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Stabilization with Decay Rate Analysis for Discrete-time Linear Systems Subject to Actuator Saturation

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Abstract— **This paper addresses stabilization problem with decay rate analysis for discrete-time linear systems subject to actuator saturation. The saturation-dependent Lyapunov function is exploited to propose new stability conditions by introducing additional slack variables. Especially, Elimination Lemma is used to show the stable property of one slack variable. If the stable slack variable is specified a priori by a systematic and simple approach, via a cone complementarity approach , a state feedback controller is then designed by using LMI-based optimization algorithm which guarantees an upper bound on the decay rate of the system. The simulation results illustrate the effectiveness of the proposed methods.**

Key words: Actuator saturation; decay rate; cone complementarity approach; LMIs.

I. INTRODUCTION

Saturation is probably the most widely encountered and most dangerous nonlinearity in control systems. It is well recognized that actuator saturation degrades the performance of the control system and may even lead to instability. The destabilizing effects of actuator saturation have been cited as contributing factors in several mishaps involving high performance aircraft [4]. As a result, actuator saturation has received increasing attention from research community (see, for example, [2], [9], [12], [23] and the references therein).

Many approaches have been developed to dealing with actuator saturation in the existing literature: for example, nested feedback design technique [21]; low-and-high gain method [19]; maximal output admissible sets approach [8]; anti-windup [13], [14], [20]; invariant subspace technique [13] and multiplier theory [14], etc. More recently, one of the most relevant approaches to the analysis of saturated systems is based on a novel polytopic model of the saturation nonlinearity which was proposed in [9]. Based on that, several interesting results have been reported by developing various Lyapunov function. For example, quadratic Lyapunov function [5], [10], [25]; Piecewise-affine Lyapunov function [15]; saturation-dependent Lyapunov function or parameter-dependent Lyapunov function(for linear discrete-time system) [3], [24]; convex hull quadratic Lyapunov function and max quadratic Lyapunov function (for linear continuous-time system) [11]. The advantages of using the polytopic model have been shown in [5], [22], etc. Moreover, several performance indexes, such as the estimation of domain of attraction, disturbance tolerance and L_2 performances, have also been studied for actuator saturation systems, see, for example, [3], [5], [16] and [24]. However, from best of our knowledge, as an important performance index, decay rate has not been discussed for discrete-time linear system with actuator saturation in all the references.

The objective of this paper is to study stabilization problem with decay rate analysis for discrete-time linear systems subject to actuator saturation. The saturation-dependent Lyapunov function is exploited to propose new stability conditions by introducing additional slack variables. Especially, Elimination Lemma is used to show the stable property of one slack variable. If the stable slack variable is specified a priori by a systematic and simple approach, via a cone complementarity algorithm involving convex optimization, a state feedback control law is then designed by utilizing LMI-based approach which guarantees an upper bound on the decay rate of the system. The effectiveness of the proposed methods is illustrated by a numerical example.

The rest of this paper is organized as follows. Section 2 introduces the problem under consideration and some preliminary results. It is followed by stability analysis with a certain decay rate in Section 3. Section 4 presents stabilization problem via state feedback, a numerical example and its simulation results are given to show the effectiveness of the proposed methods in Section 5. Conclusions are made in Section 6.

Notation: $\lambda(M)$ stands for the eigenvalue of a matrix M. M^T is the transpose of the matrix M. $M>0 (M<0)$ means that M is positive definite (negative definite). $\star$ denotes the transpose of the off diagonal element of a matrix. $\mathbf{I}$ represents the identity matrix of appropriate dimension. $\mathbf{0}$ depicts the zero matrix of appropriate dimension. For a square matrix M, $He\{M\} = M + M^T$. Denote $L_V(1) = \{x \in R^n \mid V(x) \leq 1\}$ as the level set of a Lyapunov function $V(x)$. For a matrix $F \in R^{m\times n}$, denote the ith row as f_i and define $\mathscr{L}(F) = \{x \in R^n \mid \ |f_i x| \leq 1, \ \ 1 \leq i \leq m\}$.

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II. Problem Statement and Preliminaries

Consider a discrete-time linear system subject to input saturation

$$x(k+1) = Ax(k) + Bsat(u(k)) \quad (1)$$

where $x \in R^n$ denotes the state vector and $u \in R^m$ is the control input vector. $sat(\cdot)$ is the standard saturation function. It is assumed here to be normalized, i.e., $sat(u(k)) = sign(u(k))\min\{1, |u(k)|\}$.

