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Performance Analysis for Linear Discrete-time Systems Subject to Actuator Saturation

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Abstract— **This paper investigates performance analysis problem under a given feedback law for discrete-time linear systems subject to actuator saturation. Two performance measures, the estimation of domain of attraction and L_2 performance, are considered by combining the saturation-dependent Lyapunov function method with Finsler's Lemma. New and less conservative conditions in the enlarged space containing both the state and its time difference, allowing extra degree of freedom for various performance analysis, are proposed. Furthermore, based on these results, two important lemmas and two iterative LMI-based optimization algorithms are also developed to optimize the performance indexes respectively. Numerical examples illustrate that the proposed methods improve recent results on the same problems.**

Key words: Actuator saturation; estimation of domain of attraction; L_2 performance; LMIs.

I. Introduction

Saturation is probably the most commonly encountered nonlinearity in control engineering because of the physical impossibility of applying unlimited control signals. It is well known that the input saturation is source of performance degeneration, limit cycles, different equilibrium points, and even instability. Hence, the attraction is great in the analysis and design of saturating control laws. See, for instance [7], [10], [17], and references therein.

With the absolute stability analysis tools, such as the circle and Popov criteria, various methods have been developed on controller synthesis, stability analysis and other performance analysis: the estimation of the domain of attraction, disturbance tolerance , L_2 gain analysis, etc. (see, for example, [2], [3], [5], [6], [8], [11], [13] and references therein). One of the most relevant approaches to the analysis of saturated systems is based on a novel polytopic model of the saturation nonlinearity which was proposed in [7]. Based on that, several interesting results were reported by developing various Lyapunov functions, for example, quadratic Lyapunov function [4], [7], [8]; Piecewise-affine Lyapunov function [12]; saturation-dependent Lyapunov function (parameter-dependent Lyapunov function) [3], [18],(for linear discrete-time system); convex hull quadratic Lyapunov function and max quadratic Lyapunov function [9](for linear continuous-time system). The advantages of using the polytopic model have been shown in [16], etc. However, all the existing results were obtained only in the state space by using Lyapunov function approach solely. Obviously, in this case, the degree of freedom for various analysis is restricted within narrow limits. This paper is inspired to complete the above conservativeness.

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The objective of this paper is to reinvestigate performance analysis problem under a given feedback law for discrete-time linear systems subject to actuator saturation, which is motivated by Finsler's Lemma [14], [15]. Two performance measures, the estimation of domain of attraction and L_2 performance, are considered by combining the saturation-dependent Lyapunov function method with Finsler's Lemma. The method is conceptually simple. Here, difference equations are considered as constraints and these dynamical constraints are incorporated into the stability analysis conditions through the use of matrix Lagrange multipliers. New and less conservative conditions in the enlarged space containing both the state and its time difference, allowing extra degrees of freedom for various performance analysis, are proposed. Furthermore, based on these results, two important lemmas and two iterative LMI-based optimization algorithms are also developed to optimize the performance indexes respectively. Numerical examples illustrate that the proposed methods improve recent results on the same problems.

The paper is organized as follows. Section 2 gives the problem under consideration. Two performance measures, the estimation of domain of attraction and the L_2 performance, are addressed in Section 3 by using the saturation-dependent Lyapunov method combined with Finsler's Lemma. Furthermore, two important lemmas and two iterative LMI-based optimization algorithms are also developed to optimize the performance indexes respectively in this Section. Numerical examples are given to show the effectiveness of the proposed methods in Section 4, conclusions are made in Section 5.

Notation: For a vector $\upsilon \in R^n$, we denote the standard multivariable saturation function as $\sigma(\upsilon) = [\sigma(\upsilon_1) \quad \sigma(\upsilon_2)\cdots\sigma(\upsilon_n)]^T$, where $\sigma(\upsilon_i) = sign(\upsilon_i)\min\{1,|\upsilon_i|\}$, denote its Euclidean norm as $\|\upsilon\|_2 = (\upsilon^T\upsilon)^{1/2}$. For a signal $\upsilon(k)$ defined on $[0,\infty)$, we define its l_2 norm as $\|\upsilon\|_{l_2} = (\sum_{k=0}^{\infty}\upsilon(k)^T\upsilon(k))^{1/2}$. $\star$ denotes the transpose of the off diagonal element of a matrix. $\mathbf{I}(\mathbf{0})$ represents the identity(null) matrix of appropriate dimension. Denote $L_V(1)=\{x\in R^n \mid V(x)\le 1\}$ as the level set of a Lyapunov function $V(x)$. For a matrix $F\in R^{m\times n}$, denote the ith row as f_i and define $\mathscr{L}(F)=\{x\in R^n \mid \ |f_i x|\le 1, \ \ 1\le i\le m\}$.

