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LMI Optimization Approach to Robust H_∞ Filtering for Discrete-Time Nonlinear Uncertain Systems

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***Abstract*— A new approach for the design of robust H_∞ filter for a class of discrete-time Lipschitz nonlinear systems with time-varying uncertainties is proposed based on linear matrix inequalities. Thanks to the linearity of the proposed LMIs in both the admissible Lipschitz constant of the system and the disturbance attenuation level, they can be simultaneously optimized through convex optimization. The resulting H_∞ observer guarantees exponential stability of the estimation error dynamics with guaranteed decay rate and is robust against time-varying parametric uncertainties. The proposed observer has also an extra important feature, robustness against nonlinear additive uncertainty. Explicit norm-wise and element-wise bounds on the tolerable nonlinear uncertainty are derived.**

I. INTRODUCTION

IN many practical situations, it is not possible to obtain accurate measurements of all the system states making the usage of state observers essential. In addition, due to model uncertainties and disturbances, the observer often needs to have some robustness properties. The problem of nonlinear observer design for uncertain systems has been tackled using various approaches [1], [2], [3], [4], [5]. To deal with the exogenous disturbances, the H_∞ filtering was introduced. In an H_∞ observer, the $\mathcal{L}_2$ gain from the unknown norm-bounded exogenous disturbance to the observer error is guaranteed to be less than a prespecified value. The original studies in this area go back to the works of de Souza *et. al.* where the authors considered a class of continuous-time nonlinear systems with time-varying parametric uncertainty and obtained Riccati-based sufficient conditions for the stability of the proposed observer with guaranteed disturbance attenuation level, [1], [6]. These references also present general matrix inequalities helpful in solving this type of problems. In the discrete-time domain, Xie *et. al.* proposed a Riccati equation approach to the robust H_∞ observer design [2]. The class of discrete-time systems considered was described by a linear state space model with the addition of known state dependent nonlinearly satisfying a global Lipschitz condition. In order to guarantee the robust stability of the observer in the presence of parameter uncertainties, the authors added the restrictive assumption that the "A" matrix of the linear part must be non-singular. Wang and Unbenhauen considered the robust observer design problem for the same class of discrete systems [7]. They eliminated the aforementioned restrictive assumption. However, the observer structure proposed in [7] involves parameter uncertainties, making the design of such an observer difficult in practical applications. A second shortcoming in the observer of reference [7] is that no disturbance attenuation (H_∞ performance) is guaranteed. In addition, in the Riccati approach, all the H_∞ regularity assumptions must be satisfied. The regularity assumptions in the Riccati approach can be relaxed using LMIs. An LMI solution for robust H_∞ filtering has been proposed for a class of Lipschitz nonlinear systems in which the Lipschitz constant is fixed and predetermined, [8]. The resulting observer is robust against time-varying parametric uncertainties in the linear part of the model with the guaranteed disturbance attenuation level. Recently, we have developed a new LMI optimization approach to the solution of this problem in the continues-time domain [9], [10]. In our method, the linear matrix inequalities are linear in the system Lipschitz constant making it one of the LMI variables. Therefore, the admissible Lipschitz constant can be the maximized through convex optimization. This optimization adds an important extra future to the H_∞ filter over the aforementioned features, making the proposed observer robust against some nonlinear uncertainty. In this paper, we extend the results to the discrete-time case. The discrete-time case of this problem has the merits to be studied independently since most modern control systems are implemented digitally. Besides, due to the structure of the Lyapunov difference, the LMI formulation of the solution in the discrete-time domain is more complicated. The proposed H_∞ filter is robust against time-varying parametric uncertainties as well as additive nonlinear uncertainty with the guaranteed disturbance attenuation level. We derive norm-wise and element-wise bounds on the tolerable nonlinear uncertainty.

Thanks to the linearity of our proposed LMIs in both the admissible Lipschitz constant and the disturbance attenuation

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level, it is possible to consider a combined objective function. Then, the admissible Lipschitz constant and the disturbance attenuation level are both optimized using a multiobjective optimization technique. The paper is organized as follows. In section II, the problem statement and some preliminaries are mentioned. In Section III, we propose a new method for robust H_∞ observer design for nonlinear uncertain systems. Section IV, is devoted to robustness analysis in which explicit bounds on the tolerable nonlinear uncertainty are derived. Section V, contains an illustrative example showing the high performance of our proposed method.

