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Self-triggered stabilization of homogeneous control systems

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***Abstract*— Digital implementations of feedback laws commonly consider periodic execution of control tasks. In this paper we go beyond the periodic model by developing self-triggered schedules for the execution of control tasks. These schedules guarantee asymptotic stability under sample-and-hold implementations while drastically reducing processor usage when compared with the more traditional periodic implementations. At the technical level the results rely on a homogeneity assumption on the continuous dynamics and extend to the self-triggered framework some of the advantages of event-triggered implementations recently studied by the authors. The results presented in this paper can be seen as an effort towards understanding the real-time scheduling requirements of control tasks.**

I. INTRODUCTION

Existing techniques to schedule feedback control laws on digital platforms can be divided into 3 main categories:

- Periodic time-triggered: the control task is executed periodically every T units of time. It is by far the most common and simple implementation. The main difficulty of this approach lies in the selection of a sampling period T which guarantees desired levels of control performance. Moreover, periodic implementations usually result in conservative usage of resources since T is chosen for a worst-case scenario and hence the control task is executed at the same rate regardlessly of the state of the plant. There is a vast literature about this particular implementation; still, there are many open points, specially for nonlinear systems. For linear systems, there exist several results relating sampling period and stability of the system, most of them based on the construction of an equivalent discrete model. Nonlinear systems, in general, cannot be discretized in closed form. Hence, a common approach is to find approximate discrete-time models, and then carry out the study for this set of equations [NTK99]. Other studies focused on specific structures, or tried to determine existence of a fast enough sampling rate guaranteeing stability of the sampled system [BF05]. In [ZOB90] implicit relations between the domain of attraction and sampling rates were studied, and some conservative estimates for the sampling periods were numerically computed. Therefore, as there is not yet available a solid theory to estimate sampling periods, in many applications engineers opt for a fast sampling strategy, based on some rough previous simulations.
- Event-triggered: the control task is executed according to some condition based on plant measurements. This technique was explored from a stochastic point of view in [AB02]. In [Tab07] the inter-sample behaviour was analyzed to derive a stabilizing event-triggered feedback law for nonlinear systems. Indeed, it seems natural to apply the feedback law just when something significant happens in the process. This strategy will reduce resource usage, provide more robustness, but in many cases it requires special hardware, not available in general purpose devices, to decide when the control should be executed.
- Self-triggered: the underlying idea is to merge the advantages of time and event-triggered implementations: reduce the number of times that control tasks are executed without resorting to extra hardware. In most feedback laws, the state of the plant has to be measured (or estimated) to compute the next value of the controller; hence, this information could be used to decide when the control task has to be applied again. The self-triggered task model was previously studied for linear systems in [VFM03], by using a discretization of the model of the plant (not feasible in general for nonlinear systems); and in [LCHZ07], where the computation of the state transition matrix is required, making the approach computationally inefficient.

In this paper we investigate self-triggered implementations of stabilizing control laws for homogeneous control systems. Drawing inspiration from the event-triggered framework introduced in [Tab07] we will exploit homogeneity to derive a scaling law for the execution times of the control task as a function of the state norm. This scaling law will show how we can execute the control task less frequently as the state approaches the origin while maintaining desired levels of performance. The results in this paper can be seen as a first step towards understanding the scheduling requirements for more general nonlinear control systems.

II. NOTATION AND PROBLEM STATEMENT

A. Notation

We shall use the notation $|x|$ to denote the Euclidean norm of an element $x \in \mathbb{R}^n$. A continuous function

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$\alpha : [0, a[\rightarrow \mathbb{R}_0^+$, $a > 0$, is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$. We define a ray through $x \in \mathbb{R}^n$ as the 1-parameter family $\{\lambda x : \lambda > 0\}$.

