Lemma 3 implies that

$$\begin{aligned} \theta^{n,\nu}(t+s)-\theta^{n,\nu}(t) \;=\; &\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1}\frac{1}{k}\Big[\nabla\phi(\theta_k^{n,\nu}) \\ &+b_k\Big]q^\nu(\theta_k^\nu)+o(1), \end{aligned} \tag{14}$$

where $o(1)\to 0$ in probability uniformly in t. By a truncated Taylor expansion and (A4), it is easily verified that $b_n q^\nu(\theta_n^{n,\nu}) = O(c_n^2/c_n) = O(c_n)$. Thus, $E\Big|\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1}\frac{1}{k}b_k q^\nu(\theta_k^\nu)\Big|\to 0$ as $n\to\infty$ uniform in t. Thus,

$$\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1}\frac{1}{k}b_k q^\nu(\theta_k^\nu)\to 0\;\;\text{in probability uniform in}\;\; t. \tag{15}$$

Note that there is an increasing sequence of positive integers $\{m_l\}$ and a decreasing sequence of positive real numbers $\{\delta_l\}$ such that $m(t+t_n)\leq m_l\leq m_{l+1}-1\leq m(t+t_n+s)-1$ for any $t,s>0$ and that $\frac{1}{\delta_l}\sum_{j=m_l}^{m_{l+1}-1}\frac{1}{j}\to 1$ as $n\to\infty$. Note that in the choice above, clearly, δ_l depends on n so we could write it as δ_l^n or even δ_n. However, for notation simplicity, we will write it as δ_l in what follows. For notational convenience, set $\sum_l \triangleq \sum_{l:m(t+t_n)\leq m_l\leq m_{l+1}\leq m(t+s+t_n)-1}$. Then we rewrite the first term on the right of (14) as

$$\begin{aligned} &\sum_{k=m(t+t_n)}^{m(t+s+t_n)-1}\frac{1}{k}\nabla\phi(\theta_k^{n,\nu})q^\nu(\theta_k^\nu) \\ &=\sum_l \delta_l\nabla\phi(\theta_{m_l}^{n,\nu})q^\nu(\theta_{m_l}^\nu)+o(1), \end{aligned} \tag{16}$$

where $o(1)\to 0$ in probability. Using (14), (15), and (16), for any bounded and continuous function $h(\cdot)$, continuously differentiable function $f(\cdot)$, any positive integer κ, any $t_i\leq t$ with $i\leq\kappa$, we can show that there is a sequence $\{\widetilde{e}_n\}$ of real numbers such that $\widetilde{e}_n\to 0$ as $n\to\infty$ and that

$$\begin{aligned} &Eh(\theta^{n,\nu}(t_i): i\leq\kappa)[f(\theta^{n,\nu}(t+s))-f(\theta^{n,\nu}(t))] \\ &\to Eh(\theta^\nu(t_i): i\leq\kappa) \\ &\quad\cdot\int_t^{t+s}\nabla f'(\theta^\nu(u))\nabla\phi(\theta^\nu(u))q^\nu(\theta^\nu(u))du. \end{aligned} \tag{17}$$

In the above, we have used the weak convergence of $\theta^{n,\nu}(\cdot)$ to $\theta^\nu(\cdot)$ and the Skorohod representation. On the other hand, the weak convergence and the Skorohod representation yield that, as $n\to\infty$

$$\begin{aligned} &Eh(\theta^{n,\nu}(t_i): i\leq\kappa)[f(\theta^{n,\nu}(t+s))-f(\theta^{n,\nu}(t))] \\ &\;\to Eh(\theta^\nu(t_i): i\leq\kappa)[f(\theta^\nu(t+s))-f(\theta^\nu(t))]. \end{aligned} \tag{18}$$

Then (17) and (18) lead to that $\theta^\nu(\cdot)$ is a solution of the martingale problem with operator given by $L^\nu f(\theta^\nu) = \nabla f'(\theta^\nu)\nabla\phi(\theta^\nu)q^\nu(\theta^\nu)$, or equivalently, $\theta^\nu(\cdot)$ is the solution of the truncated differential equation $\dot{\theta}^\nu = \nabla\phi(\theta^\nu)q^\nu(\theta^\nu)$.