II. Preliminaries and Problem Statement

Consider the following class of nonlinear discrete-time uncertain systems:

$$\left(\sum\nolimits_1\right): \ x(k+1) = (A+\Delta A(k))x(k) + \Phi(x,u) + Bw(k) \quad (1)$$

$$y(k) = (C+\Delta C(k))x(k) + Dw(k) \quad (2)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and $\Phi(x,u)$ contains nonlinearities of second order or higher. We assume that the system (1)-(2) is locally Lipschitz with respect to x in a region $\mathcal{D}$ containing the origin, uniformly in u, i.e.:

$$\|\Phi(0,u^*)\| = 0 \quad (3)$$

$$\|\Phi(x_1,u^*) - \Phi(x_2,u^*)\| \leqslant \gamma\|x_1 - x_2\| \quad \forall\, x(k) \in \mathcal{D} \quad (4)$$

where $\|.\|$ is the induced 2-norm, u^* is any admissible control signal and $\gamma > 0$ is called the Lipschitz constant. The region $\mathcal{D}$ is our region of interest (the operating region). If the nonlinear function Φ satisfies the Lipschitz continuity condition globally in $\mathbb{R}^n$, then the results will be valid globally. $w(k) \in \ell_2[0,\infty)$ is an unknown exogenous disturbance and $\Delta A(k)$ and $\Delta C(k)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix} \Delta A(k) \\ \Delta C(k) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F(k)N \quad (5)$$

where M_1, M_2 and N are known real constant matrices and $F(k)$ is an unknown real-valued time-varying matrix satisfying

$$\forall k, \quad F^T(k)F(k) \leq I.$$

The parameter uncertainty in the linear terms can be regarded as the variation of the operating point of the nonlinear system. It is also worth noting that the structure of parameter uncertainties in (5) has been widely used in the problems of robust control and robust filtering for both continuous-time and discrete-time systems and can capture uncertainty in several practical situations [1], [11], [8]. Considering an observer of the following form

$$\hat{x}(k+1) = A\hat{x}(k) + \Phi(\hat{x},u) + L(y - C\hat{x}) \quad (6)$$

the observer error dynamics is given by

$$e(k) \triangleq x(k) - \hat{x}(k) \quad (7)$$

$$e(k+1) = (A-LC)e(k) + \Phi(x,u) - \Phi(\hat{x},u) + (B-LD)w + (\Delta A - L\Delta C)x(k). \quad (8)$$

Suppose that

$$z(k) = He(k) \quad (9)$$

stands for the controlled output for error state where H is a known matrix. Our purpose is to design the observer parameter L such that the observer error dynamics is asymptotically stable with maximum admissible Lipschitz constant and the following specified H_∞ norm upper bound is simultaneously guaranteed.

$$\|z\| \leq \mu\|w\|. \quad (10)$$

In the following, we mention some useful lemmas that will be used later in the proof of our results.

Lemma 1. [8] *For any $x, y \in \mathbb{R}^n$ and any positive definite matrix $P \in \mathbb{R}^{n\times n}$, we have*

$$2x^Ty \leq x^TPx + y^TP^{-1}y.$$

Lemma 2. [6] *Let A, D, E, F and P be real matrices of appropriate dimensions with $P > 0$ and F satisfying $F^TF \leq I$. Then for any scalar $\epsilon > 0$ satisfying $P^{-1} - \epsilon^{-1}DD^T > 0$, we have*

$$(A+DFE)^TP(A+DFE) \leq A^T(P^{-1} - \epsilon^{-1}DD^T)^{-1}A + \epsilon E^TE.$$

A. Notation

The following matrix notation will be used throughout the paper. For matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, $A \preceq B$ means $a_{i,j} \leq b_{ij}\ \forall\, 1 \leq i \leq m, 1 \leq j \leq n$ and $diag(A,B)$ is a block diagonal matrix having A and B on the main diagonal. For square A, $diag(A)$ is a vector containing the elements on the main diagonal of A and $diag(x)$ where x is a vector is a diagonal matrix with the elements of x on the main diagonal. $|A|$ is the element-wise absolute value of A, i.e. $[|a_{ij}|]_n$. $A \circ B$ stands for the element-wise product *(Hadamard product)* of A and B.