B. Problem statement

We consider a control system:

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \tag{II.1}$$

for which a feedback controller:

$$u = k(x) \tag{II.2}$$

has been designed rendering the closed loop system:

$$\dot{x} = f(x, k(x + e)) \tag{II.3}$$

Input-to-State Stable (ISS) with respect to measurement errors $e \in \mathbb{R}^n$. We shall not need the definition[1] of ISS in this note but rather the following characterization.

Definition 2.1: A smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ is said to be an ISS Lyapunov function for the closed loop system (II.3) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \alpha$ and γ satisfying:

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{II.4}$$

$$\frac{\partial V}{\partial x} f(x, k(x + e)) \leq -\alpha(|x|) + \gamma(|e|) \tag{II.5}$$

Closed loop system (II.3) is said to be ISS with respect to measurement errors $e \in \mathbb{R}^n$ if there exists an ISS Lyapunov function for (II.3).

The implementation of the feedback law (II.2) on an embedded processor is typically done by sampling the state at time instants t_i, computing $k(x(t_i))$ and updating the actuator values at time instants $t_i + \Delta_i$, where $\Delta_i \geq 0$ represents the time required to read the state from the sensors, compute the control law and update the actuators. Furthermore, the sequence of times t_i is typically periodic meaning that $t_{i+1} - t_i = T$, where $T > 0$ is the period. In this paper we drop the periodicity assumption in favour of less frequent aperiodic executions. In particular we will solve the following problem:

Problem 2.2: Let $\dot{x} = f(x, u)$ be a homogeneous control system for which a homogeneous control law $u = k(x)$ rendering the closed loop system $\dot{x} = f(x, k(x))$ globally asymptotically stable has been designed. Identify a class of aperiodic execution schedules for the computation of $k(x)$ guaranteeing stability while reducing processor usage.

To tackle this problem, we will explore the inter-sample behaviour of the plant under the event-triggered implementation introduced in [Tab07] and reviewed in the next section.

III. Event-triggered stabilization of linear systems

Although the results of this paper apply to nonlinear systems, we shall review the event-triggered stabilization in a linear context for simplicity of presentation. Let our control system be described by:

$$\dot{x} = Ax + Bu \tag{III.1}$$

and globally asymptotically stabilized by a linear feedback:

$$u = Kx \tag{III.2}$$

The dynamics of the closed loop system under the controller $u = Kx(t_i)$ is given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKx(t_i) \\ &= Ax(t) + BKx(t) - BKx(t) + BKx(t_i) \\ &= (A + BK)x(t) + BKe(t) \end{aligned} \tag{III.3}$$

where the measurement error e is defined by:

$$t \in [t_i + \Delta, t_{i+1} + \Delta[\quad \Longrightarrow \quad e(t) = x(t_i) - x(t) \tag{III.4}$$

Thus we can rewrite the state space representation of the linear system including the measurement error as another state variable:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & BK \\ -A - BK & -BK \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{III.5}$$

since $\dot{e} = -\dot{x}$. For simplicity, we will assume $\Delta = 0$ since the results can be generalized for nonzero Δ by following the procedure described in [Tab07]. Since $A+BK$ is a stable matrix we have a quadratic Lyapunov function V whose derivative along (III.3) satisfies:

$$\dot{V} \leq -a|x|^2 + b|x||e| \tag{III.6}$$

If we restrict the error to satisfy:

$$b|e| \leq \sigma a|x| \tag{III.7}$$

the dynamics of V is bounded by:

$$\dot{V} = \frac{\partial V}{\partial x}\big((A + BK)x(t) + BKe(t)\big) \leq (\sigma - 1)a|x|^2$$

thus guaranteeing that V decreases provided that $\sigma < 1$. Inequality (III.7) can be enforced by executing the control task whenever:

$$|e| = \sigma \frac{a}{b}|x| \tag{III.8}$$

And every time the control task is executed the current state is measured, making $x(t_i) = x(t)$ which implies $e(t) = x(t_i) - x(t) = 0$. An event-triggered implementation based on this equality would require dedicated hardware to test (III.8) frequently. This constitutes the main disadvantage of the event-triggered policy introduced in [Tab07]. To overcome this drawback, we could implicitly implement an approximation of this strategy by computing offline the sequence of inter-execution times and then execute the control task in a time-triggered fashion, using the precomputed schedule. Obviously, this new strategy will lack the inherent robustness properties of the event-triggered implementation, but on the other hand no additional hardware is required. The approach we will follow in this paper lies in between pre-computing a schedule off-line and testing (III.8) on-line:

[1] See, for example, [Son05] for an introduction to ISS and related notions.

the control task will use the current state measurement to determine its next deadline.