II. Problem Statement

Consider a discrete-time linear system subject to input saturation

$$x(k+1)=Ax(k)+B\sigma(u(k)) \tag{1}$$

where $x\in R^n$ denotes the state vector and $u\in R^m$ is the control input vector.

This paper considers two performance measures of system (1) under a given linear state feedback law

$$u(k)=Fx(k) \tag{2}$$

Two performance measures are the estimation of domain of attraction and the L_2 performance respectively. Designing optimal control strategies to obtain the optimized performance indexes is the objective of this paper.

Now, let Ξ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in Ξ. Suppose that each element of Ξ is labeled as $D_s,\ s=1,2,\cdots,2^m$, and denote $D_s^-=\mathbf{I}-D_s$. Clearly, D_s^- is also an element of Ξ if $D_s\in\Xi$.

By means of the following well-known Lemma [8]: Let $F,H\in R^{m\times n}$ be given. For $x\in R^n$, if $x\in\mathscr{L}(H)$, then there exist $\eta_s\ge 0,\ s\in[1,2^m]$ satisfying $\sum_{s=1}^{2^m}\eta_s=1$ such that

$$\sigma(Fx)=\sum_{s=1}^{2^m}\eta_s(D_sF+D_s^-H)x. \tag{3}$$

the closed-loop system (1-2) can be rewritten as

$$x(k+1)=\hat{A}(\eta)x(k), \quad \forall\ x\in\mathscr{L}(H) \tag{4}$$

where

$$\hat{A}(\eta)=\sum_{s=1}^{2^m}\eta_s\hat{A}_s=\sum_{s=1}^{2^m}\eta_s(A+B(D_sF+D_s^-H)) \tag{5}$$

and $\eta=[\eta_1\ \eta_2\ \cdots\ \eta_{2^m}]$ is a function of x that satisfies (3). Here and later in this paper, we use $\eta_s(k)$ to denote $\eta_s(x_k)$. It is easy to see that the parameters $\eta(k)$ reflect the severity of actuator saturation (see details in [3]).

III. Performance Analysis

In this section, two performance measures are considered: the estimation of domain of attraction and the L_2 performance.

To check asymptotical stability of system (4), a saturation-dependent Lyapunov function was used [3]:

$$\begin{aligned} V(k,x(k)) &= x^T(k)P(\eta(x(k)))x(k) \\ &= x^T(k)(\textstyle\sum_{s=1}^{2^m}\eta_s(x(k))P_s)x(k), \qquad P_s>0 \end{aligned} \tag{6}$$

if such a positive-definite Lyapunov function exists and

$$\begin{aligned} \Delta V(k,x(k)) &= x^T(k+1)P(\eta(k+1))x(k+1) \\ &\quad -x(k)^TP(\eta(k))x(k) \end{aligned} \tag{7}$$

is negative definite along the solutions of (4), then the origin of the saturated system (4) is asymptotically stable for $\forall x_0\in\mathscr{L}(H)$.

In what follows, new performance test criterion are obtained by combining saturation-dependent Lyapunov function method with Finsler's Lemma [14], [15]. These conditions are more general than ones in existing references and no matrix inversion is involved in the construction of saturation-dependent Lyapunov function.

Before the main results are given, the following Finsler's lemma is needed.

Lemma 3.1(*Finsler's Lemma*): Let $x\in R^n$, symmetric matrix $P\in R^{n\times n}$, and $\Phi\in R^{m\times n}$ such that $rank(\Phi)=r<n$. Then the following statements are equivalent:

i) $x^TPx<0,\ \forall\ \ \Phi x=0, x\neq 0$.

ii) $\exists\ X\in R^{n\times m}:\ \ P+X\Phi+\Phi^TX^T<0$.

A. Estimation of Domain of Attraction

Theorem 3.1.1: Consider the closed-loop system (4) under a given state feedback control matrix F. If there exist matrices H,M,G and $P_s>0, s=1,2,\cdots,2^m$, such that $\forall s,l\in[1,2^m]$

$$\begin{bmatrix}\Theta+\Theta^T-P_s & -M^T+AG+B(D_sFG+D_s^-HG)\\ \star & -G-G^T+P_l\end{bmatrix}<0 \tag{8}$$

$$\begin{bmatrix}1 & h_j\\ \star & P_s\end{bmatrix}\ge 0 \tag{9}$$

where

$$\Theta=AM+B(D_sFM+D_s^-HM)$$

then the closed-loop system is asymptotically stable at the origin with the level set $L_V(1)$ contained in the domain of attraction.