B. Guaranteed Decay Rate

In the continuous-time domain, the decay rate of the error dynamics is defined to be the largest $\beta > 0$ such that

$$\lim_{t\to\infty} \exp(\beta t)\|e(t)\| = 0$$

holds for all trajectories e. We can use the quadratic Lyapunov function $V(e) = e^TPe$ to establish a lower bound on the decay rate of the error dynamics. If $\frac{dV(e(t))}{dt} \leqslant -2\beta V(e(t))$ for all trajectories, then $V(e(t)) \leqslant \exp(-2\beta t)V(e(0))$, so that $\|e(t)\| \leqslant \exp(-\beta t)\kappa(P)^{\frac{1}{2}}\|e(0)\|$ for all trajectories, where $\kappa(P)$ is the condition number of P and therefore the decay rate of the (8) is at least β. In fact, decay rate is a measure of observer speed of convergence. Here we introduce its discrete-time counterpart. Consider the nominal system ($\sum_1$) with $\Delta A = \Delta C = 0$ and $w(k) = 0$. The decay rate of the error dynamics (8) is defined to be the largest $\beta > 0$ such that

$$\lim_{k\to\infty} \exp(\beta k)\|e(k)\| = 0$$

holds for all trajectories e. Suppose that

$$\Delta V_k = V_{k+1} - V_k \leq -(1-\exp(-2\beta))V_k. \quad (11)$$

Then, we have

$$\begin{aligned} V_{k+1} \leq \exp(-2\beta)V_k \Rightarrow V_k \leq \exp(-2k\beta)V_0 \\ \Rightarrow \|e(k)\| \leq \exp(-k\beta)\kappa(P)^{\frac{1}{2}}\|e(0)\|. \end{aligned} \quad (12)$$

III. Nonlinear H_∞ Observer Synthesis

In this section a new method of robust H_∞ observer design for systems of class $\sum_1$ is proposed based on LMIs. Consider system ($\sum_1$) and observer (6). We first prove a lemma about robust asymptotic stability in the presence of exogenous disturbance.

Lemma 3. *Consider the following nonlinear uncertain system*

$$\left(\sum\nolimits_2\right): \; x(k+1) = (A+\Delta A(k))x(k) + \Phi(x,u) + Bw \quad (13)$$

$$z(k) = Hx(k). \quad (14)$$

This system is asymptotically stable with $\|z\| \leq \mu\|w\|$ and maximum admissible Lipschitz constant γ^*, if there exist scalars $\alpha > 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and a matrix $P > 0$ such that the following LMI optimization problem has a solution:

$$\min(\alpha + \epsilon_1)$$

$$\begin{bmatrix} \Lambda_1 & I & A^TP & 0 & 0 \\ \star & -\alpha I & 0 & 0 & 0 \\ \star & \star & -\frac{1}{2}P & P & PM_1 \\ \star & \star & \star & P-2\epsilon_1 I & 0 \\ \star & \star & \star & \star & -\epsilon_2 I \end{bmatrix} < 0 \quad (15)$$

$$\begin{bmatrix} -\mu^2 I & B^TP & B^TP \\ \star & -\frac{1}{2}P & 0 \\ \star & \star & -I \end{bmatrix} < 0 \quad (16)$$

where $\Lambda_1 = H^TH - P + \epsilon_2 N^TN$. Once the problem is solved

$$\alpha^* \triangleq \min(\alpha), \quad \epsilon_1^* \triangleq \min(\epsilon_1)$$

$$\gamma^* \triangleq \max(\gamma) = \frac{1}{\sqrt{\alpha^*(1+\epsilon_1^*)}}$$

Proof: Consider the Lyapunov function candidate:

$$V = x^TPx. \quad (17)$$

We have

$$\begin{aligned} \Delta V_k &= x^T(A+\Delta A)^TP(A+\Delta A)x \\ &+ 2x^T(A+\Delta A)^TP\Phi + \Phi^TP\Phi - x^TPx + 2w^TB^TP\Phi \\ &+ 2w^TB^TP(A+\Delta A)x + w^TB^TPBw. \end{aligned} \quad (18)$$

Based on Lemma 1 and (3)-(4),

$$\begin{aligned} &2x^T(A+\Delta A)^TP\Phi + \Phi^TP\Phi = 2x^T(A+\Delta A)^TP\Phi \\ &- \Phi^TW\Phi + \epsilon_1\Phi^T\Phi \leq x^T(A+\Delta A)^TPW^{-1}P(A+\Delta A)x \\ &+ \epsilon_1\gamma^2x^Tx \end{aligned}$$

where $W \triangleq \epsilon_1 I - P$. Thus,

$$\begin{aligned} &x^T(A+\Delta A)^TP(A+\Delta A)x + 2x^T(A+\Delta A)^TP\Phi \\ &+ \Phi^TP\Phi - x^TPx \leq x^T[(PA+PM_1FN)^T \cdots \\ &(P^{-1}+W^{-1})(PA+PM_1FN) - P + \epsilon_1\gamma^2 I]x. \end{aligned} \quad (19)$$

