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Closed-Form Solution of Pure Proportional Navigation with Tracker-in-Loop

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Abstract- **The analytical closed form solution of the equations of motion of a missile tracking a non-maneuvering target using the Pure Proportional Navigation (PPN) law based on an estimate of line-of-sight (LOS) rate is presented. The solution is obtained in the form of a uniformly convergent series of functions in polar coordinates for navigation constants $N \geq 2$. The effect of error bounds of the estimator on the closed loop solution is presented, in line with the approach of Becker [1].**

Keywords: Closed-form solution, PPN, Guidance, Estimator

I. INTRODUCTION

The Proportional Navigation (PN) guidance law has been widely reported in the literature [1-5, 12]. Closed form solutions of two variants of the PN guidance law have been reported in literature for an ideal missile pursuing a non-maneuvering target. The closed form solution of the True Proportional Navigation (TPN) law for an ideal missile has been derived by Guelman [2] and then generalized by Yang et al [3] for an arbitrary angle between missile acceleration and LOS. The corresponding solution for PPN has been derived for particular cases by Wang et al [4] and in more generalized form by Becker [1]. Most of the works reported in the literature focus on the capturability aspect of the PN law and the corresponding derivations from the equations of motion [2, 5].

Interception of aerospace targets using Proportional Navigation (PN) guidance law needs line-of-sight (LOS) rate and closing velocity information between the target and the interceptor [1-7, 12]. Interestingly, all of them assume a perfect feedback of the LOS rate and closing velocity to the guidance law, free of lag and noise. The onboard seeker which acts as the guidance sensor is known to give accurate information of the closing velocity from the Doppler measurements. But the LOS rate measurements are highly noisy due to noise components caused by RCS, Glint, eclipsing in the onboard seeker [6, 7]. So LOS rates have to be estimated from the noisy measurements, typically using some variant of the Kalman Filter. It is well known that the Kalman Filter adjusts its time lag depending on the SNR provided to it through the Q and R matrices [8, 9]. The lower the SNR, the higher the filter lag. Thus for a noisy sensor the time lag can be considerable. Such a filter introduces a significant time lag in the guidance loop. Inclusion of the dynamics of the estimator is therefore important for obtaining realistic solution.

In this study we determine the solution of the set of differential equations describing the trajectories of a missile guided according to PPN law with an estimator in the loop. The effects of the error bounds of the estimator on the closed loop solution is illustrated and discussed in line with the approach of Becker [1]. The uniformly convergent infinite product series has been derived for case of a PPN guidance loop with an estimator. The closed loop solution provides interesting insight into PPN guidance behavior.

II. EQUATIONS OF MOTION

A non-maneuvering target T and a missile M have been considered to be point masses in a plane with velocities V_T and V_M respectively. The interception geometry of PPN is shown in Fig.1. The system can be described in a relative system of coordinates with axis of reference on the missile as shown.