The inter-execution time implicitly defined by (III.8) is the time it takes for $\frac{|e|}{|x|}$ to evolve from[2] 0 to $\sigma\frac{a}{b}$. An estimate of this time can be obtained by first constructing the following estimate of $\frac{d}{dt}\frac{|e|}{|x|}$ (see [Tab07] and [TW06] for details):

$$\frac{d}{dt}\frac{|e|}{|x|} \leq \alpha_0 + \alpha_1\frac{|e|}{|x|} + \alpha_2\left(\frac{|e|}{|x|}\right)^2$$

with $\alpha_0 = |A+BK|$, $\alpha_1 = \big(|A+BK| + |BK|\big)$ and $\alpha_2 = |BK|$. Hence, an upper bound for $\frac{|e|}{|x|}$ is given by:

$$\frac{|e(t)|}{|x(t)|} \leq \phi(t,\phi_0) \tag{III.9}$$

where $\phi(t,\phi_0)$ satisfies the following Riccati differential equation:

$$\dot{\phi} = \alpha_0 + \alpha_1\phi + \alpha_2\phi^2 \tag{III.10}$$

with $\phi(0,\phi_0) = \phi_0$. The desired lower bound for the inter-execution is thus obtained as the time τ satisfying:

$$\phi\big(\tau, 0\big) = \sigma\frac{a}{b} \tag{III.11}$$

The explicit solution for (III.10) is:

$$\phi(t,\phi_0) = -\frac{1}{2\alpha_2}\left(\alpha_1 - \Theta\tan\left(\frac{1}{2}\Theta(t+\Psi)\right)\right)$$

with Ψ and Θ defined as:

$$\Psi = -\frac{2}{i\Theta}\left(\arctan\left(\frac{\alpha_1 + 2\phi_0\alpha_2}{-i\Theta}\right)\right)$$

$$\Theta = \sqrt{4\alpha_2\alpha_0 - \alpha_1^2}$$

Finally, we can explicitly compute the value of τ:

$$\tau = -\Psi - \frac{2}{i\Theta}\left(\arctan\left(\frac{\alpha_1 + 2\frac{a}{b}\sigma\alpha_2}{-i\Theta}\right)\right) \tag{III.12}$$

As shown in [Tab07], the bounds herein obtained are sufficiently accurate to be useful in practical situations, and they are valid for any $x(t_i) \in \mathbb{R}^n$, as we are dealing with a linear system. We should point out that these times are just bounds, as we cannot find an explicit expression defining the evolution of $|e|/|x|$. However, by exploiting linearity we can obtain the inter-execution times $\tau\big(x(t_i)\big)$ for different initial conditions $x(t_i)$.

Proposition 3.1: Let $\dot{x} = Ax + Bu$ be a control system for which a feedback control law $u = Kx$ has been designed. The inter-execution times implicitly defined by the execution rule $|e| = c|x|$ with $c > 0$ coincide for any $x(t_i)$ lying along the same ray:

$$\tau\big(\lambda(x(t_i))\big) = \tau\big(x(t_i)\big) \tag{III.13}$$

Proof: Both e and x depend on $e(t_i)$ and $x(t_i)$, but since we are only interested in the case where $e(t_i) = 0$, we will simply regard e and x as functions of time and $x(t_i)$. From (III.5), we see that $e(t, x(t_i))$ is linear in $x(t_i)$, so the following holds:

$$\frac{|e(t,\lambda x(t_i))|}{|x(t,\lambda x(t_i))|} = \frac{|\lambda e(t,x(t_i))|}{|\lambda x(t,x(t_i))|} = \frac{|e(t,x(t_i))|}{|x(t,x(t_i))|} \tag{III.14}$$