Using Lemma 1, it can be written

$$\begin{aligned} &2w^TB^TP(A+\Delta A)x + 2w^TB^TP\Phi + w^TB^TPBw \\ &\leq 2w^TB^TP(A+\Delta A)x + w^TB^TPPBw + w^TB^TPBw \\ &+ \Phi^T\Phi \leq 2w^TB^TP(A+\Delta A)x + w^TB^TPPBw \\ &+ w^TB^TPBw + \gamma^2x^Tx \leq x^T(A+\Delta A)^TP(A+\Delta A)x \\ &+ w^T[B^TPB + B^TPPB + B^TPB]w + \gamma^2x^Tx \\ &= x^T[(A+\Delta A)^TPP^{-1}P(A+\Delta A) + \gamma^2]x^T \\ &+ w^T[2B^TPP^{-1}PB + B^TPPB]w. \end{aligned}$$

Substituting (19) and the above into (18) yields to

$$\begin{aligned} &\Delta V_k \leq x^T[(PA+PM_1FN)^T(2P^{-1}+W^{-1})\cdots \\ &(PA+PM_1FN) - P + (1+\epsilon_1)\gamma^2 I]x + 3w^TB^TPP^{-1}PBw \end{aligned}$$

Now, define $J \triangleq \sum_{k=0}^{\infty}\left[z(k)^Tz(k) - \mu^2 w(k)^Tw(k)\right]$. Thus,

$$J < \sum_{k=0}^{\infty}\left[z(k)^Tz(k) - \mu^2 w(k)^Tw(k) + \Delta V\right]. \quad (20)$$

So, a sufficient condition for $J \leq 0$ is that

$$\begin{aligned} &x^T[(PA+PM_1FN)^T(2P^{-1}+W^{-1})(PA\cdots \\ &+ PM_1FN) + H^TH - P + (1+\epsilon_1)\gamma^2 I]x \\ &+ w^T[3B^TPP^{-1}PB - \mu^2 I]w \leq 0. \end{aligned} \quad (21)$$

It can be concluded from (15) that $\frac{1}{2}P - (2\epsilon_1 I - P)^{-1}P^2 - \epsilon_2^{-1}PM_1M_1^TP > 0$ and $2\epsilon_1 I - P > 0$. Therefore, since $P > 0$, the condition $W > 0$ is already included in (15). Defining the new variable

$$\alpha \triangleq \frac{1}{(1+\epsilon_1)\gamma^2} \Rightarrow \gamma = \frac{1}{\sqrt{\alpha(1+\epsilon_1)}}, \quad (22)$$

In order to maximize γ, both α and ϵ_1 must be minimized. Combining the two minimization problems, we will minimize the *scalarized* linear objective function $\alpha + \epsilon_1$. On the other hand, after some matrix manipulations, we can show that [12]:

$$2P^{-1} + (\epsilon_1 I - P)^{-1} < [\frac{1}{2}P - (2\epsilon_1 I - P)^{-1}P^2]^{-1}. \quad (23)$$

Hence, according to (21), a sufficient condition for $J \leq 0$ is that

$$x^T\{(PA+PM_1FN)^T[\frac{1}{2}P-(2\epsilon_1 I-P)^{-1}P^2]^{-1}(PA\cdots +PM_1FN)+H^TH-P+\alpha^{-1}I\}x+w^T[2B^TPP^{-1}PB\cdots + B^TPPB-\mu^2 I]w<0.$$

According to Lemma 2,

$$\begin{aligned} & x^T\{A^TP[\frac{1}{2}P-(2\epsilon_1 I-P)^{-1}P^2-\epsilon_2^{-1}PM_1M_1^TP]^{-1}PA \\ & \quad +\epsilon_2 N^TN+H^TH-P+\alpha^{-1}I\}x \\ & \quad +w^T[2B^TPP^{-1}PB+B^TPPB-\mu^2 I]w<0 \end{aligned}$$

which is by Schur complements equivalent to LMIs (15) and (16). ■

Remark 1. Lemma 3, provides a tool for robust stability analysis of the aforementioned class of nonlinear systems. Maximization of γ guarantees the stability of the system for any Lipschitz nonlinear function with Lipschitz constant less than or equal γ^*. It is clear that if the stability of a systems with a given fixed Lipschitz constant is to analyzed, the proposed LMI optimization problem will reduce to an LMI feasibility problem and there will be no need to the change of variable (22) anymore.

Remark 2. The proposed LMIs are linear in α, ϵ_1 and $\zeta(=\mu^2)$. Thus, either can be a fixed constant or an optimization variable or they might be simultaneously optimized.

Using the above Lemma, in the following we propose a new method for nonlinear H_∞ observer design for the aforementioned class of nonlinear discrete-time systems.