Consider a constant state feedback law of the form:

$$u(k) = Fx(k) \quad (2)$$

The objective of this paper is to stabilize (1) with a certain decay rate by designing the state feedback control law (2).

The following preliminaries will be used in the sequel.

Let $\aleph$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\aleph$. Suppose that each element of $\aleph$ is labeled as D_s, $s = 0, 1, \cdots, 2^m - 1$, and denote $D_s^- = \mathbf{I} - D_s$. Clearly, D_s^- is also an element of $\aleph$ if $D_s \in \aleph$.

Lemma 2.1: [10] Let $F, H \in R^{m\times n}$ be given. For an $x \in R^n$, if $x \in \mathscr{L}(H)$, then there exist $\eta_s \geq 0$, $s \in [0, 2^m - 1]$ satisfying $\sum_{s=0}^{2^m-1} \eta_s = 1$ such that

$$sat(Fx) = \sum_{s=0}^{2^m-1} \eta_s (D_s F + D_s^- H)x. \quad (3)$$

We note that the parameters η_s in (3) are functions of the state x.

Lemma 2.2: (*Elimination Lemma*)[17] Let $x \in R^n$, symmetric matrix $P \in R^{n\times n}$, and $Q \in R^{m\times n}$ such that $rank(Q) = r < n$. Then the following statements are equivalent:

i) $x^T P x < 0$, $\forall\ Qx = 0, x \neq 0$.

ii) $\exists\ \Xi \in R^{n\times m}$: $P + \Xi Q + Q^T \Xi^T < 0$.

III. Stability Analysis

Given a state feedback control matrix F in (2), by means of Lemma 2.1, then the closed-loop system (1-2) can be rewritten as

$$x(k+1) = A(\eta(k))x(k), \quad \forall\ x \in \mathscr{L}(H) \quad (4)$$

where

$$A(\eta(k)) = \sum_{s=0}^{2^m-1} \eta_s(k) A_s = \sum_{s=0}^{2^m-1} \eta_s(k)(A + B(D_s F + D_s^- H)) \quad (5)$$

and $\eta = [\eta_0\ \eta_1\ \cdots\ \eta_{2^m-1}]$ is a function of x that satisfies (3). Here and later in this paper, we use $\eta_s(k)$ to denote $\eta_s(x_k)$.

Now, an extended stability condition based on the saturation-dependent Lyapunov function [3] is derived in the following. The main idea was originally proposed in [1] for the study of robust analysis and synthesis of linear polytopic discrete-time periodic systems.

Theorem 3.1: If there exist symmetric matrices $0 < P_s \in R^{n\times n}$, matrices $H \in R^{m\times n}$ and $\Lambda \in R^{n\times 2n}$ such that $\forall s, l \in [0, 2^m - 1]$

$$\begin{bmatrix} -P_s & \mathbf{0} \\ \mathbf{0} & P_l \end{bmatrix} + He\left\{ \begin{bmatrix} A + B(D_s F + D_s^- H) \\ -\mathbf{I} \end{bmatrix} \Lambda \right\} < 0 \quad (6)$$

and $L_V(1) \subset \mathscr{L}(H)$ hold, then (4) is asymptotically stable at the origin in level set $L_V(1)$.

Proof: Consider the following saturation-dependent Lyapunov function [3]:

$$V(k, x(k)) = x(k)^T P(\eta(x(k)))x(k) = x(k)^T \left(\sum_{s=0}^{2^m-1} \eta_s(x(k)) P_s\right) x(k), \quad (7)$$

where $P_s > 0$.

If above positive-definite Lyapunov function (7) exists and

$$\begin{aligned} \Delta V(k, x(k)) &= V(k+1, x(k+1)) - V(k, x(k)) \\ &= x(k)^T [A^T(\eta(k)) P(\eta(k+1)) A(\eta(k)) \\ &\quad - P(\eta(k))] x(k) \end{aligned} \quad (8)$$

is negative definite along the trajectories of system (4), then the origin of the system (4) is asymptotically stable $\forall\ x_0 \in L_V(1) \subset \mathscr{L}(H)$.