(iii) The convergence of the untruncated sequence $\{\theta^n(\cdot)\}$. So far we have worked with a fixed but otherwise arbitrary integer ν. In this step, we examine the asymptotic properties as $\nu\to\infty$. The details are similar to that of [18, p. 284] and are omitted for brevity. □

Next we present a corollary. It ensures the convergence to the stable point of the ODE. We omit the verbatim proof.

Corollary 4: Suppose that in addition to the conditions of Theorem 2, $\{\theta_n\}$ is tight. Suppose also that (7) has a unique stationary point θ^* that is asymptotically stable in the sense of Liapunov. Then there is a sequence of positive real numbers $\{s_n\}$ satisfying $s_n\to\infty$ as $n\to\infty$, such that, $\theta^n(\cdot+s_n)$ converges weakly to θ^* as $n\to\infty$.

IV. Numerical Demonstration

In this section, we demonstrate numerically the performance of the proposed algorithm. First, we carry out the simulation of an example that was considered in [24]. Thus we are able to compare the numerical results with the studies in [24]. Then we investigate the performance of the algorithm using real data. Several stocks with mean reversion features are considered in the numerical experiments.

Example 5: In this example, the mean reversion stochastic differential equation is represented by $dX(t) = 0.8(2-X(t))dt+0.5dW(t)$, $X(0)=2$.

The asset price is given by $S(t)=\exp(X(t))$. The mean-reverse model tries to capture price movements that tend to move towards an "equilibrium" level $X(t)=2$, and $S(t)$ fluctuates around the level $e^2 = 7.388$. We simulate a sample

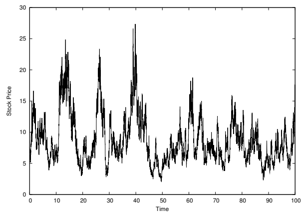

Fig. 1. Demonstration of mean-reverting type stock price (Example 5)

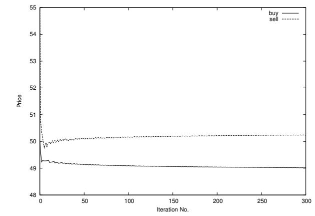

Fig. 4. Buy and Sell Price with Real Data (Example 6-1)

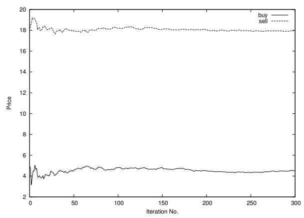

Fig. 2. Buy and Sell Price (Example 5-6)

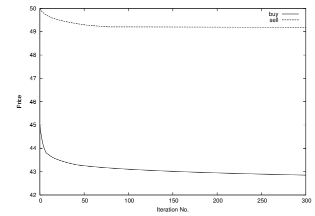

Fig. 5. Buy and Sell Price with Real Data (Example 6-2)

trajectory of the solution of the SDE by Euler method of 10000 steps with step size $h = 0.01$. See Figure 1. The purpose of the graph is for demonstration purpose.

In what follows, we take $\rho = 0.01$ and $K = 0.01$. We start with initial guesses $b_0 = 5$ and $s_0 = 18$, perform stochastic approximation (4) until the iteration number becomes 300.

It ends up with $b = 4.50324$, $s = 17.9413$ after 300 iterations. The value of the utility function at the final iterate is $\phi(b, s) = 10.8542$; see Figure 2.

We use different random seeds to perform above stochastic approximation. By using 100 replications (with the same initial guess and iteration numbers), we take the sample mean and sample standard deviation. The results are recorded in Table 1.