Theorem 1. *Consider the Lipschitz nonlinear system* $(\sum_1)$ *along with the Luenberger type observer in* (6). *The observer error dynamics is (globally) asymptotically stable with guaranteed decay rate* β*, maximum admissible Lipschitz constant,* γ^**, and guaranteed* $\mathfrak{L}_2(w \to z)$ *gain,* μ*, if there exist fixed scalars* $\beta>0$ *and* $\mu>0$*, scalars* $\epsilon_1>0$*,* $\epsilon_2>0$ *and* $\alpha>0$*, and matrices* $P_1>0$*,* $P_2>0$ *and* G*, such that the following LMI optimization problem has a solution.*

$$\min(\alpha+\epsilon_1)$$

$$\left[\begin{array}{cc} \Psi_1 & \Omega \\ \star & \Psi_2 \end{array}\right]<0 \tag{24}$$

$$\Xi_1 \triangleq \left[\begin{array}{ccccc} -\mu^2 I & \Lambda_2 & B^TP_2 & \Lambda_2 & B^TP_2 \\ \star & -\frac{1}{2}P_1 & 0 & 0 & 0 \\ \star & \star & -\frac{1}{2}P_2 & 0 & 0 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -I \end{array}\right]<0 \tag{25}$$

where $\Lambda_2=B^TP_1-D^TG^T$*,* $\Lambda_3=P_1M_1-GM_2$ *and*

$$\Psi_1=\left[\begin{array}{ccc} H^TH-\exp(-2\beta)P_1 & 0 & I \\ \star & \epsilon_2 N^TN-P_2 & 0 \\ \star & \star & -\alpha I \end{array}\right]$$

$$\Psi_2=\left[\begin{array}{ccccc} -\frac{1}{2}P_1 & 0 & P_1 & 0 & \Lambda_3 \\ \star & -\frac{1}{2}P_2 & 0 & P_2 & P_2M_1 \\ \star & \star & P_1-2\epsilon_1 I & 0 & 0 \\ \star & \star & \star & P_2-2\epsilon_1 I & 0 \\ \star & \star & \star & \star & -\epsilon_2 I \end{array}\right]$$

$$\Omega=\left[\begin{array}{ccccc} A^TP_1-C^TG^T & 0 & 0 & 0 & 0 \\ 0 & A^TP_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Once the problem is solved

$$L=P_1^{-1}G \tag{26}$$

$$\alpha^* \triangleq \min(\alpha), \quad \epsilon_1^* \triangleq \min(\epsilon_1) \tag{27}$$

$$\gamma^* \triangleq \max(\gamma)=\frac{1}{\sqrt{\alpha^*(1+\epsilon_1^*)}} \tag{28}$$

Proof: We have

$$\hat{x}(k+1)=A\hat{x}(k)+\Phi(\hat{x},u)+L(y-C\hat{x}) \tag{29}$$

$$\begin{aligned} e(k+1)=&(A-LC)e(k)+\Phi(x,u)-\Phi(\hat{x},u) \\ &+(\Delta A-L\Delta C)x(k). \end{aligned} \tag{30}$$

Suppose $\xi(k) \triangleq \left[\begin{array}{cc} e^T(k) & x^T(k) \end{array}\right]^T$, it can be written

$$\begin{aligned} \xi(k+1)&=(\widetilde{A}+\Delta\widetilde{A})\xi(k)+\widetilde{\Phi}_k(x,\hat{x},u)+\widetilde{B}w \\ z&=He=\left[\begin{array}{cc} H & 0 \end{array}\right]\left[\begin{array}{c} e \\ x \end{array}\right] \triangleq \widetilde{H}\xi \end{aligned} \tag{31}$$

where

$$\widetilde{A}=\left[\begin{array}{cc} A-LC & 0 \\ 0 & A \end{array}\right], \ \widetilde{B}=\left[\begin{array}{c} B-LD \\ B \end{array}\right]$$

$$\begin{aligned} \Delta\widetilde{A}&=\left[\begin{array}{cc} 0 & \Delta A-L\Delta C \\ 0 & \Delta A \end{array}\right]=\left[\begin{array}{cc} 0 & M_1FN-LM_2FN \\ 0 & M_1FN \end{array}\right] \\ &=\left[\begin{array}{c} M_1-LM_2 \\ M_1 \end{array}\right]F\left[\begin{array}{cc} 0 & N \end{array}\right] \triangleq \widetilde{M_1}F\widetilde{N} \end{aligned}$$

$$\widetilde{\Phi}_k(x,\hat{x},u)=\left[\begin{array}{c} \Phi_k(x,u)-\Phi_k(\hat{x},u) \\ \Phi_k(x,u) \end{array}\right]$$