Noting that $\forall x \in L_V(1) \subset \mathscr{L}(H)$, system (1-2) can be written as (4), then it is clear $\Delta V(k, x(k)) < 0$ for any $x(k) \in L_V(1) \setminus \{0\}$ if

$$A^T(\eta(k)) P(\eta(k+1)) A(\eta(k)) - P(\eta(k)) < 0$$

Now, multiplying each inequality in (6) by $\eta_s(k)$, $\eta_l(k+1)$ respectively and summing them up for s, $l = 0, 1, \cdots, 2^m - 1$ respectively, there have

$$\begin{bmatrix} -P(\eta(k)) & \mathbf{0} \\ \mathbf{0} & P(\eta(k+1)) \end{bmatrix} + He\left\{ \begin{bmatrix} A(\eta(k)) \\ -\mathbf{I} \end{bmatrix} \Lambda \right\} < 0 \quad (9)$$

Pre-multiplying (9) by

$$[\mathbf{I}\ \ A(\eta(k))]$$

and post-multiplying it by the transpose of above matrix leads to

$$-P(\eta(k)) + A(\eta(k)) P(\eta(k+1)) A^T(\eta(k)) < 0 \quad (10)$$

since $P_s > 0$, it is obvious that the inequality (10) implies

$$\Delta V(k, x(k)) < 0, \quad \forall x \neq 0$$

So the system (4) is asymptotically stable at the origin in level set $L_V(1)$. The proof is complete. □

Remark 3.1: The matrix Λ plays the role of Lagrangian relaxation variables. The main reason to introduce slack variables consists in the decoupling between the Lyapunov matrix P_s and the system A matrix. This leads to the applicability to controller design.

Remark 3.2: According to the dimensions, the matrix Λ may be partitioned as $\Lambda = [N\quad G]$ where $N, G \in R^{n\times n}$, Noting that the bloc(2,2) in (6) requires G to be invertible. It is therefore possible to get the following reformation of Theorem 3.1.

Theorem 3.2: If there exist symmetric matrices $0 < P_s \in R^{n\times n}$ and matrices $H \in R^{m\times n}$, $G, A_0 \in R^{n\times n}$ such that

$$\begin{bmatrix} -P_s & \mathbf{0} \\ \mathbf{0} & P_l \end{bmatrix}$$

$$+He\left\{ \begin{bmatrix} A+B(D_sF+D_s^-H) \\ -\mathbf{I} \end{bmatrix}(-G)[A_0 \quad -\mathbf{I}] \right\} < 0 \tag{11}$$

hold for all $s,l \in [0,2^m-1]$ and $L_V(1) \subset \mathscr{L}(H)$, then (4) is asymptotically stable at the origin in level set $L_V(1)$.

This reformulation is *interesting* because of the interpretation of the extra variable A_0. Now let x represents each row of X and

$$P = \begin{bmatrix} -P_s & \mathbf{0} \\ \mathbf{0} & P_l \end{bmatrix}; \quad \Xi = \begin{bmatrix} A+B(D_sF+D_s^-H) \\ -\mathbf{I} \end{bmatrix}(-G); \tag{12}$$

$$X = \begin{bmatrix} \mathbf{I} \\ A_0 \end{bmatrix}; \quad Q = [A_0 \quad -\mathbf{I}]$$

then applying *Elimination Lemma* to (11), there have

$$\begin{bmatrix} \mathbf{I} \\ A_0 \end{bmatrix}^T \begin{bmatrix} -P_s & \mathbf{0} \\ \mathbf{0} & P_l \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ A_0 \end{bmatrix} < 0 \quad \forall\, s,l \in [0,2^m-1] \tag{13}$$

that is

$$A_0^T P_l A_0 - P_s < 0 \quad \forall\, s,l \in [0,2^m-1] \tag{14}$$

inequality (14) means that the matrix solution A_0 of (11) must be stable. In particular, a possible choice is $A_0 = \mathbf{0}$ for all $k \geq 0$. This leads to

Corollary 3.1: If there exist symmetric matrices $0 < P_s \in R^{n\times n}$ and matrices $H \in R^{m\times n}$, $G \in R^{n\times n}$ solutions of the following constraints:

$$\begin{bmatrix} -P_s & \mathbf{0} \\ \mathbf{0} & P_l \end{bmatrix} + He\left\{ \begin{bmatrix} A+B(D_sF+D_s^-H) \\ -\mathbf{I} \end{bmatrix}[\mathbf{0} \quad G] \right\} < 0 \tag{15}$$

for all $s,l \in [0,2^m-1]$ and $L_V(1) \subset \mathscr{L}(H)$, then (4) is asymptotically stable at the origin in level set $L_V(1)$.