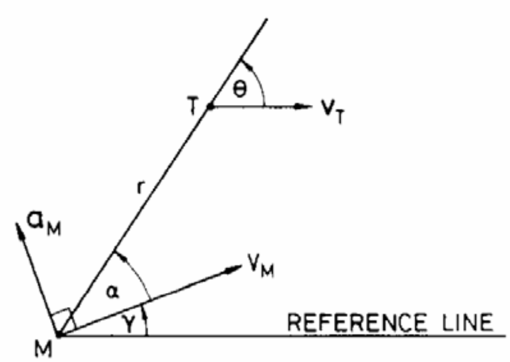

Fig 1. Engagement Geometry of PPN

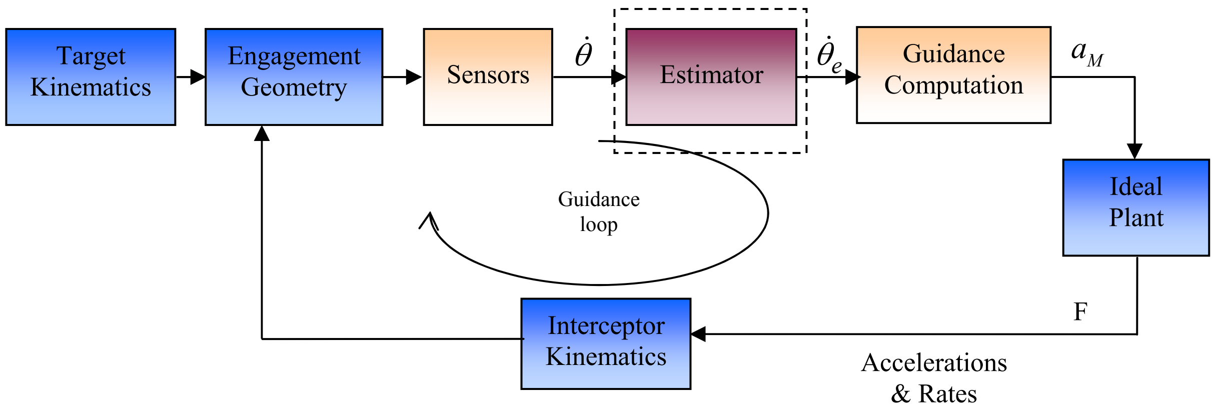

Fig 2. Schematic Block Diagram of Guidance Loop

In PPN [1, 4], it is assumed that the commanded missile acceleration a_M is applied normal to V_M.The equations of motion of the missile are derived in the following form. Letting a dot over a symbol denote differentiation with respect to time, the components of the relative velocity from the missile to target, in polar coordinates:

$$V_r = \dot{r} = V_T \cos\theta - V_M \cos(\theta - \gamma) \quad (1)$$

$$V_\theta = r\dot{\theta} = -V_T \sin\theta + V_M \sin(\theta - \gamma) \quad (2)$$

$$a_M = V_M \dot{\gamma} = N V_M \dot{\theta}_e \quad (3)$$

Fig 2 illustrates the schematic block diagram of a guidance loop. Here the autopilot, airframe subsystems are considered to be ideal. Eq. (3) is the PPN law and N, the navigation constant. Let us assume

$$\dot{\theta}_e = \dot{\theta} + \Delta\dot{\theta} \quad (4)$$

where $\Delta\dot{\theta}$ represents the error in the estimator's measured LOS rate $\dot{\theta}$ with respect to the kinematic LOS rate.

If (3) is integrated with initial values γ_0, θ_0

$$\gamma - \gamma_0 = N(\theta - \theta_0) + N\int_0^t \Delta\dot{\theta}\, dt \quad (5)$$

This may be rearranged as

$$\theta - \gamma = \varphi_0 - k\theta - \varepsilon \quad (6)$$

where

$$\varphi_0 = N\theta_0 - \gamma_0, \quad (7)$$

$$k = N - 1 \quad (8)$$

Substituting (6) into (1) and (2), the polar components of the relative velocity are:

$$V_r = \dot{r} = V_T \cos\theta - V_M \cos(\varphi_0 - k\theta - \varepsilon) \quad (9)$$

$$V_\theta = r\dot{\theta} = -V_T \sin\theta + V_M \sin(\varphi_0 - k\theta - \varepsilon) \quad (10)$$

This is the set of nonlinear differential equations which completely defines PPN law. The solution of these equations will provide the missile coordinates during flight.

III. Closed Form Solution

The solutions of the system (9), (10) have been obtained for N = 1 and N = 2, for ideal dynamics case and without an estimator in loop [1]. For N = 1, the solution of r and θ over time has been reported in [2]. For N = 2, a partial solution of r as a function of θ has been reported in [2].

From (9) and (10), eliminating time t,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{V_r(\theta)}{V_\theta(\theta)} \quad (11)$$

Integration of (11) provides r as a function of θ

$$r(\theta) = r_0 e^{I(\theta)} \quad (12)$$

where

$$I(\theta) = \int_{\theta_0}^{\theta} \frac{V_r(\theta)}{V_\theta(\theta)}\, d\theta \quad (13)$$

Thus, we observe that the integral is an explicit function of θ, if ε is assumed constant or a function of θ. Then $F(\theta)$ may be defined as

$$F(\theta) = \frac{V_r(\theta)}{V_\theta(\theta)} = \frac{V_T\cos\theta - V_M\cos(\varphi_0 - k\theta - \varepsilon)}{-V_T\sin\theta + V_M\sin(\varphi_0 - k\theta - \varepsilon)}$$
$$= \frac{\cos\theta - p\cos(\varphi_0 - k\theta - \varepsilon)}{-\sin\theta + p\sin(\varphi_0 - k\theta - \varepsilon)} \quad (14)$$

where

$$p = \frac{V_M}{V_T} \quad (15)$$

In the study, it is assumed that $p > 1$.