$$\|\widetilde{\Phi}_k(x,\hat{x},u)\| \leq \gamma\|\xi(k)\|.$$

System (31) is of the form $\sum_2$. So, Lemma 3 provides a sufficient condition for its robust asymptotic stability. Let, $P=diag(P_1,P_2)$ and $G \triangleq P_1L$. We have,

$$\epsilon_2\widetilde{N}^T\widetilde{N}-P=\left[\begin{array}{cc} -P_1 & 0 \\ 0 & \epsilon_2 N^TN-P_2 \end{array}\right] \tag{32}$$

$$P\widetilde{M_1}=\left[\begin{array}{c} P_1M_1-GM_2 \\ P_2M_1 \end{array}\right]$$

$$\widetilde{A}^TP=\left[\begin{array}{cc} A^TP_1-C^TG^T & 0 \\ 0 & A^TP_2 \end{array}\right]$$

$$P\widetilde{B}=\left[\begin{array}{c} P_1B-GD \\ P_2B \end{array}\right], \quad \widetilde{H}^T\widetilde{H}=\left[\begin{array}{cc} H^TH & 0 \\ 0 & 0 \end{array}\right].$$

Note that $V = e^T P_1 e + x^T P_2 x$. Based on (11), in order to have a guaranteed decay rate β for the error dynamics, it is needed that $\Delta V \leq -(1-\exp(-2\beta))e^T P_1 e$. This can be satisfied by replacing $-P_1$ in (32) with $-\exp(-2\beta)P_1$. Substituting the all above into (15) and (16) of Lemma 3, the LMIs in Theorem 1 are obtained. ■

IV. Robustness Against Nonlinear Uncertainty

As mentioned earlier, the maximization of Lipschitz constant makes the proposed observer robust against some Lipschitz nonlinear uncertainty. In this section this robustness feature is studied and both norm-vise and element-wise bounds on the nonlinear uncertainty are derived. The norm-wise analysis provides an upper bound on the Lipschitz constant of the nonlinear uncertainty and the norm of the Jacobian matrix of the corresponding nonlinear function. Furthermore, we will find upper and lower bounds on the elements of the *matrix-type Lipschitz constant* of the nonlinear uncertainty through a novel element-wise analysis. The discrete-time observer proposed here, enjoys the similar robustness features as those discussed in [10] for the continues-time observer. For the sake of complexness, here we study the observer robustness features in the discrete-time domain. More details on this section can be found in [12] including the proof of the results.

A. Norm-Wise Robustness

Assume a nonlinear uncertainty as follows

$$\begin{aligned} \Phi_\Delta(x,u) &= \Phi(x,u) + \Delta\Phi(x,u) && (33)\\ x(k+1) &= (A+\Delta A)x(k) + \Phi_\Delta(x,u) && (34) \end{aligned}$$

where Φ_Δ is the uncertain nonlinear function and $\Delta\Phi$ is the unknown nonlinear uncertainty. Suppose that

$$\|\Delta\Phi(x_1,u) - \Delta\Phi(x_2,u)\| \leqslant \Delta\gamma\|x_1 - x_2\|. \tag{35}$$

Proposition 1. [12] *Suppose that the actual Lipschitz constant of the system is γ and the maximum admissible Lipschitz constant achieved by Theorem 1, is γ^*. Then, the observer designed based on Theorem 1, can tolerate any additive Lipschitz nonlinear uncertainty with Lipschitz constant less than or equal $\gamma^* - \gamma$.*

B. Element-Wise Robustness

Assume that there exists a matrix $\Gamma \in \mathbb{R}^{n\times n}$ such that

$$\|\Phi(x_1,u) - \Phi(x_2,u)\| \leqslant \|\Gamma(x_1 - x_2)\|. \tag{36}$$

Γ can be considered as a *matrix-type Lipschitz constant*. Suppose that the nonlinear uncertainty is as in (34) and

$$\|\Phi_\Delta(x_1,u) - \Phi_\Delta(x_2,u)\| \leqslant \|\Gamma_\Delta(x_1 - x_2)\|. \tag{37}$$

Assuming

$$\|\Delta\Phi(x_1,u) - \Delta\Phi(x_2,u)\| \leqslant \|\Delta\Gamma(x_1 - x_2)\|, \tag{38}$$

based the proposition 1, $\Delta\Gamma$ can be any matrix with $\|\Delta\Gamma\| \leq \gamma^* - \|\Gamma\|$. Now, we look at the problem from a different angle. It is clear that $\Gamma_\Delta = [\gamma_{\Delta ij}]_n$ is a perturbed version of Γ due to $\Delta\Phi(x,u)$. The question is that how much perturbation can be tolerated on the elements of Γ without loosing the observer features stated in Theorem 1. This is important in the sense that in gives us an insight about the amount of uncertainty that can be tolerated in different directions of the nonlinear function. Here, we propose an approach to optimize the elements of Γ and provide specific upper and lower bounds on tolerable perturbations.