Example 6: In this example, we apply our algorithm to Wal-Mart Stores Inc. (WMT) stock; see Figure 3 for the daily stock price from 01/03/2000;

First, we compute optimal threshold by using *historic data* of 50 trading days: [beginning date, beginning date + 50 days]. Note that, for each iteration, we used the same trajectory in the above period. Then, using the computed

Statistics	b (buy)	s (sell)	ϕ (average return)
sample mean	3.9904	17.8688	9.66707
sample deviation	0.155591	0.761609	6.65556

TABLE I

SAMPLE MEAN AND DEVIATION FOR b, s, AND ϕ WITH 100 REPLICATIONS

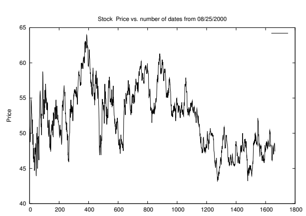

Fig. 3. WMT Stock Price after 01/03/2000 (Example 6)

threshold, we simulate the buy-low-sell-high on the *future data* of subsequent 500 trading days.

1) Based on historic data in the 50 trading days period 01/04/2000–03/16/2000, given the initial guess with $b_0 = 50$ and $s_0 = 55$, we run 600 iterations of (5). The computed result for optimal threshold is $b = 44.831$ and $s = 54.2677$. See Figure 4.
   Using the optimal threshold computed above, we practise trading strategy buy-low-sell-high in the future 500 trading days 03/16/2000– 03/18/2002. The total reward in this future period is 24.9548. The trading record is as Table II below.

TABLE II

TRADING WMT STOCK FROM 03/16/2000–03/18/2002

buy date	$\tau^{\{b_i\}}$	sell date	$\tau^{\{s_i\}}$	reward
10-12-2000	196	12-5-2000	233	8.3126
9-17-2001	425	11-6-2001	461	8.3005
7-22-2002	637	8-15-2002	655	8.34173

2) Based on different historic data in the 50 trading days period 03/16/2000–05/26/2000, given the same initial guess with $b_0 = 50$ and $s_0 = 55$, we run 600 iterations of (5). The computed result for optimal threshold is $b = 46.2061$ and $s = 58.7187$. See Figure 5.
   Using the threshold computed above, we practise buy-low-sell-high in the future 500 trading days 05/26/2000– 05/29/2002. The total reward in this future period is 21.5511. The trading record is as Table III below.

TABLE III

TRADING WMT STOCK FROM 05/26/2000– 05/29/2002

buy date	$\tau^{\{b_i\}}$	sell date	$\tau^{\{s_i\}}$	reward
10-3-2000	139	1-23-2002	463	10.7209
7-22-2002	587	8-12-2003	854	10.8302

Example 7: In this example, we apply the same procedure as in Example 6 to International Business Machines

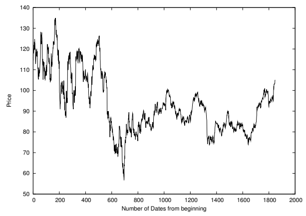

Fig. 6. IBM Stock Price vs Number of Dates since 01/03/2000 (Example 7)

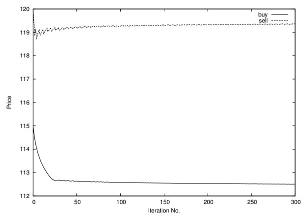

Fig. 7. Buy and Sell Price vs. Iteration Number with Real Data (Example 7-1)

Corp. (IBM); see Figure 6 for the daily stock price after 01/03/2000.

Based on historic data in the 50 trading days period 01/04/2000–03/16/2000, given the initial guess with $b_0 = 115$ and $s_0 = 120$, after 600 iterations of (5), we have computed result for optimal threshold with $b = 110.39$ and $s = 118.752$. See Figure 7.

Using the threshold computed above, we practise buy-low-sell-high in the future 500 trading days 03/16/2000–03/18/2002. The total reward in this future period is 28.8968. The trading record is as Table IV below.

TABLE IV

TRADING IBM STOCK FROM 03/16/2000–03/18/2002

buy date	$\tau^{\{b_i\}}$	sell date	$\tau^{\{s_i\}}$	reward
3-16-2000	50	3-24-2000	56	6.04398
4-14-2000	71	6-7-2000	107	5.90787
6-27-2000	121	8-8-2000	150	5.93909
10-12-2000	196	5-21-2001	347	5.39343
7-6-2001	379	12-5-2001	481	5.61238

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