Remark 3.3: In fact, Corollary 3.1 is equivalent to Theorem 1 in [3] while the proof here is more straightforward. Imposing the above structure $\Lambda = [\mathbf{0} \quad G]$ surely introduces additional conservation for the condition (15) with respect to (6) or (11) but does not prevent (15) to be always better than the quadratic stability one which has been shown while the estimation problem of the domain of attraction is concerned [3]. The hierarchy of stability test is therefore the following.

$$Quadratic \quad stability \Rightarrow (15) \Rightarrow (6) or (11) \tag{16}$$

Furthermore, one of the advantages of above results is to allow us to formulate an LMI-based optimization problem to find a sharp estimate of exponential convergence rate of (4) as illustrated in the next Section. The latter part in (16) will be demonstrated again by a numerical example in Section 5.

IV. Stabilization with Decay Rate Analysis

From Section 3, the asymptotical stability of a linear system subject to actuator saturation can be checked with the feasibility tests (6) or (11) or (15). Indeed, we can say more: asymptotical stability of system (4) implies that it is also exponentially stable about the origin. Namely, $\exists \kappa > 0$ and $0 \leq \xi \leq 1$, such that $\forall\, x(0) \in L_V(1), \quad \forall\, k > 0$

$$\|x(k)\| \leq \kappa \cdot \xi^k \|x(0)\| \tag{17}$$

In this section, we will show how it is possible to compute the gain F of stabilizing state feedback controller (2) such that (1) (or (4)) is exponentially stabilized with a certain decay rate. Before the main results are given, the following well-known fact on exponential stability is needed.

Lemma 4.1:[18], For system (4), The sequence $x(k)$ is exponentially stable about the origin if there exists a Lyapunov function $V(k,x(k))$ such that $\forall k \geq 0$

$$\eta\|x(k)\|^2 \leq V(k,x(k)) \leq \rho\|x(k)\|^2 \tag{18}$$

$$V(k+1,x(k+1)) - V(k,x(k)) \leq -\nu\|x(k)\|^2 \tag{19}$$

$\eta,\rho,\nu > 0$. Then $\|x(k)\| \leq \kappa \cdot \xi^k\|x(0)\|$, where $\kappa^2 = \rho/\eta$ and $\xi^2 = 1-\nu/\rho$.

Then sufficient conditions on designing a stabilizing state feedback controller with a certain decay rate are stated in the following.

Theorem 4.1: If there exist $\nu > 0, \eta > 0, \rho > 0, P_s > 0, Q_s > 0, G, Y, Z$, fulfilling the following constraints: $\forall s,l \in [0,2^m-1]$

$$\eta\mathbf{I} \leq P_s \leq \rho\mathbf{I}, \tag{20}$$

$$\begin{bmatrix} -P_s+\nu\mathbf{I} & AG+B(D_sY+D_s^-Z) \\ \star & -G-G^T+P_l \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} 1 & z_j \\ \star & G^T+G-Q_s \end{bmatrix} \geq 0, \tag{22}$$

$$P_sQ_s = \mathbf{I}. \tag{23}$$

then the state feedback controller given by (2) with

$$F = YG^{-1} \tag{24}$$

exponentially stabilizes the system (1) $\forall x_0 \in L_V(1)$. The decay rate of the system is given by $\xi = (1-\nu/\rho)^{1/2}$.

Proof: Noting that satisfying (21) implies $-G-G^T+P_l < 0$ and matrix G is non-singular. Hence the following feedback gain $F = YG^{-1}$ is always available whenever (21) is feasible.