Corollary 1. *Consider Lipschitz nonlinear system $(\sum_1)$ satisfying* (36)*, along with the observer* (6)*. The observer error dynamics is (globally) asymptotically stable with the matrix-type Lipschitz constant $\Gamma^* = [\gamma^*_{ij}]_n$ with maximized admissible elements, decay rate β and $\mathfrak{L}_2(w \to z)$ gain, μ, if there exist fixed scalars $\beta > 0$, $\mu > 0$ and $c_{ij} > 0 \ \forall\ 1 \leq i,j \leq n$, scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and matrices $\mathcal{A} = [\alpha_{ij}]_n \succ 0$, $P_1 > 0$, $P_2 > 0$ and G, such that the following LMI optimization problem has a solution.*

$$\min \left(\epsilon_1 - \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\alpha_{ij} \right)$$

$$\Xi_1 < 0$$

$$\begin{bmatrix} \Psi_1 & \Omega \\ \star & \Psi_2 \end{bmatrix} < 0$$

where

$$\Psi_1 = \begin{bmatrix} H^T H - \exp(-2\beta)P_1 & 0 & \mathcal{A} \\ \star & \epsilon_2 N^T N - P_2 & 0 \\ \star & \star & -I \end{bmatrix}$$

and Ξ_1, Ψ_2 and Ω are as in Theorem 1. Once the problem is solved

$$L = P_1^{-1}G, \quad \alpha^*_{ij} \triangleq \max(\alpha_{ij}), \quad \epsilon^*_1 \triangleq \min(\epsilon_1)$$
$$\mathcal{A}^* \triangleq [\alpha^*_{ij}]_n, \quad \Gamma^* \triangleq \frac{1}{\sqrt{(1+\epsilon^*_1)}}\mathcal{A}^*.$$

Proof: The proof is similar to the proof of Theorem 1, replacing γI with Γ and using the change of variables $(1+\epsilon_1)\Gamma^T\Gamma = \mathcal{A}^T\mathcal{A}$. ■

Remark 3. In fact, we are maximizing every single element of the positive matrix $\mathcal{A}$ by suboptimaly maximizing a weighted sum of its elements. By appropriate selection of the weights $c_{i,j}$, it is possible to put more emphasis on the directions in which the tolerance against nonlinear uncertainty is more important. To this goal, one can take advantage of the knowledge about the structure of the nonlinear function $\Phi(x,u)$.

According to the norm-vise analysis, it is clear that $\Delta\Gamma$ in (38) can be any matrix with $\|\Delta\Gamma\| \leq \|\Gamma^*\| - \|\Gamma\|$. We will now proceed by deriving bounds on the elements of Γ_Δ.

Lemma 4. [12] *For any $Q = [q_{i,j}]_n$ and $R = [r_{i,j}]_n$, if $|Q| \preceq R$, then $QQ^T \leq RR^T \circ nI$.*

Now we are ready to state the element-wise robustness result. It is worth mentioning that the bound given in the following proposition is better than that given in [10] and yet is valid for the continues-time filter in [10] as well.

Proposition 2. [12] *Suppose that the actual matrix-type Lipschitz constant of the system is Γ and the maximized admissible matrix-type Lipschitz constant achieved by Corollary 1, is Γ^*. Then, $\Delta\Phi$ can be any additive nonlinear uncertainty such that $|\Gamma_\Delta| \preceq n^{-\frac{1}{2}}\Gamma^*$.*

Therefore, denoting the element of Γ_Δ as $\gamma_{\Delta i.j} = \gamma_{i,j} + \delta_{i,j}$, the following bound on the element-vise perturbations is obtained

$$-n^{-\frac{1}{2}}\gamma^*_{i,j} - \gamma_{i,j} \leq \delta_{i,j} \leq n^{-\frac{1}{2}}\gamma^*_{i,j} - \gamma_{i,j}. \qquad (39)$$

V. Numerical Example

Consider a system of the form $(\sum_1)$ where

$$A = \begin{bmatrix} 0.1 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.2 \\ -0.1 & 0.3 & 0.2 \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} 0.1x_1x_2 \\ 0.3x_2^2 \\ 0.3sin(x_1) \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0.2 & 0 \\ 0 & 0.2 & 0.1 \end{bmatrix}, \quad N = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.2 \end{bmatrix}.$$