Hence, the inter-execution times are exactly the same for any $x(t_i)$ lying along a ray. ■

In the next section we generalize Proposition (3.1) to the nonlinear case under a homogeneity assumption. As we will see, in the nonlinear case the inter-execution times will no longer be constant along rays, but will still satisfy a simple relationship.

[2]Recall that if $\Delta = 0$ we have $e(t) = x(t_i) - x(t) = 0$ at the execution instant $t = t_i$ and thus $\frac{|e|}{|x|} = 0$.

IV. Homogeneous systems and dilations

Homogeneous systems appear as local approximations for general nonlinear systems [Her91] since we can always decompose an analytic function in an infinite sum of homogeneous functions. Moreover, many physical systems such as the rigid body can be described as homogeneous systems (see [Bai80] for more examples). To define homogeneity we first review the notion of dilation.

Definition 4.1: Given an n-tuple $r = (r_1,..,r_n) \in (\mathbb{R}_0^+)^n$, a dilation map $\delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n$ is defined by:

$$\delta_\lambda^r(x) = (\lambda^{r_1}x_1, ..., \lambda^{r_n}x_n), \quad \lambda > 0 \tag{IV.1}$$

We call a dilation standard if $r_i = 1$ for any $i = 1,..,n$. A homogeneous ray is defined as the 1-parameter family of dilations $\{\delta_\lambda^r(x) : \lambda > 0\}$.

Definition 4.2: A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous of order d if for all $\lambda > 0$, there exist $r_i > 0$, $i = 1..n$ such that:

$$f_i(\delta_\lambda^r(x)) = \lambda^d\lambda^{r_i}f_i(x) \tag{IV.2}$$

where $d > -\min_i r_i$. With this definition, we see that linear functions are homogeneous of degree $d = 0$ with respect to the standard dilation.

Consider now a differential equation:

$$\dot{x} = f(x) \tag{IV.3}$$

whose right-hand side is homogeneous. Then, the solution $\phi(t,\phi_0)$ satisfies:

$$\phi(t, \delta_\lambda^r(x_0)) = \delta_\lambda^r \circ \phi(\lambda^d t, x_0) \tag{IV.4}$$

In fact, since ϕ is a solution of (IV.3) we have $\dot{\phi} = f(\phi(t,x_0))$ and we can check that the right hand side of (IV.4) is also a solution:

$$\begin{aligned}\frac{d\,\delta_\lambda^r \circ \phi_i(\lambda^d t, x_0)}{dt} &= \frac{d\,\delta_\lambda^r \circ \phi_i(\lambda^d t, x_0)}{d\,\phi_i}\frac{d\,\phi_i(\lambda^d t, x_0)}{d\,\lambda^d t}\frac{d\,\lambda^d t}{dt} \\ &= \lambda^{r_i}\cdot f_i(\phi(\lambda^d t, x_0))\cdot\lambda^d \\ &= f_i(\delta_\lambda^r \circ \phi(\lambda^d t, x_0))\end{aligned} \tag{IV.5}$$

Let $\psi(t) := \delta_\lambda^r \circ \phi(\lambda^d t, x_0)$. Then $\psi(t)$ will satisfy the differential equation. Since uniqueness of solutions is a consequence of homogeneity [MA00] we conclude:

$$\begin{aligned}&\psi(t) = \phi(t,\psi(0)) \\ &\psi(0) = \delta_\lambda^r(\phi(0,x_0)) = \delta_\lambda^r(x_0) \Rightarrow \\ &\psi(t) = \delta_\lambda^r \circ \phi(\lambda^d t, x_0) = \phi(t, \delta_\lambda^r(x_0))\end{aligned} \tag{IV.6}$$

as desired.