Since

$$\begin{bmatrix} 1 & h_j \\ \star & P_s \end{bmatrix} \geq 0 \Leftrightarrow L_V(1) \subset \mathscr{L}(H)$$

$\forall x_0 \in L_V(1)$, system (1-2) can be written as (4) if

$$\begin{bmatrix} 1 & h_j \\ \star & P_s \end{bmatrix} \geq 0 \;\forall\; j \in [1,m] \tag{25}$$

holds. Now, let $HG = Z$, pre-multiplying (25) by $diag\{\mathbf{I},\ G^T\}$ and post-multiplying it by its transpose leads to

$$\begin{bmatrix} 1 & z_j \\ \star & G^TP_sG \end{bmatrix} \geq 0 \tag{26}$$

since $P_s > 0$, it follows

$$(G-P_s^{-1})^T P_s (G-P_s^{-1}) \geq 0$$

it is equivalent to

$$G^TP_sG \geq G^T+G-P_s^{-1}$$

thus, (26) holds if the following inequality holds

$$\begin{bmatrix} 1 & z_j \\ \star & G^T + G - P_s^{-1} \end{bmatrix} \geq 0$$

By means of (23), the above inequality is nothing more than (22). That is to say, inequalities (22-23) guarantee that system (1-2) can be rewritten as (4) $\forall x_0 \in L_V(1)$.

In addition, it is not difficult to conclude that the Lyapunov function (7) is positive definite, decrescent, and radially unbounded since

i. $V(k,0) = 0,\ \forall k > 0$

ii. $\forall x(k) \in R^n$, if we choose positive scalars $\eta = \min_{s\in[0,2^m-1]} \lambda_{\min}(P_s)$ and $\rho = \max_{s\in[0,2^m-1]} \lambda_{\max}(P_s)$, there have

$$\eta \|x(k)\|^2 \leq V(k,x(k)) \leq \rho \|x(k)\|^2 \tag{27}$$

iii. if let $\nu = \min_{s,l\in[0,2^m-1]} \lambda_{\min}(P_s - (A + B(D_sF + D_s^- H))^T P_l (A + B(D_sF + D_s^- H))) < \rho$, then

$$\Delta V(k,x(k)) \leq -\nu \|x(k)\|^2 \tag{28}$$

With the above observations, the exponential stability of (1-2) (or (4)) about the origin immediately follows from Lemma 4.1. Furthermore, by means of Corollary 3.1, the asymptotical stability with the decay rate ξ for the system (1-2) (or (4)) can be obtained if the following constraints are satisfied: $\forall\ s,l \in [0,2^m-1]$

$$\begin{aligned} &(a). && \eta \mathbf{I} \leq P_s \leq \rho \mathbf{I} \\ &(b). && \begin{bmatrix} -P_s + \nu \mathbf{I} & \mathbf{0} \\ \mathbf{0} & P_l \end{bmatrix} \\ & && + He\left\{ \begin{bmatrix} A + B(D_sF + D_s^- H) \\ -\mathbf{I} \end{bmatrix} [\mathbf{0} \quad G] \right\} < 0 \\ &(c). && \begin{bmatrix} 1 & z_j \\ \star & G^T + G - Q_s \end{bmatrix} \geq 0 \\ &(d). && P_s Q_s = \mathbf{I}., \quad \nu > 0, \quad \eta > 0, \quad \rho > 0, \end{aligned} \tag{29}$$

where $\xi = (1-\nu/\rho)^{1/2}$. Now via $FG = Y, HG = Z$, it is clear that (29) is nothing more than the conditions in Theorem 4.1. So the proof is completed. □

Remark 4.1: In the case of verifying exponential stability of (4), it may be desirable not only find feasible solutions to (20-23) but also to search for solutions that give an estimate of the decay rate ξ. To this end, we can normalize ρ to 1, then the estimation problem of ξ can be converted to the estimation problem of ν easily.

Remark 4.2: It is noted that the resulting conditions in Theorem 4.1 are no more LMI conditions due to (23). However, with the result of [7], we can solve this non-convex feasibility problem by formulating it into an optimization problem subject to LMI constraints. In order to deal $P_s^{-1} = Q_s$, using a cone complementarity approach [7], we suggest the following nonlinear minimization problem involving LMI conditions instead of the original non-convex feasibility problem formulated in Theorem 4.1.

$$\begin{cases} \min Tr(\sum_{s=0}^{2^m-1} P_s Q_s) \quad s.\ t. \\ (20-22)\ and\ \begin{bmatrix} P_s & \mathbf{I} \\ \mathbf{I} & Q_s \end{bmatrix} \geq 0, \end{cases} \tag{30}$$

If the solution of the above minimization problem is $2^m n$ (n is the dimension of $x(k)$), that is $Tr(\sum_{s=0}^{2^m-1} P_s Q_s) = 2^m n$, then the conditions in Theorem 4.1 are solvable. Although it is still impossible to always find the global optimal solution, the proposed nonlinear minimization problem is easier to solve than the original non-convex feasibility problem. Actually, we can readily modify Algorithm 1 in [7] to solve the above problems. For similar algorithms, see also [6], etc.