We assume $\mu = 1.25, \beta = 0.05, B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, D = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^T, H = 0.25I_3$. The system is locally Lipschitz. The Lipschitz constant is region-based. Using Theorem 1, we get

$$\epsilon_1^* = 0.7715, \ \alpha^* = 2.9982, \ \gamma^* = 0.4339$$

$$L = \begin{bmatrix} 0.3220 & 1.0764 \\ 0.3426 & 1.1104 \\ -0.1205 & 1.5892 \end{bmatrix}$$

For any initial conditions in the region $\gamma \leq \gamma^*$, the observer error dynamics is asymptotically stable. The region in which $\gamma \leq \gamma^*$ is a tube in the three dimensional space, going to infinity from both sides with respect to the axis x_3 (i.e. there is no limit on the values of x_3). Figure 1, shows the true and estimated values of states and the cross-section of our region of interest, the region in which $\gamma \leq \gamma^*$, with the $x_1 - x_2$ plane.

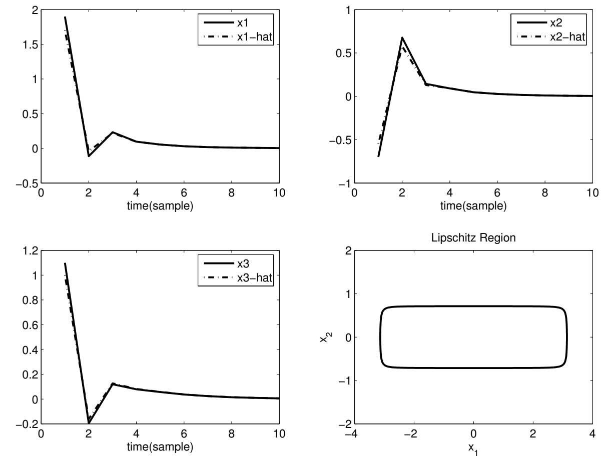

Fig. 1. The true and estimated states of the Example

VI. Conclusions

A new LMI optimization approach to the robust H_∞ observer design for nonlinear discrete-time uncertain is systems is proposed. The considered class of nonlinear systems contains norm-bounded time-varying model uncertainties as well as additive Lipschitz nonlinear model uncertainties. Explicit bounds on the tolerable uncertainty were derived via norm-wise and element-wise robustness analysis.

References

[1] C. E. de Souza, L. Xie, and Y. Wang, "H_∞ filtering for a class of uncertain nonlinear systems," *Systems and Control Letters*, vol. 20, no. 6, pp. 419–426, 1993.

[2] L. Xie, C. E. de Souza, and Y. Wang, "Robust filtering for a class of discrete-time uncertain nonlinear systems: An H_∞ approach," *International Journal of Robust and Nonlinear Control*, vol. 6, no. 4, pp. 297–312, 1996.

[3] M. Arcak and P. Kokotovic, "Nonlinear observers: A circle criterion design and robustness analysis," *Automatica*, vol. 37, no. 12, pp. 1923–1930, 2001.

[4] Y. M. Cho and R. Rajamani, "A systematic approach to adaptive observer synthesis for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 42, no. 4, pp. 534–537, 1997.

[5] Q. P. Ha and H. Trinh, "State and input simultaneous estimation for a class of nonlinear systems," *Automatica*, vol. 40, no. 10, pp. 1779–1785, 2004.

[6] Y. Wang, L. Xie, and C. E. de Souza, "Robust control of a class of uncertain nonlinear systems," *Systems and Control Letters*, vol. 19, no. 2, pp. 139–149, 1992.

[7] Z. Wang and H. Unbehauen, "A class of nonlinear observers for discrete-time systems with parametric uncertainty," *International Journal of Systems Science*, vol. 31, no. 1, pp. 19–26, 2000.

[8] S. Xu, "Robust H_∞ filtering for a class of discrete-time uncertain nonlinear systems with state delay," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 12, pp. 1853 – 1859, 2002.

[9] M. Abbaszadeh and H. J. Marquez, "A robust observer design method for continuous-time Lipschitz nonlinear systems," *Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, U.S.A.*, pp. 3895–3900, 2006.

[10] ——, "Robust H_∞ observer design for a class of nonlinear uncertain systems via convex optimization," *Proceedings of the 2007 American Control Conference, New York, U.S.A.*, pp. 1699–1704.

[11] P. P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ control theory," *IEEE Transactions on Automatic Control*, vol. 35, no. 3, pp. 356–361, 1990.

[12] M. Abbaszadeh and H. J. Marquez, "LMI optimization approach to robust H_∞ observer design and static output feedback stabilization for discrete-time nonlinear uncertain systems," *Accepted for publication in the Int. J. of Robust and Nonlinear Control, DIO: 10.1002/rnc.1310*, 2008.