In the next section we will consider homogeneous control systems with respect to the standard dilation, since the general case can be reduced to this one, as explained in [Grü00].

V. Inter-execution time scaling laws for homogeneous systems

The main technical contribution of the paper is the following scaling law for the inter-execution times that generalizes Proposition (3.1) to the nonlinear homogeneous case.

Theorem 5.1: Let $\dot{x} = f(x,u)$ be a control system for which a feedback control law $u = k(x)$ rendering the closed loop homogeneous of order d with respect to the standard dilation has been designed. The inter-execution times implicitly defined by the execution rule $|e| = c|x|$ with $c > 0$ scale according to:

$$\tau(\delta^r_\lambda(x(t_i,x_0))) = \lambda^{-d}\tau(x(t_i,x_0)) \tag{V.1}$$

Proof: To clarify the argument, we define $x_a = x(t_i, x_0)$, the initial condition for the inter-sample behaviour. As the closed loop is homogeneous, the trajectories of the system will satisfy (IV.4), i.e., $x(t,\delta^r_\lambda(x_a)) = \delta^r_\lambda \circ x(\lambda^d t, x_a)$. And this condition holds for all t, so $x(t_i,\delta^r_\lambda(x_a)) = \delta^r_\lambda \circ x(\lambda^d t_i, x_a)$. Hence if we consider the initial condition to be $\delta^r_\lambda(x_a)$ the inter-sample behaviour will be:

$$\begin{aligned}\frac{|e(t,\delta^r_\lambda(x_a))|}{|x(t,\delta^r_\lambda(x_a))|} &= \frac{|x(t_i,\delta^r_\lambda(x_a)) - x(t,\delta^r_\lambda(x_a))|}{|x(t,\delta^r_\lambda(x_a))|}\\ &= \frac{|\delta^r_\lambda \circ x(\lambda^d t_i, x_a) - \delta^r_\lambda \circ x(\lambda^d t, x_a)|}{|\delta^r_\lambda \circ x(\lambda^d t, x_a)|}\\ &= \frac{|\lambda x(\lambda^d t_i, x_a) - \lambda x(\lambda^d t, x_a)|}{|\lambda x(\lambda^d t, x_a)|}\\ &= \frac{|e(\lambda^d t, x_a)|}{|x(\lambda^d t, x_a)|}\end{aligned} \tag{V.2}$$

So the inter-sample dynamics for dilations of x_a is λ^d times faster than for x_a. Therefore, the inter-execution times will be λ^d shorter, as shown in equation (V.1). ∎

Similar results were obtained in [Tun05] for hybrid homogeneous systems.

Remark 5.2: The execution rule $|e| = c|x|$ stabilizes the system whenever we have an ISS Lyapunov function for the system of the type:

$$\frac{\partial V}{\partial x} f(x, k(x+e)) \leq -\alpha(|x|) + \gamma(|e|)$$

for α, γ being $\mathcal{K}_\infty$ functions, α^{-1}, γ Lipschitz continuous on compact sets (see [Tab07]) and k appropriately chosen. However, in many cases these functions might not be Lipschitz continuous at the origin. To overcome this issue, we can define a set where the aforementioned execution rule guarantees practical stability. For instance, we could pick the set $\{x \in \mathbb{R}^n \big| |x| \geq r_a\}$, with r_a as small as desired (to achieve practical stability).