Proof: Obviously, inequality (9) is equivalent to $h_jP_s^{-1}h_j^T\le 1$, it follows $|h_ix|\le 1,\ \forall x\in L_V(1), i\in[1,m]$, that is, $L_V(1)\in\mathscr{L}(H)$. So system (1-2) can be written as (4).

Recall that the requirement $\Delta V(k,x(k))<0$ for any $x(k)\in L_V(1)\setminus\{0\}$ can be stated as

$$\begin{bmatrix}x^T(k) & x^T(k+1)\end{bmatrix}\begin{bmatrix}-P(\eta(k)) & \mathbf{0}\\ \mathbf{0} & P(\eta(k+1))\end{bmatrix}\begin{bmatrix}x(k)\\ x(k+1)\end{bmatrix}<0 \tag{10}$$

$$\forall\ \begin{bmatrix}\hat{A}(\eta(k)) & -\mathbf{I}\end{bmatrix}\begin{bmatrix}x(k)\\ x(k+1)\end{bmatrix}=0, \qquad \begin{bmatrix}x(k)\\ x(k+1)\end{bmatrix}\neq\mathbf{0}$$

Apply Lemma 3.1 with

$$\begin{bmatrix}x(k)\\ x(k+1)\end{bmatrix}\to x, \qquad \begin{bmatrix}-P(\eta(k)) & \mathbf{0}\\ \mathbf{0} & P(\eta(k+1))\end{bmatrix}\to P$$

$$\begin{bmatrix}\hat{A}(\eta(k)) & -\mathbf{I}\end{bmatrix}\to\Phi, \qquad \begin{bmatrix}M^T\\ G^T\end{bmatrix}\to X$$

Then system (4) is asymptotically stable if the following inequality holds:

$$\Bigg[\ \begin{matrix} M^T\hat{A}(\eta(k))+\hat{A}^T(\eta(k))M-P(\eta(k)) \\ G^T\hat{A}(\eta(k))-M \end{matrix}$$

$$\begin{bmatrix} \cdots & -M^T+\hat{A}^T(\eta(k))G \\ & -G^T-G+P(\eta(k+1)) \end{bmatrix}<0 \quad (11)$$

by transposing $\hat{A}(\eta(k))$, (11) can also be written as

$$\sum_{s=1}^{2^m}\eta_s(k)\sum_{l=1}^{2^m}\eta_l(k+1)\begin{bmatrix} M^T\hat{A}_s^T+\hat{A}_sM-P_s & -M^T+\hat{A}_sG \\ G^T\hat{A}_s^T-M & -G^T-G+P_l \end{bmatrix}<0 \quad (12)$$

obviously, the above inequality holds if

$$\begin{bmatrix} M^T\hat{A}_s^T+\hat{A}_sM-P_s & -M^T+\hat{A}_sG \\ G^T\hat{A}_s^T-M & -G^T-G+P_l \end{bmatrix}<0, \quad \forall s,l\in[1,2^m] \quad (13)$$

via (5), (13) is nothing more than (8). then the closed-loop system (4) is asymptotically stable at the origin with the level set $L_V(1)$ contained in the domain of attraction. So the proof is complete. □

Remark 3.1.2: The key idea behind Theorem 3.1.1 is to increase the dimension of the inequalities and to introduce new matrix variables M and G, here identified as Lagrange multipliers, allowing some degree of freedom. With special choice $M=\mathbf{0}$, it is not difficult to see that Theorem 3.1.1 is essentially equivalent to Theorem 1 in [3] while the proof here is more straightforward.

Theorem 3.1.1 provides conditions under which the level set $L_V(1)$ is inside the domain of attraction. In general, the size of $L_V(1)$ can be measured with respect to a given shape reference set $\mathscr{X}_R$ which is a polyhedron defined as $\mathscr{X}_R = co\{x_1,x_2,\cdots,x_q\}$, where $x_1,x_2,\cdots,x_q$ are given points in R^n a priori. we can optimize a scalar $\alpha>0$ such that

$$\alpha\mathscr{X}_R\subset L_V(1)|_{x_t},\ x_t\in R^n,\ t\in\{1,2,\cdots,q\}$$

it is

$$\alpha^2x_t^TP_sx_t\le 1\Leftrightarrow\begin{bmatrix}\alpha^{-2} & x_t^TG^T \\ Gx_t & GP_s^{-1}G^T\end{bmatrix}\ge 0 \quad (14)$$

noting that

$$GP_s^{-1}G^T\ge G+G^T-P_s,$$

then (14) is satisfied if the the following inequality holds

$$\begin{bmatrix}\alpha^{-2} & x_t^TG^T \\ Gx_t & G+G^T-P_s\end{bmatrix}\ge 0 \quad (15)$$

Thus, the estimation of domain of attraction can be reduced to the following optimization problem:

$$\begin{aligned} &maximize_{P_s>0,H,M,G,\alpha>0} \quad \alpha \\ &s.t. \quad (8),(9),(15) \end{aligned} \quad (16)$$

It is noted that the condition (8) in above optimization problem is not convex and cannot be solved directly. To facilitate solving this non-convex problem, the following useful lemma will be presented.

