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Output Feedback Stabilization for Switched Systems Subject to Saturation Nonlinearity

Yong-Mei Ma and Guang-Hong Yang

Abstract— **This paper studies the stabilization problem via output feedback for a class of switched discrete-time linear systems subject to saturation nonlinearity. A switched output feedback controller in the quasi-LPV form is designed which guarantees the closed-loop system is locally asymptotically stable. New sufficient LMI conditions related to the number of system modes are proposed for control synthesis, which are developed by introducing a new congruence transformation. The design problem of controller (coefficient matrices) that maximizes an estimation of the domain of attraction is then reduced to optimization problems with LMI constraints. A numerical example is given to illustrate that the proposed method is less conservative than existing results on the same problem.**

Key words: Switched systems; saturation nonlinearity; output feedback; quasi-LPV; LMIs.

I. Introduction

A switched system is an important class of hybrid dynamic systems consisting of a family of linear time-invariant subsystems and a switching law specifying the switching between them. In the last decade, the study of switched systems has received growing attention in control theory and its application, see, for example: [3], [5], [10], [16], [25]. On the more general topic and recent research progress in the field of switched systems, we refer readers to [16], [20]. As pointed out in [16], one of the interesting problems in switched systems is to find non (or less)-conservative conditions to guarantee the stability of the systems under arbitrary switching rules. Many analytical approaches and techniques regarding this issue have been reported in literatures, for instance: common Lyapunov function approach, switched Lyapunov function approach, multiple Lyapunov function approach, etc.,(see [3], [6], [14], [17] and the references therein).

Despite much progress, the principal drawback of above papers for switched systems is they can not deal with constraint problem. Significant research remains to be done in the direction of constrained control of switched systems, especially since the control action is often subject to hard actuator constraints. For actuator constraint systems, it is great the interest in the analysis and design of saturating control laws and various methods have been developed(see, for example, [4], [9], [11], [12], [13], [15], [23], [24] and references therein). One of the most relevant approaches is based on a novel polytopic model of the saturation nonlinearity which was proposed in [12]. The advantages of using the polytopic model have been shown in [22], etc. More recently, some study results concerning switched systems with input saturation could also be found in [1], [2], [18], [26], etc., for example, in [1](or [26]), a constant state feedback law was obtained via LMIs to stabilize(robust stabilize) the considered system; [2] presented a static output feedback law involving equality constraints; a hybrid control methodology via state feedback with high computational burden was proposed in [18].

In addition, as pointed out in [21], if the control input is subject to certain constraints, the switched system is only locally reachable provided that the unconstrained switched system is completely reachable. However, in the event of local stabilization, the exact determination of the domain of attraction for actuator saturation system is possible only in very special cases [13]. So one of the interesting issues in switched systems with actuator saturation is to obtain a large enough domain of attraction which ensures asymptotic stability for the switched systems despite the presence of actuator saturation.

The goal of this paper is to design a stabilizing controller for a class of switched discrete-time linear systems subject to actuator saturations via output feedback. By utilizing the polytopic model of a saturating linear feedback law, a switched nonlinear output feedback controller in the quasi-LPV form is proposed which guarantees the closed-loop system is locally asymptotically stable. Our main contribution consists in new sufficient LMI conditions related to the number of system modes for control synthesis, which are developed by introducing a new congruence transformation and are less conservative than existing results on the same problem. The design problem of controller (coefficient matrices) that maximizes an estimation of the domain of attraction of the considered systems is then reduced to optimization problems with LMI constraints. Finally, the effectiveness of the proposed method is illustrated by a numerical example. We note that controllers in quasi-LPV form were also used in other works [8], [19], [24].

The rest of this paper is organized as follows. Section 2 introduces the problem under consideration and some

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Yong-Mei Ma is with the College of Information Science and Engineering, Northeastern University, Shenyang, China; she is also with the College of Science, Yanshan University, Qinhuangdao, China. Email: mym@ysu.edu.cn

Guang-Hong Yang is with the College of Information Science and Engineering, Northeastern University, Shenyang, China. Email: yangguanghong@ise.neu.edu.cn

preliminary results. It is followed by the controller design via switched nonlinear output feedback in the quasi-LPV form for the considered system in Section 3. Section 4 proposed optimization problems concerning the estimation of domain of attraction. A numerical example and its simulation results are given to show the effectiveness of the proposed method in Section 5, Section 6 draws a conclusion.

Notation: Given a matrix E, E^T and E^{-1} denote its transpose, and inverse when it exists, respectively. For a square matrix Q, $Q>0 (Q<0)$ means that Q is positive definite (negative definite); $\star$ denotes the transpose of the off diagonal element of a matrix. For two integers $k_1,\ k_2,\ k_1<k_2,\ [k_1,k_2]=\{k_1,\ k_1+1,\ \cdots,\ k_2\}$. Let $0<P\in R^{n\times n}$, denote $\varepsilon(P,\rho)=\{x\in R^n \mid x^\top Px\le\rho,\ \rho>0\}$.