Algorithm 4.1:

Step 1: Let $\rho = 1$, choose a sufficiently small initial $\nu_{ini} > 0$ such that there exists a feasible solution to (20-22) and (29). Set $\nu = \nu_{ini}$.

Step 2: Find a feasible set $(P_s, Q_s, Y, Z, G, \eta)^0$ satisfying (23-25) and (30). Set $k = 0$.

Step 3: Solve the following LMI problem

$$\min Tr(\sum_{s=0}^{2^m-1} P_s Q_s) \quad s.\ t. \quad (22-22)\ and\ (30)$$

Set $P_s^{k+1} = P_s, Q_s^{k+1} = Q_s$, $\forall s \in [0,2^m-1]$.

Step 4: If the condition (23) is satisfied, then set $\nu = \nu_{ini}$ and return to Step 2 after increasing ν_{ini} to some extent. If the condition (23) is not satisfied within a specified number of iterations, say k_{max}, then exit. Otherwise, set $k = k+1$ and go to Step 3.

The above algorithm gives a suboptimal decay rate bound $\xi = (1-\nu)^{\frac{1}{2}}$ such that the system (1) can be stabilized. Later, in Section 5, we shall illustrate via a numerical example that the above algorithm can provide satisfactory results.

The sufficient conditions (20-23) of exponentially stabilizability are defined for $A_0 = \mathbf{0}$ for all $k \geq 0$. By means of (11) in Theorem 3.2, for different choices of stable matrices A_0, different sufficient conditions of exponentially stabilizability via state-feedback may be given.

Theorem 4.2: For a given stable matrix A_0, if there exist symmetric matrices $0 < P_s, 0 < Q_s$, matrices G, Y, Z and positive scalar η, ν fulfilling the following constraints:

$$\eta \mathbf{I} \leq P_s \leq \mathbf{I}, \tag{31}$$

$$\begin{bmatrix} \Theta & AG + BD_sY + BD_s^- Z + A_0^T G^T \\ \star & -G - G^T + P_l \end{bmatrix} < 0, \tag{32}$$

$$\begin{aligned} \Theta \ &= -AGA_0 - B(D_sY + D_s^- HG)A_0 - A_0^T G^T A^T \\ &\quad -A_0^T (D_sY + D_s^- HG)^T B^T - P_s + \nu \mathbf{I} \end{aligned}$$

$$\begin{bmatrix} 1 & z_j \\ \star & G^T + G - Q_s \end{bmatrix} \geq 0, \tag{33}$$

$$P_s Q_s = \mathbf{I}. \tag{34}$$

for all $s,l \in [0,2^m-1]$. then the state feedback controller given by (2) with

$$F = YG^{-1} \tag{35}$$

exponentially stabilizes the system (4) $\forall x_0 \in L_V(1)$. The decay rate of the system is given by $\xi = (1-\nu)^{1/2}$.

Proof: let $FG = Y,\ HG = Z$, via the result in Theorem 3.2, the proof is similar to the one of Theorem 4.1. So it is omitted here. □

Remark 4.3: Similar to Remark 4.2, in order to deal with $P_s^{-1} = Q_s$, the following nonlinear minimization problem involving LMI conditions and Algorithm 4.2 are proposed:

$$\begin{cases} \min Tr(\sum_{s=0}^{2^m-1} P_s Q_s) \ s.\ t. \\ (31-33) \ and \ \begin{bmatrix} P_s & \mathbf{I} \\ \mathbf{I} & Q_s \end{bmatrix} \geq 0, \end{cases} \quad (36)$$

Algorithm 4.2:

Step 1: Choose a stable matrix A_0 and a sufficiently small initial $\nu_{ini} > 0$ such that there exists a feasible solution to (31-33) and (36). Set $\nu = \nu_{ini}$.