Lemma 3.1.3: For matrix variables $P_s>0,H,M,G,H_0,M_0,G_0,\forall s,l\in[1,2^m]$, the following statements hold: (8) holds if and only if the following inequality holds

$$\begin{bmatrix}\tilde{\Lambda} & -M^T+AG+BD_sFG & U & U+M^T \\ \star & -G-G^T+P_l+\Theta_2 & G^T & \mathbf{0} \\ \star & \star & -\mathbf{I} & \mathbf{0} \\ \star & \star & \star & -\mathbf{I}\end{bmatrix}<0 \quad (17)$$

where

$$\begin{aligned} U &= BD_s^{-1}H, \quad \Lambda=AM+BD_sFM \\ \tilde{\Lambda} &= \Lambda+\Lambda^T-P_s+2\Theta_1+\Theta_3 \\ \Theta_1 &= -U(BD_s^{-1}H_0)^T-BD_s^{-1}H_0U^T+BD_s^{-1}H_0(BD_s^{-1}H_0)^T \\ \Theta_2 &= -G_0G^T-GG_0^T+G_0G_0^T \\ \Theta_3 &= -M_0M^T-MM_0^T+M_0M_0^T \end{aligned}$$

Proof: Denote

$$W=\begin{bmatrix}\Lambda+\Lambda^T-P_s & -M^T+AG+BD_sFG \\ \star & -G-G^T+P_l\end{bmatrix}$$

then, inequality (8) can be rewritten as

$$W+\begin{bmatrix}UM+(UM)^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{0} & UG \\ \star & \mathbf{0}\end{bmatrix}<0 \quad (18)$$

Obviously, for any two matrices X and Y, the following equality always holds

$$XY+Y^TX^T=(X+Y^T)(X+Y^T)^T-XX^T-Y^TY \quad (19)$$

Using equality (19), we have

$$\begin{aligned}\begin{bmatrix}UM+(UM)^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} &= \begin{bmatrix}U \\ \mathbf{0}\end{bmatrix}\begin{bmatrix}M & \mathbf{0}\end{bmatrix}+\begin{bmatrix}M & \mathbf{0}\end{bmatrix}^T\begin{bmatrix}U \\ \mathbf{0}\end{bmatrix}^T \\ &= \begin{bmatrix}U+M^T \\ \mathbf{0}\end{bmatrix}\begin{bmatrix}U+M^T \\ \mathbf{0}\end{bmatrix}^T+\begin{bmatrix}-UU^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}-M^TM & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\end{aligned} \quad (20)$$

and

$$\begin{aligned}\begin{bmatrix}\mathbf{0} & UG \\ \star & \mathbf{0}\end{bmatrix} &= \begin{bmatrix}U \\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0} & G\end{bmatrix}+\begin{bmatrix}\mathbf{0} & G\end{bmatrix}^T\begin{bmatrix}U \\ \mathbf{0}\end{bmatrix}^T \\ &= \begin{bmatrix}U \\ G^T\end{bmatrix}\begin{bmatrix}U \\ G^T\end{bmatrix}^T+\begin{bmatrix}-UU^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & -G^TG\end{bmatrix}\end{aligned} \quad (21)$$

As is known to all that for any matrix V , there always exists a matrix V_0 such that the following inequality holds

$$(V-V_0)(V-V_0)^T\ge 0 \quad (22)$$

So, there exist matrices H_0; M_0 and G_0 such that (18) holds if the following inequality holds

$$W+\begin{bmatrix}2\Theta_1+\Theta_3 & \mathbf{0} \\ \mathbf{0} & \Theta_2\end{bmatrix}+\begin{bmatrix}U \\ G^T\end{bmatrix}\begin{bmatrix}U \\ G^T\end{bmatrix}^T+\begin{bmatrix}U+M^T \\ \mathbf{0}\end{bmatrix}\begin{bmatrix}U+M^T \\ \mathbf{0}\end{bmatrix}^T<0 \quad (23)$$