II. Problem Statement and Preliminaries

In this section, we give a more precise problem statement for the class of systems under consideration and some preliminary results.

We consider systems described by

$$\begin{cases} x(k+1)=A_ix(k)+B_isat(u(k)) \\ \quad y(k)=C_ix(k) \end{cases} \tag{1}$$

where $x(k)\in R^n$ is the state, $u(k)\in R^p$ is the control, $y(k)\in R^l$ is the output and the matrix triple (A_i,B_i,C_i) are stabilizable and detectable. $sat(\cdot)$ is the standard saturation function and i is a switching rule which takes its values in the finite set $\aleph=\{1,\ \cdots,N\}$. The saturation function is assumed here to be normalized, i. e., $sat(u(k))=sign(u(k))\min\{1,|u(k)|\}$.

Remark 2.1: We assume that the switching rule i is not known a priori, but its instantaneous value is available at each sampling period, that is each k. As reported in [6], the assumption corresponds to practical implementations where the switched system is supervised by a discrete-event system or operator allowing for the value of i to be known at only each sampling period in real time. In addition, we note that it is without loss of generality to assume an unity saturation level, as level of saturation can always be scaled to unity by scaling B_i and u.

Consider an output feedback law of the form:

$$\begin{cases} x_c(k+1)=A_{ci}(x_c(k),\ y(k))x_c(k)+B_{ci}(x_c(k),\ y(k))y(k) \\ \quad u(k)=C_{ci}x_c(k)+D_{ci}y(k) \end{cases} \tag{2}$$

where $x_c\in R^{n_c}$, n_c is dimension of the controller, C_{ci} and D_{ci} are constant matrices of appropriate dimensions.

The objective of this paper is to design an output feedback law of the form (2) that locally asymptotically stabilizes the plant (1) at the origin with a domain of attraction as large as possible for all sequences of switching $i(k)$.

To this end, the following preliminaries are needed.

For $H_C\in R^{p\times n_c}$ and $H_D\in R^{p\times l}$, denote :

$$\mathscr{L}(H_C,H_D) \quad =\{(x_c,y)\in R^{n_c+l}:|H_{Cr}x_c+H_{Dr}y|\le 1,\ r\in\{1,2,\cdots,p\}\}$$

where H_{Cr} and H_{Dr} represent the *rth* rows of matrices H_C and H_D, respectively. We note that $\mathscr{L}(H_C,H_D)$ represents the region in R^{n_c+l} where the auxiliary feedback $H_Cx_c+H_Dy$ does not saturate.

Let Ξ be the set of $p\times p$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^p elements in Ξ. Suppose that each element of Ξ is labeled as $E_s,\ s=0,1,\cdots,2^p-1$, and denote $E_s^-=I_p-E_s$. Clearly, E_s^- is also an element of Ξ if $E_s\in\Xi$.

Lemma 2.1: [12] [24] For any $(x_c,y)\in\mathscr{L}(H_C,H_D)$, there exist $0\le\eta_s\le 1,\ s\in[0,2^p-1]$ satisfying $\sum_{s=0}^{2^p-1}\eta_s=1$ such that

$$sat(C_cx_c+D_cy)=\sum_{s=0}^{2^p-1}\eta_s[E_s(C_cx_c+D_cy)+E_s^-(H_Cx_c+H_Dy)], \tag{3}$$

Remark 2.2: We note here that the values of the parameters $\eta=[\eta_0\ \eta_1\ \cdots\ \eta_{2^p-1}]$ are dependent on x_c and y and are available for real time use in gain-scheduling control. In fact, it is easy to see that the parameters $\eta(k)$ reflect the severity of actuator saturation (see details in [4]). Furthermore, a formula for computing these variant parameters $\eta(k)$ can be found in [24].

III. Output Feedback Controller Design

In this section, the design problem of a switched nonlinear output feedback controller in the quasi-LPV form for the considered systems is studied.

Firstly, if $(x_c,y)\in\mathscr{L}(H_C,H_D)$, the closed-loop systems $(1)-(2)$ can be rewritten by Lemma 2.1:

$$\begin{cases} x(k+1)=A_ix(k)+B_i\sum_{s=0}^{2^p-1}\eta_s(x_c(k),y(k)) \\ \qquad (E_{is}(C_{ci}x_c(k)+D_{ci}y(k)) \\ \qquad +E_{is}^-(H_{Ci}x_c(k)+H_{Di}y(k))) \\ \quad y(k)=C_ix(k) \end{cases} \tag{4}$$

In addition, we also use the functions $\eta_s(x_c,y)'s$ to parameterize the output feedback control (2) into the following form:

$$\begin{cases} x_c(k+1)=(\sum_{s=0}^{2^p-1}\eta_s(x_c(k),y(k))A_{cis})x_c(k) \\ \qquad +(\sum_{s=0}^{2^p-1}\eta_s(x_c(k),y(k))B_{cis})y(k) \\ \quad u(k)=C_{ci}x_c(k)+D_{ci}y(k) \end{cases} \tag{5}$$

where the variant parameters $\eta_s(x_c(k),y(k))$ can be computed in real time by utilizing Lemma 2 in [24], the coefficient matrices $A_{cis},\ B_{cis},\ C_{ci},\ D_{ci},\ i\in\aleph,\ s\in[0,2^p-1]$ are to be designed.