Theorem 5.1 allows us to use the estimate of the inter-execution times at some x in order to estimate the times for the whole ray through x. Therefore, it is enough to find estimates of these times on any $n-1$ sphere, and then extend the results along homogeneous rays. Moreover, since estimates for τ are easily computed for linear systems, as described in Section III, we can always choose a $n-1$ sphere where a linear over-approximation for the control system can be easily obtained. To do so, we recall that a homogeneous function g of order d satisfies:

$$(d + r_i)g_i(x) = \sum_{i=1}^{n} r_i x_i \frac{\partial g_i}{\partial x_i}$$

Hence, for the closed loop system $\dot{x} = \tilde{f}(x,e) = f(x, k(x+e))$ we can find a bound for $|\tilde{f}(x,e)|$ linear in $|x|$ and $|e|$:

$$\begin{aligned}|\tilde{f}(x,e)| &= |H(x,e)x + G(x,e)e|\\ &\leq |H(x,e)||x| + |G(x,e)||e|\\ &\leq |H(x^*_a,e^*_a)||x| + |G(x^*_b,e^*_b)||e|\end{aligned}$$

where:

$$H := \begin{bmatrix} \frac{r_1}{d+r_1}\frac{\partial \tilde{f}_1}{\partial x_1} & \cdots & \frac{r_1}{d+r_1}\frac{\partial \tilde{f}_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{r_n}{d+r_n}\frac{\partial \tilde{f}_n}{\partial x_1} & \cdots & \frac{r_n}{d+r_n}\frac{\partial \tilde{f}_n}{\partial x_n}\end{bmatrix}$$

$$G := \begin{bmatrix} \frac{r_1}{d+r_1}\frac{\partial \tilde{f}_1}{\partial e_1} & \cdots & \frac{r_1}{d+r_1}\frac{\partial \tilde{f}_1}{\partial e_n}\\ \vdots & \ddots & \vdots\\ \frac{r_n}{d+r_n}\frac{\partial \tilde{f}_n}{\partial e_1} & \cdots & \frac{r_n}{d+r_n}\frac{\partial \tilde{f}_n}{\partial e_n}\end{bmatrix} \tag{V.3}$$

and (x^*_a, e^*_a) and (x^*_b, e^*_b) are such that $|H(x,e)| \leq |H(x^*_a,e^*_a)|$ and $|G(x,e)| \leq |G(x^*_b,e^*_b)|$ for all (x,e) in a neighbourhood Ω around the origin. So given this set Ω we can find where the norm of these weighted Jacobians attain its maximum values and then work with the linear model

$$\dot{x} = H_* x + G_* e \tag{V.4}$$

It is important to emphasize that we are not trying to find a linearized model, as it would not guarantee stability for the original nonlinear system. To summarize, the computation of aperiodic sampling strategy is made in 4 steps:

1) Define an invariant set Ω around the equilibrium point, for instance a level set of the Lyapunov function.
2) Find H and G as defined in (V.3). Then, we compute the point(s) $\{x^*, e^*\} \in \Omega$ where $|H|$ and $|G|$ are maximized.
3) Compute the inter-execution time τ^* for the linear model (V.4) as described in (III.12), by identifying $A+BK$ with H_* and BK with G_*. τ^* is a stabilizing sampling period of our original system for any initial condition lying in Ω.
4) Let Γ be the largest ball inside Ω, and let p be its radius. Relate the current state $x(t_i)$ with some point in the boundary[3] of Γ via homogeneous rays, that is, find λ such that $\delta^r_\lambda(y) = x(t_i)$ for some y in the boundary of Γ. Since we are working with the standard dilation

[3] Note that the boundary of Γ is an $n-1$ sphere.

and since we have an estimate τ^* valid for any point in the boundary of Γ we can compute the next deadline $\tau(x(t_{i+1}))$ of the control task by using (V.1):

$$\tau(x(t_{i+1})) = \left(\frac{|x|}{p}\right)^{-d} \tau^* \quad \text{(V.5)}$$

As τ^* can be precomputed offline, the evaluation of (V.5) can be performed online in a very short time. It is important to notice that the conservativeness of this technique relies entirely on the accuracy of the sampling rate guaranteeing stability for the linear system. That is, no conservativeness is added when the scaling law is applied. This implies as well that we could use any technique available to find a lower bound τ^* for the linear system, and then nicely extend it to any point via homogeneous rays.