On the other hand, when $H_0=H;M_0=M$ and $G_0=G$, then (23) holds if (18) holds. Now, by the Schur complement, (23) is equivalent to (17). So the proof is complete. □

Now, we can solve problem (16) involving the inequalities (9), (15) and (17). But the inequality (17) is not convex with respect to the matrix variables $H_0;G_0$ and M_0, it is difficult to solve them directly. However, when $H_0;G_0$ and M_0 are given, the inequalities (9), (15) and (17) are convex with respect to $P_s>0,H,M,G,H_0,M_0,G_0,\mu,\ \forall s,l\in[1,2^m]$ where $\mu=\alpha^{-2}$, and can be solved by using MATLAB via the LMI Control Toolbox [1]. So, based on this property, we give an iterative LMI-based optimization algorithm in the following to optimize α:

Let $\mu = \alpha^{-2}$, the following algorithm is presented to minimize μ.

Algorithm 3.1.4: Let $\varepsilon > 0$ be a given small constant specifying a convergence criterion.

Step 1: By the result in [3], we can obtain an initial value H_{ini}. Let $H = H_{ini}$ be a given value, optimization problem (16) is converted to a convex problem, then the corresponding feasible solution G, M can be chosen as the initial values G_{ini}, M_{ini}. Let $\mu_0 = 1$, go to Step 2.

Step 2: Let $H_0 = H_{ini}$, $M_0 = M_{ini}$, $G_0 = G_{ini}$, minimize μ subject to the LMIs (9), (15) and (17), H_{opt}, M_{opt}, G_{opt} and μ_{opt} denote the optimized solutions, then go to Step 3.

Step 3: If $|\mu_{opt} - \mu_0| < \varepsilon$, stop, then let $\mu = \mu_{opt}$. Else, let $H_{ini} = H_{opt}$, $M_{ini} = M_{opt}$, $G_{ini} = G_{opt}$ and $\mu_0 = \mu_{opt}$, return to Step 2.

The above algorithm gives a suboptimal estimation α of domain of attraction. Later, in Section 4, we shall illustrate via a numerical example that the above algorithm can provide quite satisfactory result.

B. L2 Performance Analysis

Consider systems

$$\begin{cases} x(k+1) & = Ax(k) + B\sigma(u(k)) + E\omega(k) \\ z(k) & = Cx(k) + D\omega(k) \end{cases} \tag{24}$$

where $z(k) \in R^p$ is an output vector, $\omega(k) \in R^q$ is the disturbance and assume that $\omega \in \mathscr{W} = \{\omega \in R^q : \ \|\omega\|_{l_2}^2 \le \beta\}$, $\beta > 0$ is a given constant. Firstly, for a given $\gamma > 0$, the L_2 performance index is defined as follows:

Definition 3.2.1: Consider the saturated system (24), let $\gamma > 0$ be a given constant, then the system (24) is said to be with a L_2 performance index less than γ, if there exists a saturation-dependent Lyapunov function (6) such that:

i) when $\omega(k) = 0$, $x(k) \neq 0$, $dV/dt < 0$,

ii) when initial condition $x(0) = 0$, the performance index

$$J(\omega) = \sum_{k=0}^{\infty} z^T(k)z(k) - \gamma^2 \sum_{k=0}^{\infty} \omega^T(k)\omega(k) < 0 \tag{25}$$

for all nonzero $\omega(k) \in \mathscr{W}$.

Then a sufficient condition under which system (24) has L_2 performance γ can be stated in the following Theorem:

Theorem 3.2.2: Consider the system (24) under a given state feedback control matrix F. If there exist matrices H, M, G, N and $P_s > 0$, such that $\forall s, l \in [1, 2^m]$

$$\begin{bmatrix} \Theta & -M^T + \Theta_2 & M^T E + \Theta_3 + C^T D \\ \star & -G - G^T + P_l & G^T E - N \\ \star & \star & N^T E + E^T N + D^T D - \gamma^2 \mathbf{I} \end{bmatrix} < 0 \tag{26}$$

$$\begin{bmatrix} 1/\beta & h_j \\ \star & P_s \end{bmatrix} \ge 0 \tag{27}$$

where

$$\begin{aligned} \Theta &= \Theta_1 + \Theta_1^T - P_s + C^T C \\ \Theta_1 &= AM + B(D_s FM + D_s^- HM) \\ \Theta_2 &= AG + B(D_s FG + D_s^- HG) \\ \Theta_3 &= AN + B(D_s FN + D_s^- HN) \end{aligned}$$

then there have $J(\omega) < 0$, i.e.,

$$\sum_{k=0}^{\infty} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} \omega^T(k)\omega(k), \quad \forall w(k) \in \mathscr{W} \tag{28}$$