If $\Omega(\mathsf{P},1)\subset\mathscr{L}(H_DC_i,H_{Ci})$ is an invariant level set, where

$$\Omega(\mathsf{P},1)=\{\xi\in R^{n+n_c}:\quad \xi^\top(k)\mathsf{P}\xi(k)\le 1\},\quad \mathsf{P}>0$$

then combining (4) and (5), we obtain the following closed-loop system,

$$\xi(k+1)=\mathsf{A}_i(\eta(k))\xi(k) \tag{6}$$

where $\xi(k)=(x^\top(k),x_c^\top(k))^\top$, $\eta=[\eta_0\ \eta_1\ \cdots\ \eta_{2^p-1}]$ and

$$\Gamma=\{\eta(k)\in R^{2^p}:\ \sum_{s=0}^{2^p-1}\eta_s(k)=1,0\le\eta_s(k)\le 1\}$$

$$\mathsf{A}_i(\eta(k))=\sum_{s=0}^{2^p-1}\eta_s(x_c(k),y(k))$$

$$\begin{bmatrix} A_i + B_i(E_{is}D_{ci} + E_{is}^- H_{Di})C_i & B_i(E_{is}C_{ci} + E_{is}^- H_{Ci}) \\ B_{cis}C_i & A_{cis} \end{bmatrix}$$

$$= \sum_{s=0}^{2^p-1} \eta_s(x_c(k), y(k)) A_{is}$$

The following theorem presents sufficient conditions on the controller coefficient matrices under which the plant (1) is asymptotically stable with a switched nonlinear output feedback controller in the quasi-LPV form (5).

Theorem 3.1 : Consider the plant (1), if there exist symmetric positive-definite matrices $R_{mi} > 0$, $S_{mi} > 0$, symmetric matrices R_{min}, S_{min} and matrices $\bar{A}_{cis}$, $\bar{B}_{cis}$, $\bar{C}_{ci}$, $\bar{D}_{ci}$, $\bar{H}_{Ci}$, $\bar{H}_{Di}$, T_i, M_{mi}, M_{min}, X_{i11} , W_{i11}, such that the following inequalities hold $\forall m,\ i \in \{1, 2, \cdots, N\},\ m < n\ ,\ n \in \{2, \cdots, N\},\ s \in [0, 2^p - 1]$

$$\begin{bmatrix} X_{i11} + X_{i11}^\top - \frac{1}{2}R_{ii} & \star & \star & \star \\ T_i + I - \frac{1}{2}M_{ii} & W_{i11} + W_{i11}^\top - \frac{1}{2}S_{ii} & \star & \star \\ NA_iX_{i11} + NB_i\bar{U}_{cis} & NA_i + NB_i\bar{V}_{cis}C_i & \Gamma_1 & \star \\ \bar{A}_{cis} & \sum_{m=1}^N W_{m11}^\top A_i + \bar{B}_{cis}C_i & \Gamma_2 & \Gamma_3 \end{bmatrix} > 0 \tag{7}$$

$$\begin{bmatrix} R_{mi} & \star \\ M_{mi} & S_{mi} \end{bmatrix} > 0 \tag{8}$$

$$\begin{bmatrix} 1 & \bar{H}_{Cir} & \bar{H}_{Dir}C_i \\ \star & X_{i11} + X_{i11}^\top - \frac{1}{2}R_{ii} & T_i^\top + I - \frac{1}{2}M_{ii}^\top \\ \star & \star & W_{i11} + W_{i11}^\top - \frac{1}{2}S_{ii} \end{bmatrix} \geq 0 \tag{9}$$

where N indicates the number of system modes and

$$\bar{U}_{cis} = E_{is}\bar{C}_{ci} + E_{is}^- \bar{H}_{Ci},$$

$$\bar{V}_{cis} = E_{is}\bar{D}_{ci} + E_{is}^- \bar{H}_{Di}$$

$$T_i = W_{i11}^\top X_{i11} + W_{i21}^\top X_{i21}$$

$$\Gamma_1 = \frac{1}{2}\sum_{m=1}^N R_{mj} + \sum_{m<n, m\in\{1,\cdots,N-1\}, n\in\{2,\cdots,N\}} R_{mjn}$$

$$\Gamma_2 = \frac{1}{2}\sum_{m=1}^N M_{mj} + \sum_{m<n, m\in\{1,\cdots,N-1\}, n\in\{2,\cdots,N\}} M_{mjn}$$

$$\Gamma_3 = \frac{1}{2}\sum_{m=1}^N S_{mj} + \sum_{m<n, m\in\{1,\cdots,N-1\}, n\in\{2,\cdots,N\}} S_{mjn}$$