VI. Example

To illustrate the previous results, we consider the following nonlinear system:

$$\begin{aligned} \dot{x_1} &= -x_1^3 + x_1 x_2^2 \\ \dot{x_2} &= x_1 x_2^2 + u - x_1^2 x_2 \end{aligned} \quad \text{(VI.1)}$$

which is homogeneous with $r_1 = r_2 = 1$ and $d = 2$. First, we find a state feedback controller rendering the closed loop ISS with respect to measurement errors. One such law is given by:

$$u = -x_2^3 - x_1 x_2^2 \quad \text{(VI.2)}$$

and using $V = \frac{1}{2}(x_1^2 + x_2^2)$ as the Lyapunov function we obtain:

$$\begin{aligned} \dot{V} = & - \; x_1^4 - x_2^4 + x_2(-e_1(x_2+e_2)^2 \\ & - \; x_1(e_2^2 + 2x_2 e_2) - e_2(e_2^2 + 3x_2^2 + 3x_2 e_2)) \end{aligned}$$

This system is ISS as we can find two $\mathcal{K}_\infty$ functions α and γ satisfying (II.5). In order to find a stabilizing execution rule of the type $|e| = c|x|$, we can define a compact set where the system will achieve practical semi-global stability. For instance, let's define an annulus $\{x \in \mathbb{R}^n | r_a \leq |x| \leq r_b\}$. If we pick $r_a = 10^{-3}$ and $r_b = 10$ we obtain the following bound (valid only in the annulus):

$$\dot{V} \leq k_1 |x|^4 + k_2 |x|^2 |e|^2 \quad \text{(VI.3)}$$

with $k_1 = -1/2$ and $k_2 = 3112.2$. The corresponding execution rule to guarantee stability is:

$$3112.2|e|^2 = \frac{1}{2}\sigma |x|^2 \quad \text{(VI.4)}$$

We select a value of $\sigma < 1$ guaranteeing stability under rule (III.8), for instance $\sigma = 0.1$. We will find an estimate of the aperiodic time sequence following the 4 steps mentioned before. Let the initial condition be $x_0 = (0.1, 0.4)$

1) The equilibrium point is $(0,0)$. We define a ball $\Omega = \{|x| \leq p\}$, for $p = 10^{-1}$. For this particular Lyapunov function, the sets Ω and Γ coincide.
2) Find the weighted Jacobians for the closed loop system $\dot{x} = f(x,e)$:

$$H = \frac{1}{3}\begin{bmatrix} -3x_1^2 + x_2^2 & 2x_1 x_2 \\ -x_2^2 - 2x_1 x_2 - (x_2+e_2)^2 & -x_1^2 + 2x_1 x_2 + l \end{bmatrix}$$

$$G = \frac{1}{3}\begin{bmatrix} 0 & 0 \\ -(x_2+e_2)^2 & l \end{bmatrix}$$

with $l = -3(x_2+e_2)^2 - 2(x_2+e_2)(x_1+e_1)$. We search for the maximum of H and G (independently) under the constraints $|x| \leq p$ and $|e| \leq \sigma p^4$.[4]
The maximum is attained at $x_a^* = (0.0187, 0.0982)$, $e_a^* = 10^{-3}(0.099, 0.388)$, $x_b^* = (-0.0248, -0.0968)$ and $e_b^* = 10^{-3}(-0.099, -0.388)$.
3) Compute the inter-execution time for linear model: $\tau^* = 0.308s$.
4) Compute the next deadline using (V.5):

$$\begin{aligned} \tau(x_0) &= \lambda^{-d}\tau^* = \left(\frac{|x_0|}{p}\right)^{-d}\tau^* = \\ &= \left(\frac{0.412}{0.1}\right)^{-2} 0.308 = 0.018s. \end{aligned}$$