Proof: Similar to the proof of Theorem 3.1.1, we know the condition (27) guarantees that the considered closed-loop system can be written as

$$x(k+1) = \hat{A}(\eta)x(k) + E\omega(k), \quad \forall\ x \in \mathscr{L}(H) \tag{29}$$

where

$$\hat{A}(\eta) = \sum_{s=1}^{2^m} \eta_s \hat{A}_s = \sum_{s=1}^{2^m} \eta_s (A + B(D_s F + D_s^- H)) \tag{30}$$

Define the same Lyapunov function (6), then, when the system (24) has a L_2 performance index less than γ, there have the modified Lyapunov stability conditions

$$\begin{aligned} &\Delta V(k, x(k)) + z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k) < 0, \quad \forall \omega(k) \in \mathscr{W} \\ &\forall (x(k), x(k+1), \omega(k)) \ \text{ satisfying } \ (24), \\ &(x(k), x(k+1), \omega(k)) \neq 0. \end{aligned} \tag{31}$$

If (31) is feasible, then it follows

$$V(k, x(k)) - V(0, x(0)) < -\sum_{i=0}^{k-1} z^T(i)z(i) + \gamma^2 \sum_{i=0}^{k-1} \omega^T(i)\omega(i), \quad \forall k > 0$$

since $x(0) = 0, V(k, x(k)) > 0$ if $x \neq 0$, we can conclude that the L_2 performance index $J(\omega) < 0$, $\forall\ \omega(k) \in \mathscr{W}, x(k) \in L_V(\beta), k > 0$.

Then (31) can be formulated

$$\begin{bmatrix} x(k) \\ x(k+1) \\ \omega(k) \end{bmatrix}^T \Omega \begin{bmatrix} x(k) \\ x(k+1) \\ \omega(k) \end{bmatrix} < 0$$

$$\forall \ \begin{bmatrix} \hat{A}(\eta(k)) & -\mathbf{I} & E \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \\ \omega(k) \end{bmatrix} = 0, \qquad \begin{bmatrix} x(k) \\ x(k+1) \\ \omega(k) \end{bmatrix} \neq \mathbf{0}$$

where

$$\Omega = \begin{bmatrix} -P(\eta(k)) + C^T C & \mathbf{0} & C^T D \\ \mathbf{0} & P(\eta(k+1)) & \mathbf{0} \\ D^T C & \mathbf{0} & D^T D - \gamma^2 \mathbf{I} \end{bmatrix}$$

Assign

$$x \leftarrow \begin{bmatrix} x(k) \\ x(k+1) \\ \omega(k) \end{bmatrix},$$

$$P \leftarrow \begin{bmatrix} -P(\eta(k)) + C^T C & \mathbf{0} & C^T D \\ \mathbf{0} & P(\eta(k+1)) & \mathbf{0} \\ D^T C & \mathbf{0} & D^T D - \gamma^2 \mathbf{I} \end{bmatrix}$$

$$\Phi \leftarrow \begin{bmatrix} \hat{A}(\eta(k)) & -\mathbf{I} & E \end{bmatrix}, \qquad X \leftarrow \begin{bmatrix} M^T \\ G^T \\ N^T \end{bmatrix}$$

applying Lemma 3.1, it follows

$$\begin{bmatrix} \Gamma_1 & -M^T+\hat{A}^T(\eta(k))G & M^TE+\hat{A}^T(\eta(k))N+C^TD \\ \star & -G^T-G+P(\eta(k+1)) & G^TE-N \\ \star & \star & N^TE+E^TN+D^TD-\gamma^2\mathbf{I} \end{bmatrix} < 0 \tag{32}$$

where

$$\Gamma_1 = M^T\hat{A}(\eta(k))+\hat{A}^T(\eta(k))M-P(\eta(k))+C^TC$$

by transposing $\hat{A}(\eta(k))$, (32) can be rewritten as

$$\sum_{s=1}^{2^m}\eta_s(k)\sum_{l=1}^{2^m}\eta_l(k+1)\Upsilon < 0 \tag{33}$$

where

$$\Upsilon = \begin{bmatrix} \hat{\Gamma}_1 & -M^T+\hat{A}_sG & M^TE+\hat{A}_sN+C^TD \\ \star & -G^T-G+P_l & G^TE-N \\ \star & \star & N^TE+E^TN+D^TD-\gamma^2\mathbf{I} \end{bmatrix}$$

$$\hat{\Gamma}_1 = M^T\hat{A}_s^T+\hat{A}_sM-P_s+C^TC$$

obviously, the above inequality holds if

$$\begin{bmatrix} \hat{\Gamma}_1 & -M^T+\hat{A}_sG & M^TE+\hat{A}_sN+C^TD \\ \star & -G^T-G+P_l & G^TE-N \\ \star & \star & N^TE+E^TN+D^TD-\gamma^2\mathbf{I} \end{bmatrix} < 0, \tag{34}$$

$$\forall s,l \in [1,2^m]$$

via (30), (34) is nothing more than (26). Then the closed-loop system (24) has an L_2 performance index less than γ. So the proof is complete. □