X_{i21}, W_{i21} can be found by computing a singular value decomposition (SVD) of $T_i - W_{i11}^\top X_{i11}$, then, with the controller coefficient matrices for each $s \in \{0, 1, \cdots 2^p - 1\}, i,\ m \in \aleph$:

$$\begin{cases} D_{ci} &= \bar{D}_{ci} \\ H_{Di} &= \bar{H}_{Di} \\ C_{ci} &= (\bar{C}_{ci} - D_{ci}C_iX_{i11})X_{i21}^{-1} \\ H_{Ci} &= (\bar{H}_{Ci} - H_{Di}C_iX_{i11})X_{i21}^{-1} \\ B_{cis} &= (\sum_{m=1}^N W_{m21}^\top)^{-1}(\bar{B}_{cis} - \sum_{m=1}^N W_{m11}^\top B_iV_{cis}) \\ A_{cis} &= (\sum_{m=1}^N W_{m21}^\top)^{-1}(\bar{A}_{cis} - \sum_{m=1}^N W_{m11}^\top A_iX_{i11} \\ & \quad - \sum_{m=1}^N W_{m11}^\top B_iV_{cis}C_iX_{i11} - \sum_{m=1}^N W_{m11}^\top B_iU_{cis}X_{i21} \\ & \quad - \sum_{m=1}^N W_{m21}^\top B_{cis}C_iX_{i11})X_{i21}^{-1} \end{cases} \tag{10}$$

where

$$U_{cis} = E_{is}C_{ci} + E_{is}^- H_{Ci},$$

$$V_{cis} = E_{is}D_{ci} + E_{is}^- H_{Di},$$

then, the output feedback controller (5) with the controller coefficient matrices given by (10) locally asymptotically stabilizes the plant (1) at the origin with the invariant level set $\Omega(\mathsf{P}, 1)$ contained in the domain of attraction.

Proof: Choose the following switched quadratic Lyapunov function

$$V(\xi(k), \delta(k)) = \xi^\top(k)\mathsf{P}\xi(k)$$

where $\mathsf{P} = \sum_{i=1}^N \delta_i(k)(Q_i)^{-1},\ Q_i > 0,\quad i \in \aleph$. $\delta_i(k)$ is the indicator function [6]:

$$\delta(k) = \{\delta_1(k), \delta_2(k), \cdots, \delta_N(k)\}$$

where $\delta_i(k) = 1$ if the switched system is in mode i and $\delta_i(k) = 0$ if it is in a different mode.

In the following part, we will find $\forall\ i \in \aleph$

$$Q_i = \begin{bmatrix} I & W_{i11} \\ 0 & W_{i21} \end{bmatrix}^{-\top} \begin{bmatrix} \frac{1}{2}R_{ii} & \star \\ \frac{1}{2}M_{ii} & \frac{1}{2}S_{ii} \end{bmatrix} \begin{bmatrix} I & W_{i11} \\ 0 & W_{i21} \end{bmatrix}^{-1}$$

$Q_i > 0$ is a natural result from (8).

We know that the closed-loop system (6) was obtained under the assumption $(x_c, y) \in \mathscr{L}(H_{Ci}, H_{Di})$. In what follows, we will firstly demonstrate when $(x_c, y) \in \mathscr{L}(H_{Ci}, H_{Di})$ holds, the conditions (7) and (8) imply system (6) is asymptotically stable at the origin. Finally, we will illustrate condition (9) indicates $(x_c, y) \in \mathscr{L}(H_{Ci}, H_{Di})$.

By means of Theorem 2 in [6], we know that system (6) is asymptotically stable at the origin if there exist matrices X_i such that $\forall\ i,\ j \in \aleph$

$$\begin{bmatrix} X_i + X_i^\top - Q_i & \star \\ \mathsf{A}_i(\eta(k))X_i & Q_j \end{bmatrix} > 0 \tag{11}$$

It is obvious (11) holds if the following inequality holds

$$\begin{bmatrix} X_i + X_i^\top - Q_i & \star \\ A_{is}X_i & Q_j \end{bmatrix} > 0\ \forall\ i,\ j \in \aleph,\ s \in [0, 2^p - 1] \tag{12}$$

where

$$A_{is} = \begin{bmatrix} A_i + B_i(E_{is}D_{ci} + E_{is}^- H_{Di})C_i & B_i(E_{is}C_{ci} + E_{is}^- H_{Ci}) \\ B_{cis}C_i & A_{cis} \end{bmatrix}$$