As we get closer to the equilibrium point, $|x(t_i)|$ will decrease and so will $(|x(t_i)|/p)$, leading to larger inter-execution times. In the next figures, we compare periodic and aperiodic strategies. Both of them exhibit a similar behaviour for the Lyapunov function (see Figure (1)). Figure (2) shows the evolution of the input for the homogeneous system. At the beginning, both the periodic and aperiodic use the same inter-execution time, but as the system tends to the equilibrium point the periodic policy updates the controller at the same rate, whereas the aperiodic policy increases the time between executions. The right side of Figure (2) zooms the last part of the simulation, where the inter-execution times for the aperiodic strategy is nine times larger than the periodic. Hence the aperiodic time-triggered implementation leads to a much smaller number of executions, while achieving similar performance. The number of executions required for periodic, aperiodic and event-triggered implementation are shown in Table (I): the aperiodic policy executes the controller 74% less times than the periodic, and 67% more than the event-triggered (due to the conservativeness of τ^*) for a simulation time of 40sec. The aperiodic implementation will be even more efficient when the systems works in a large compact sets or when the degree of homogeneity of the system is large (so that inter-execution times vary widely).

Additionally, the aperiodic sampling will be robust with respect to measurement noise, as expected because of the existence of an ISS Lyapunov function. The last figure shows the behaviour of the Lyapunov function for both periodic and aperiodic strategies when sensor noise is considered (noise power being 2% of the signal power). Again, the aperiodic strategy achieves a similar rate of decay with a much smaller number of executions. The periodic behaviour without noise is included as a reference.

[4] Execution rule (VI.4) will guarantee that $|e|$ will not violate this bound.

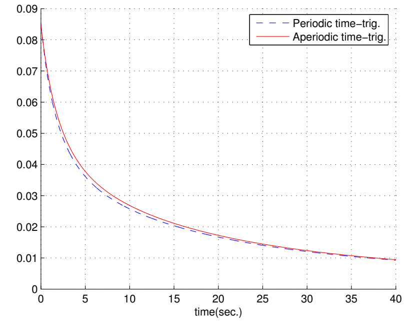

Fig. 1. Comparison of the Lyapunov functions for periodic and aperiodic implementation

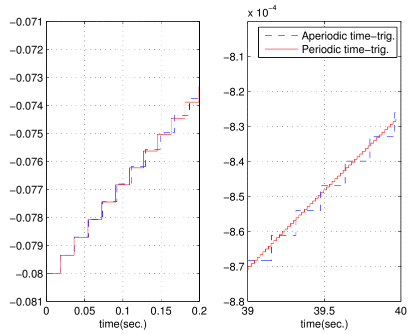

Fig. 2. Control input for periodic and aperiodic implementation

VII. Conclusions and Further Work

In this paper we showed that it is possible to implement a stabilizing control law in an aperiodic fashion while guaranteeing not only asymptotic stability but also a given rate of decay for the Lyapunov function of the system. The proposed implementation is based on a scaling law for the deadlines of the control task that was derived under an homogeneity assumption on the continuous dynamics. This scaling law shows that less frequent executions of the control tasks are required when the state approaches the origin and thus showing that periodic implementations do not result in an efficient use of processor time. An example illustrated in detail the benefits of self-triggered vs periodic time-triggered implementations. We are currently extending these results to the more general case of polynomial control systems.

σ	periodic	aperiodic	event-triggered
0.1	2208	581	347
0.2	1564	411	246
0.3	1278	336	201

TABLE I

Number of executions for a simulation time of 40sec.

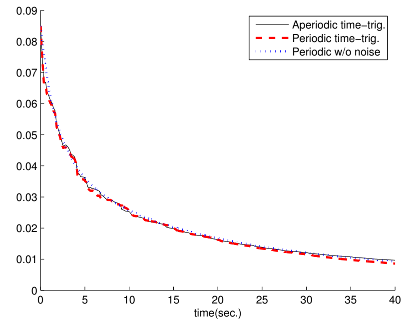

Fig. 3. Comparison of the Lyapunov functions for periodic and aperiodic implementation in presence of noise

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