Remark 3.2.3: By setting $M=\mathbf{0}, N=\mathbf{0}$, we can recover a condition which is equivalent to Theorem 1 in [18] while the proof here is more straightforward, Moreover, with no matrix inversion involved in the Lyapunov function.

A natural idea is to optimize the L_2 performance index γ which can be formulated as:

$$\begin{aligned} &minimize_{P_s>0,H,M,G,N,\gamma>0} \quad \gamma \\ &s.t. \quad (26),(27) \end{aligned} \tag{35}$$

Noticing that the condition (26) in above optimization problem is not convex and cannot be solved directly. Similar to the method of dealing with problem (16), a lemma and an optimal algorithm will be stated in the following.

Lemma 3.2.4: For matrix variables $P_s > 0, H, M, G, N, H_0, M_0, G_0, N_0, \forall s,l \in [1,2^m]$, the following statements hold: (26) holds if and only if the following inequality holds

$$\begin{bmatrix} \Lambda & -M^T+\Lambda_2 & M^TE+\Lambda_3 & C^T & U & U & U+M^T \\ \star & -G-G^T+P_l+\Theta_2 & G^TE-N & \mathbf{0} & G^T & \mathbf{0} & \mathbf{0} \\ \star & \star & N^TE+E^TN-\gamma^2\mathbf{I}+\Theta_4 & D^T & \mathbf{0} & N^T & \mathbf{0} \\ \star & \star & \star & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\mathbf{I} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -\mathbf{I} \end{bmatrix} < 0 \tag{36}$$

where

$$\begin{aligned} U &= BD_s^{-1}H, \quad \Lambda_1 = AM+BD_sFM \\ \Lambda_2 &= AG+BD_sFG, \quad \Lambda_3 = AN+BD_sFN \\ \Lambda &= \Lambda_1+\Lambda_1^T-P_s+3\Theta_1+\Theta_3 \\ \Theta_1 &= -U(BD_s^{-1}H_0)^T-BD_s^{-1}H_0U^T+BD_s^{-1}H_0(BD_s^{-1}H_0)^T \\ \Theta_2 &= -G_0G^T-GG_0^T+G_0G_0^T \\ \Theta_3 &= -M_0M^T-MM_0^T+M_0M_0^T \\ \Theta_4 &= -N_0^TN-N^TN_0+N_0^TN_0 \end{aligned}$$

Algorithm 3.2.5: Let $\rho=\gamma^2$, $\varepsilon>0$ be a given small constant specifying a convergence criterion.

Step 1: Via the result in [18], we can obtain an initial value H_{ini}. Let $H=H_{ini}$ be a given value, optimization problem (35) is converted to a convex problem, then the corresponding feasible solution G,M,N can be chosen as the initial values G_{ini}, M_{ini}, N_{ini}. Let $\rho_0=1$, go to Step 2.

Step 2: Let $H_0=H_{ini}$, $M_0=M_{ini}$, $G_0=G_{ini}$, $N_0=N_{ini}$, minimize ρ subject to the LMIs (27) and (36), H_{opt}, M_{opt}, G_{opt} N_{opt} and ρ_{opt} denote the optimized solutions, then go to Step 3.

Step 3: If $|\rho_{opt}-\rho_0|<\varepsilon$, stop, then let $\rho=\rho_{opt}$. Else, let $H_{ini}=H_{opt}$, $M_{ini}=M_{opt}$, $G_{ini}=G_{opt}$, $N_{ini}=N_{opt}$ and $\rho_0=\rho_{opt}$, return to Step 2.

The above algorithm gives a suboptimal L_2 performance index γ. A numerical example will be provided to show the above algorithm can provide more improved result than one on the same problem in Section 4.

Remark 3.2.6: By viewing the feedback gain F as an additional free parameter, based on Theorem 3.1.1 and Theorem 3.2.2, both the corresponding Lemmas and algorithms can be presented easily for controller design.