Condition (12) is not linear with respect to the unknowns $(X_i, Q_i, D_{ci}, H_{Di}, C_{ci}\ H_{Ci},\ A_{cis}, B_{cis})$. In order to linearize this condition and get a well tractable condition in terms of LMIs, we propose a new change of variable inspired from [7], denote matrix X_i and its inverse as $\forall\ i \in \aleph$

$$X_i = \begin{bmatrix} X_{i11} & X_{i12} \\ X_{i21} & X_{i22} \end{bmatrix},\quad W_i = X_i^{-1} = \begin{bmatrix} W_{i11} & W_{i12} \\ W_{i21} & W_{i22} \end{bmatrix}$$

Let transformation matrix

$$\Pi_i = \begin{bmatrix} I & W_{i11} \\ 0 & W_{i21} \end{bmatrix}\ \forall\ i \in \aleph$$

Then there have the relations $\forall\ i \in \aleph$

$$X_i\Pi_i = \tilde{\Pi}_i,\quad W_i\tilde{\Pi}_i = \Pi_i,\quad \tilde{\Pi}_i = \begin{bmatrix} X_{i11} & I \\ X_{i21} & 0 \end{bmatrix}$$

Perform the *new congruence transformation* for (12) by $diag\{\Pi_i,\ \sum_{m=1}^{N}\Pi_m\}$, it follows that $\forall\ i,\ j\in\aleph,\ s\in[0,2^p-1]$

$$\begin{bmatrix}\Pi_i^\top(X_i+X_i^\top-Q_i)\Pi_i & \star\\ \sum_{m=1}^{N}\Pi_m^\top A_{is}X_i\Pi_i & (\sum_{m=1}^{N}\Pi_m^\top)Q_j(\sum_{m=1}^{N}\Pi_m)\end{bmatrix}>0 \tag{13}$$

For $\forall\ m,\ n,\ j\in\{1,2,\cdots,N\}$, let

$$\Pi_m^\top Q_j\Pi_n+\Pi_n^\top Q_j\Pi_m=\begin{bmatrix}R_{mjn} & M_{mjn}^\top\\ M_{mjn} & S_{mjn}\end{bmatrix}, \tag{14}$$

Especially, there have

$$\Pi_m^\top Q_j\Pi_m=\frac{1}{2}\begin{bmatrix}R_{mjm} & \star\\ M_{mjm} & S_{mjm}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}R_{mj} & \star\\ M_{mj} & S_{mj}\end{bmatrix}>0,$$
$$R_{mj}>0,\quad S_{mj}>0.$$

By computing, we can obtain $\forall\ i,\ j\in\aleph,\ s\in[0,2^p-1]$

$$\Pi_i^\top(X_i+X_i^\top)\Pi_i=\begin{bmatrix}X_{i11}+X_{i11}^\top & \star\\ W_{i11}^\top X_{i11}+W_{i21}^\top X_{i21}+I & W_{i11}+W_{i11}^\top\end{bmatrix}$$

$$\sum_{m=1}^{N}\Pi_m^\top A_{is}X_i\Pi_i=\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}$$

$$(\sum_{m=1}^{N}\Pi_m^\top)Q_j(\sum_{m=1}^{N}\Pi_m)=\begin{bmatrix}\Gamma_1 & \star\\ \Gamma_2 & \Gamma_3\end{bmatrix}$$

where

$$\begin{aligned}
U_{cis} &= E_{is}C_{ci}+E_{is}^{-}H_{Ci},\\
V_{cis} &= E_{is}D_{ci}+E_{is}^{-}H_{Di},\\
Y_{11} &= NA_iX_{i11}+NB_iU_{cis}X_{i21}+NB_iV_{cis}C_iX_{i11},\\
Y_{12} &= NA_i+NB_iV_{cis}C_i\\
Y_{21} &= \textstyle\sum_{m=1}^{N}(W_{m11}^\top A_iX_{i11}+W_{m11}^\top B_iV_{cis}C_iX_{i11}\\
&\quad +W_{m11}^\top B_iU_{cis}X_{i21}+W_{m21}^\top B_{cis}C_iX_{i11}+W_{m21}^\top A_{cis}X_{i21})\\
Y_{22} &= \textstyle\sum_{m=1}^{N}(W_{m11}^\top A_i+W_{m11}^\top B_iV_{cis}C_i+W_{m21}^\top B_{cis}C_i)\\
\Gamma_1 &= \tfrac{1}{2}\textstyle\sum_{m=1}^{N}R_{mj}+\sum_{m<n,m\in\{1,\cdots,N-1\},n\in\{2,\cdots,N\}}R_{mjn}\\
\Gamma_2 &= \tfrac{1}{2}\textstyle\sum_{m=1}^{N}M_{mj}+\sum_{m<n,m\in\{1,\cdots,N-1\},n\in\{2,\cdots,N\}}M_{mjn}\\
\Gamma_3 &= \tfrac{1}{2}\textstyle\sum_{m=1}^{N}S_{mj}+\sum_{m<n,m\in\{1,\cdots,N-1\},n\in\{2,\cdots,N\}}S_{mjn}
\end{aligned}$$