Step 2: Find a feasible set $(P_s, Q_s, Y, Z, G, \eta)^0$ satisfying (31-33) and (36). Set $k = 0$.

Step 3: Solve the following LMI problem

$$\min Tr(\sum_{s=0}^{2^m-1} P_s Q_s) \quad s.\ t. \quad (31-33) \ and \ (36)$$

Set $P_s^{k+1} = P_s, Q_s^{k+1} = Q_s$, $\forall s \in [0, 2^m - 1]$.

Step 4: If the condition (34) is satisfied, then set $\nu = \nu_{ini}$ and return to Step 2 after increasing ν_{ini} to some extent or rechoosing stable matrix A_0. If the condition (34) is not satisfied within a specified number of iterations, say k_{max}, then exit. Otherwise, set $k = k+1$ and go to Step 3.

The above algorithm also gives a suboptimal decay rate bound $\xi = (1-\nu)^{\frac{1}{2}}$ such that the system (1) can be stabilized. We shall illustrate, via a numerical example, that the above algorithm 4.2 can provide more satisfactory results than algorithm 4.1 by choosing A_0 appropriately in Section 5, that is, the latter part in (16) is verified.

Remark 4.4: The main drawback of the algorithm 4.2 comes from the difficulty to choose A_0 appropriately. One simple and systematic way is to specialize A_0 as $\mu * \mathbf{I}, -1 < \mu < 1, \mu \neq 0$. As will be shown by a numerical example in next Section, for a very simple choice of A_0, better results can be obtained.

V. Numerical Example

The following example is borrowed from [3]. Considering the system (1) with the following coefficient matrices.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}.$$

By proposed method, the following satisfied results can be obtained:

	Algorithm	A_0	decay rate ξ	state feedback gain F
case 1	4.1	-	0.9230	$[-0.3900 \quad -1.1950]$
case 2	4.2	$0.15 * \mathbf{I}$	0.8602	$[-0.4721 \quad -1.2346]$
case 3	4.2	$-0.45 * \mathbf{I}$	0.6633	$[-0.4540 \quad -1.1655]$

TABLE I

Simulation Results

Furthermore,

case 1 :

$$P_1 = P_2 = \begin{bmatrix} 0.9586 & -0.0925 \\ -0.0925 & 0.7932 \end{bmatrix},$$

$$Q_1 = Q_2 = \begin{bmatrix} 1.0550 & 0.1230 \\ 0.1230 & 1.2751 \end{bmatrix}.$$

case 2 :

$$P_1 = P_2 = \begin{bmatrix} 1.0000 & -0.0000 \\ -0.0000 & 1.0000 \end{bmatrix},$$

$$Q_1 = Q_2 = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}.$$

case 3 :

$$P_1 = P_2 = \begin{bmatrix} 0.9894 & -0.0180 \\ -0.0180 & 0.9692 \end{bmatrix},$$

$$Q_1 = Q_2 = \begin{bmatrix} 1.0110 & 0.0188 \\ 0.0188 & 1.0321 \end{bmatrix}.$$

Simulation results verify that the constraints $P_s Q_s = \mathbf{I}$. In addition, by means of $\sum_{s=0}^{2^m-1} \eta_s = 1$, an interesting phenomenon is found from the simulation results: the saturation-dependent Lyapunov function is equivalent to a quadratic Lyapunov function in this example.

From Table 1, it can be seen that Algorithm 4.1 is effective and Algorithm 4.2 can provide much better decay rate bound ξ which justifies that the conditions of Theorem 4.2 are less conservative than that of Theorem 4.1 by choosing A_0 appropriately, i.e., the latter part in (16) is demonstrated again. So the Algorithm 4.2 gives an alternative approach for better decay rate bound.

VI. Conclusion

This paper addresses stability analysis and controller design problems with a certain decay rate bound for discrete-time linear systems subject to actuator saturation. The saturation-dependent Lyapunov function is exploited to develop new stability conditions by introducing additional slack variables. Especially, Elimination Lemma is utilized to show the stable property of one slack variable. If the stable slack variable is specified a priori by a systematic and simple approach, the state feedback controller can be designed, via two algorithms involving convex optimization respectively, which guarantees a certain upper bound on the decay rate of the system. The results are reduced to LMI-based optimization problems. The effectiveness of the proposed methods is illustrated by a numerical example.

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