IV. Numerical Example

Several numerical examples borrowed from the literature are now presented to illustrate the effectiveness of proposed approaches.

Example 1: Let us consider the same system in [3] to estimate its domain of attraction

$$A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \quad B=\begin{bmatrix}0.5\\ 1.0\end{bmatrix}.$$

Design the state feedback control law by the LQR approach with $Q=\mathbf{I}$ and $R=0.1$. For the above system, we obtain the following controller: $F=[-0.6167 \quad -1.2703]$. As in [3], we use the shape reference set of the form $\mathscr{X}_R = \left\{\begin{bmatrix}\sin\theta\\ \cos\theta\end{bmatrix}\right\}$, $\theta\in[0,2\pi]$. For this example, when $\theta = 0.4\pi$, by means of the result in [3], the optimized $\alpha = 4.5235$,

$H_{ini} = [-0.1019 \quad -0.2929]$, let $H = H_{ini}$ in problem (16), we obtain

$$M_{ini} = \begin{bmatrix} -0.36087 & 1.0489 \\ 0.3576 & -1.2615 \end{bmatrix}, \quad G_{ini} = \begin{bmatrix} 0.2291 & -0.4405 \\ -0.6433 & 1.5335 \end{bmatrix}$$

In contrast, by Algorithm 3.1.4 with $H_0 = H_{ini}, G_0 = G_{ini}, M_0 = M_{ini}$, $\varepsilon = 0.00001$ and $\mu_0 = 1$, after 5 iterations, the optimized solutions obtained as $\alpha_{opt} = 363.5185$.

$$H_{opt} = [0.3251 \quad -1.0032];$$

$$M_{opt} = \begin{bmatrix} -0.3717 & 1.1507 \\ 0.6449 & -1.9994 \end{bmatrix}, \quad G_{opt} = \begin{bmatrix} 0.3972 & -1.2273 \\ -1.2291 & 3.7979 \end{bmatrix}$$

From the numerical value results, we can see that the result is quite satisfied because it is much better than the one in [3].

Example 2: The following example is borrowed from [18]. Considering

$$A = \begin{bmatrix} 0 & 1 \\ -0.58 & -0.6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0.$$

let $\beta = 10$, we obtain the best L_2 performance $\gamma = 1.6564$ with state feedback gain $F = [0.5801 \quad 0.3510]$ via the Corollary 2 in [18]. In addition, $H_{ini} = [0.2440 \;\; 0.1362]$ can also be obtained, by solving the convex optimization problem (35) under $H = H_{ini}$, there have

$$M_{ini} = \begin{bmatrix} 4.9286 & 0.5089 \\ -3.7523 & 1.7105 \end{bmatrix}, \quad G_{ini} = \begin{bmatrix} 27.9442 & -7.3413 \\ -7.2827 & 8.2145 \end{bmatrix}$$

$$N_{ini} = \begin{bmatrix} 7.1237 \\ -8.1095 \end{bmatrix}$$

Whereafter, via Algorithm 3.2.5 under $H_0 = H_{ini}, G_0 = G_{ini}, M_0 = M_{ini}, N_0 = N_{ini}$ $\varepsilon = 0.001$ and $\rho_0 = 1$, after 1 iteration, the optimized solutions obtained as $\gamma_{opt} = 9.5742e-007$.

$$H_{opt} = [0.5026 \quad -0.1219];$$

$$M_{opt} = \begin{bmatrix} 4.5452 & 1.7948 \\ -4.0507 & 1.4097 \end{bmatrix}, \quad G_{opt} = \begin{bmatrix} 29.1520 & -7.1526 \\ -7.6788 & 8.9061 \end{bmatrix}$$

$$N_{opt} = \begin{bmatrix} 7.3825 \\ -8.1397 \end{bmatrix}$$

From the numerical value results, the proposed method in this paper improve recent results in [18] very much, that is, the result is very satisfactory.

V. Conclusion

This paper studies performance analysis problem for discrete-time linear systems with input saturations by combining the saturation-dependent Lyapunov function method with Finsler's Lemma. Two performance measures, the estimation of domain of attraction and the L_2 performance, are concerned in this paper. The used method is conceptually simple. Here, difference equations are considered as constraints and these dynamical constraints are incorporated into the stability analysis condition through the use of matrix Lagrange multipliers. New and less conservative conditions in the enlarged space containing both the state and its time difference, allowing extra degree of freedom for various performance analysis, are proposed. Furthermore, based on these results, two important lemmas and two iterative LMI-based optimization algorithms are also developed to optimize the performance indexes respectively. Numerical examples illustrate that the proposed methods improve recent results on the same problems.

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