via the following change of variables:

$$\begin{aligned}
T_i &= W_{i11}^\top X_{i11}+W_{i21}^\top X_{i21},\\
\bar{D}_{ci} &= D_{ci},\\
\bar{H}_{Di} &= H_{Di},\\
\bar{C}_{ci} &= C_{ci}X_{i21}+D_{ci}C_iX_{i11},\\
\bar{H}_{Ci} &= H_{Ci}X_{i21}+H_{Di}C_iX_{i11},\\
\bar{U}_{cis} &= E_{is}\bar{C}_{ci}+E_{is}^{-}\bar{H}_{Ci},\\
\bar{V}_{cis} &= E_{is}\bar{D}_{ci}+E_{is}^{-}\bar{H}_{Di},\\
\bar{B}_{cis} &= \textstyle\sum_{m=1}^{N}(W_{m11}^\top B_iV_{cis}+W_{m21}^\top B_{cis}),\\
\bar{A}_{cis} &= \textstyle\sum_{m=1}^{N}(W_{m11}^\top A_iX_{i11}+W_{m11}^\top B_iV_{cis}C_iX_{i11}\\
&\quad +W_{m11}^\top B_iU_{cis}X_{i21}+W_{m21}^\top B_{cis}C_iX_{i11}+W_{m21}^\top A_{cis}X_{i21}).
\end{aligned}$$

then the condition (7) can be obtained. To complete the proof, one has to show that it is always possible to invert above relation in order to build a feasible controller. Assume that condition (7) is feasible. This means that $\forall\ i\in\aleph$

$$\begin{bmatrix}X_{i11}+X_{i11}^\top & T_i^\top+I\\ T_i+I & W_{i11}+W_{i11}^\top\end{bmatrix}>\frac{1}{2}\begin{bmatrix}R_{ii} & M_{ii}^\top\\ M_{ii} & S_{ii}\end{bmatrix}>0 \tag{15}$$

and X_{i11}, W_{i11} are nonsingular. Multiplying the above inequalities by $[X_{i11}^{-\top},\ -I]$ on the left and by $[X_{i11}^{-\top},\ -I]^\top$ on the right we have

$$(T_i-W_{i11}X_{i11})X_{i11}^{-1}+X_{i11}^{-\top}(T_i-W_{i11}X_{i11})^\top<0$$

which implies that there always exist matrices W_{i21} and X_{i21} both nonsingular satisfying $\forall\ i\in\aleph$

$$T_i=W_{i11}^\top X_{i11}+W_{i21}^\top X_{i21}$$

Thus it is always possible to invert the above relation and get the controller coefficient matrices (10).

In what follows, we will illustrate condition (9) indicates $(x_c,y)\in\mathscr{L}(H_{Ci},H_{Di})$. Noting that

$$(x_c,y)\in\mathscr{L}(H_{Ci},H_{Di})\quad\forall\ i\in\aleph$$

holds if the following holds

$$(x,x_c)\in\mathscr{L}(H_{Di}C_i,H_{Ci})\ \forall\ i\in\aleph$$

where

$$\mathscr{L}(H_{Di}C_i,H_{Ci})=\{(x,x_c):|H_{Dir}C_ix+H_{Cir}x_c|\le 1\}$$
$$i\in\aleph,\ r\in\{1,2,\cdots,p\}$$

It thus follows from the above that the closed-loop system is locally asymptotically stable at the origin with $\Omega(\mathsf{P},1)$ contained in its domain of attraction if

$$\Omega(\mathsf{P},1)\subset\mathscr{L}(H_{Di}C_i,H_{Ci})\ \forall\ i\in\aleph$$

which is equivalent to

$$\begin{bmatrix}H_{Dir}C_i & H_{Cir}\end{bmatrix}(\mathsf{P})^{-1}\begin{bmatrix}C_i^\top H_{Dir}^\top\\ H_{Cir}^\top\end{bmatrix}\le 1\ \forall\ i\in\aleph,\ r\in\{1,2,\cdots,p\}$$

it follows that

$$\begin{bmatrix}1 & \begin{bmatrix}H_{Dir}C_i & H_{Cir}\end{bmatrix}\\ \begin{bmatrix}C_i^\top H_{Dir}^\top\\ H_{Cir}^\top\end{bmatrix} & Q_i^{-1}\end{bmatrix}\ge 0\quad\forall\ i\in\aleph,\ r\in\{1,2,\cdots,p\} \tag{16}$$

by multiplying (16) from the left by block-diag$\{1,X_i^\top\}$ and from the right by block-diag$\{1,X_i\}$, we obtain $\forall\ i\in\aleph,\ r\in\{1,2,\cdots,p\}$

$$\begin{bmatrix}1 & \begin{bmatrix}H_{Dir}C_i & H_{Cir}\end{bmatrix}X_i\\ X_i^\top\begin{bmatrix}C_i^\top H_{Dir}^\top\\ H_{Cir}^\top\end{bmatrix} & X_i^\top Q_i^{-1}X_i\end{bmatrix}\ge 0 \tag{17}$$

since

$$X_i^\top Q_i^{-1}X_i\ge X_i+X_i^\top-Q_i \tag{18}$$

(17) holds if the following inequality holds $\forall\ i\in\aleph,\ r\in\{1,2,\cdots,p\}$

$$\begin{bmatrix}1 & \begin{bmatrix}H_{Dir}C_i & H_{Cir}\end{bmatrix}X_i\\ X_i^\top\begin{bmatrix}C_i^\top H_{Dir}^\top\\ H_{Cir}^\top\end{bmatrix} & X_i+X_i^\top+Q_i\end{bmatrix}\ge 0 \tag{19}$$

noting that $\forall\ i\in\aleph,\ r\in\{1,2,\cdots,p\}$

$$\begin{bmatrix}H_{Di}C_i & H_{Ci}\end{bmatrix}X_i\Pi_i=\begin{bmatrix}\bar{H}_{Ci} & \bar{H}_{Di}C_i\end{bmatrix}$$

multiplying (19) from the left by block-diag$\{1,\Pi_i^\top\}$ and from the right by block-diag$\{1,\Pi_i\}$, we can obtain (9). So the proof is complete. □

Remark 3.1: If the considered system is a general linear time-invariant system with actuator saturation, the result in this paper can be viewed as an extension to the discrete-time systems of the work presented in [24].

Remark 3.2: In the course of linearizing (12), a congruence transformation was proposed after specifying X_i as X in (12) [7]. However, in this paper, we don't specialize X_i as X but develop a new congruence transformation (13) to deal with more general X_i. So compared with the result in [7], the result in this paper is less conservative.

IV. Estimation of Domain of Attraction

Theorem 3.1 provides conditions on the coefficient matrices of the output feedback controller under which the level set $\Omega(\mathsf{P},1)$ is inside the domain of attraction. In this section, our objective is to maximize the size of the domain of attraction on the closed-loop system, that is, the size of the projection of a level set $\Omega(\mathsf{P},1)$ onto the states of the plant (i.e. x).

In [1](or [2]), the level set $\Omega(\mathsf{P},1)$ is estimated by $\bigcap_{i=1}^{N}\varepsilon(Q_i^{-1},\ \rho_i)$(or $\bigcup_{i=1}^{N}\varepsilon(Q_i^{-1},\ \rho_i)$) firstly, then the estimation problem of the domain of attraction is performed by maximizing $trace(Q_i^{-1})$. Since the shape of the level set $\Omega(\mathsf{P},1)$ isn't explicit beforehand, Here, we will follow the idea in [12] and take its shape into consideration. In general, the size of the projection of a level set $\Omega(\mathsf{P},1)$ onto the states of the plant (i.e. x) can be measured with respect to a given shape reference set $\mathscr{X}_R$ which is an ellipsoid

$$\mathscr{X}_R = \{x \in R^n :\ \ x^T X_r x \le 1\}, \quad X_r > 0 \tag{20}$$

or, a polyhedron defined as

$$\mathscr{X}_R = co\{x_1, x_2,\ \cdots\ , x_q\}, \tag{21}$$

where $x_1, x_2,\ \cdots\ , x_q$ are given points in R^n a priori. we can maximize a scalar $\alpha > 0$ such that, let $x_c = 0$

$$\alpha\mathscr{X}_R \subset \Omega(\mathsf{P},1)|_{(x,0)} \tag{22}$$

If $\mathscr{X}_R$ is an ellipsoid, then (22) is equivalent to

$$\begin{bmatrix} \alpha^{-2}X_r & I & W_{i11} \\ \star & \frac{1}{2}R_{ii} & \frac{1}{2}M_{ii}^\top \\ \star & \star & \frac{1}{2}S_{ii} \end{bmatrix} \ge 0 \tag{23}$$

If $\mathscr{X}_R$ is a polyhedron, then, by Schur complement, (22) can also be converted into an LMI constraint easily

$$\begin{bmatrix} \alpha^{-2} & x_t^\top & x_t^\top W_{i11} \\ \star & \frac{1}{2}R_{ii} & \frac{1}{2}M_{ii}^\top \\ \star & \star & \frac{1}{2}S_{ii} \end{bmatrix} \ge 0 \tag{24}$$

Thus, the determination of the controller coefficient matrices which maximize an estimation of domain of attraction can be reduced to the following LMI optimization problems:

$$maximize \quad \alpha$$

$$s.t. \quad (7),(8),(9),(23) \tag{25}$$

or

$$maximize \quad \alpha$$

$$s.t. \quad (7),(8),(9),(24) \tag{26}$$

V. Numerical Example

A numerical example borrowed from the literature is now presented to illustrate the effectiveness of proposed approaches.

Consider the same system in [1] to estimate its domain of attraction:

$$A_1 = \begin{bmatrix} -0.7 & 1 \\ -0.5 & -1.5 \end{bmatrix};\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ C_1 = \begin{bmatrix} -1 & 1 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} 0.9 & -1 \\ 1.7 & -1.5 \end{bmatrix};\ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix};\ C_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Here, $\mathscr{X}_R$ is taken as an ellipsoid with

$$X_r = \begin{bmatrix} 2500 & 0 \\ 0 & 10000 \end{bmatrix};$$

Solving the optimization problem (25), we obtain the controller coefficient matrices given by

$$A_{c10} = \begin{bmatrix} 29.4488 & 53.8974 \\ -87.4350 & -178.9554 \end{bmatrix};\ B_{c10} = \begin{bmatrix} -0.1420 \\ 0.4799 \end{bmatrix} \times 10^{-5},$$

$$A_{c11} = \begin{bmatrix} -18.5387 & -45.6160 \\ 46.4439 & 98.8401 \end{bmatrix};\ B_{c11} = \begin{bmatrix} 0.1301 \\ -0.2787 \end{bmatrix} \times 10^{-5},$$

$$A_{c20} = \begin{bmatrix} 3.4163 & -49.6799 \\ -5.8491 & 89.5535 \end{bmatrix};\quad B_{c20} = \begin{bmatrix} 0.2346 \\ -0.4215 \end{bmatrix} \times 10^{-5},$$

$$A_{c21} = \begin{bmatrix} 6.2709 & -82.7058 \\ -7.1312 & 104.9359 \end{bmatrix};\ B_{c21} = \begin{bmatrix} 0.4124 \\ -0.5022 \end{bmatrix} \times 10^{-5},$$

$$C_{c1} = \begin{bmatrix} 6.7600 & 4.4091 \end{bmatrix} \times 10^{-6};\ D_{c1} = -0.6996;$$

$$C_{c2} = \begin{bmatrix} 3.3612 & -0.9560 \end{bmatrix} \times 10^{-6};\quad D_{c2} = 1.5337;$$

The resulting maximum value of α is 1.7343e+005, with auxiliary matrices H_{Ci} and H_{Di} given by

$$H_{C1} = \begin{bmatrix} 0.3314 & -0.4533 \end{bmatrix};\ H_{D1} = -3.8370 \times 10^{-8};$$

$$H_{C2} = \begin{bmatrix} -0.4728 & 0.0009 \end{bmatrix};\quad H_{D2} = 1.2996 \times 10^{-7}.$$

Fig 1 presents the domain of attraction estimated by different methods. The inner two ellipsoids overlapped each other are obtained by state feedback law via optimization problem 2 (*Pb.*2) in [1]; while the outer ellipsoid is the cross-section of $\Omega(\mathsf{P},1)$ at $x_c = 0$ by our output feedback controller. For simplicity, only $\alpha = 1.7343e+002$ is plotted. It is clear our result is much better.

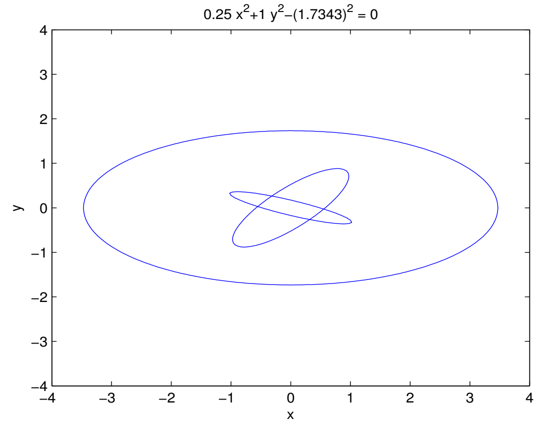

Fig. 1. The domain of attraction estimated by different methods.

VI. Conclusion

In this paper, we propose a method to design stabilizing controllers for a class of switched discrete-time linear systems subject to actuator saturations via output feedback. By utilizing the polytopic model of a saturating linear feedback law, a nonlinear output feedback controller in a quasi-LPV form is presented which guarantees the closed-loop system is locally asymptotically stable. Our main contribution consists in new sufficient LMI conditions related to the number of system modes for control synthesis, which are developed by introducing a new congruence transformation. The design problem of controller (coefficient matrices) that maximizes an estimation of the domain of attraction of the considered systems is then reduced to optimization problems with LMI constraints. Finally, the effectiveness of the proposed method is illustrated by a numerical example.

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