A closed form expression of $r(\theta)$ will require a closed-form solution of the integral in (13). Note that the integration is complicated by the simultaneous presence of θ and $k\theta$ in the harmonic functions.

From the Theory of Complex Analysis [10, and 11] and that of PPN guidance law [1], the integral (13) can be represented as a uniformly convergent infinite series of rational functions. This is due to the Mittag-Leffler's expansion theorem for partial fraction expansion for meromorphic functions [10].

The extension of (14) to the complex plane $z=\theta+i\,y$ yields:

$$F(z)=\frac{\cos z-p[\cos(\varphi_0-kz)+\varepsilon\sin(\varphi_0-kz)]}{-\sin z+p[\sin(\varphi_0-kz)-\varepsilon\cos(\varphi_0-kz)]} \tag{16}$$

The singularities of $F(z)$ are the zeros of the denominator

$$H(z)=p[\sin(\varphi_0-kz)-\varepsilon\cos(\varphi_0-kz)]-\sin z \tag{17}$$

Let the zeros of $H(z)$ be denoted as z_ν, $\nu=1,2,...,\infty$. It can be shown, following the approach in [1] that $H(z)$ has infinite isolated simple and real zeros at z_ν. A detailed proof of the same is given in Appendix I.

Thus, $F(z)$ is a meromorphic function with infinite simple and real poles at θ_ν. The residue of $F(z)$ at θ_ν is given by

$$A_\nu=\lim_{z\to\theta_\nu}(z-\theta_\nu)\,F(z)$$
$$=\frac{p\{\cos(\varphi_0-k\theta_\nu)\;+\varepsilon\sin(\varphi_0-k\theta_\nu)\}-\cos\theta_\nu}{kp\{\cos(\varphi_0-k\theta_\nu)+\varepsilon\sin(\varphi_0-k\theta)\}+\cos\theta_\nu} \tag{18}$$

From Appendix I, it may be concluded that $A_\nu>0$

Cauchy's Integral Formula requires evaluation of the function (16) over a closed loop contour [10]. We consider the contour C_n as a square with vertices at

$$z_n=\frac{\varphi_0}{k}\pm\left(2n+\frac{1}{2}\right)\frac{\pi}{k}\pm i\left(2n+\frac{1}{2}\right)\frac{\pi}{k} \tag{19}$$

It is shown in Appendix II that C_n does not pass through poles at θ_ν and on all C_n, $|F(z)|\le M$. This proof lies in line with that shown in [1].

Assume further without loss of generalization that the poles θ_ν are arranged in order of increasing values: $|\theta_1|<|\theta_2|<...|\theta_n|$.

Then $\lim_{\nu\to\infty}|\theta_\nu|=\infty$. If $\varphi_0=n\pi$, $\theta_1=0$; else $\theta_1\ne0$.

For $\theta_1\ne0$, a series representation of $F(z)$ is given by

$$F(z)=F(0)+\sum_{\nu=1}^{\infty}\left(\frac{A_\nu}{z-\theta_\nu}+\frac{A_\nu}{\theta_\nu}\right) \tag{20}$$

where

$$F(0)=\frac{1-p\{\cos(\varphi_0)\;+\varepsilon\sin(\varphi_0)\}}{p\{\sin(\varphi_0)+\varepsilon\cos(\varphi_0)\}} \tag{21}$$

The $\theta_1=0$ case can be reduced to the case $\theta_1\ne0$ by considering the function $G(z)=F(z)-A_1/z$. Except at $\theta_1=0$, $G(z)$ and $F(z)$ are identical as $G(z)$ has removable singularity at $\theta_1=0$. Hence, for $\theta_1=0$,

$$F(z)=\frac{A_1}{z}+\sum_{\nu=2}^{\infty}\left(\frac{A_\nu}{z-\theta_\nu}+\frac{A_\nu}{\theta_\nu}\right) \tag{22}$$

Integrating term by term one has:

$$r(\theta)=\begin{cases} r_0e^{F(0)(\theta-\theta_0)}\prod_{\nu=1}^{\infty}\left|\dfrac{\theta-\theta_\nu}{\theta_0-\theta_\nu}\right|^{A_\nu} e^{A_\nu(\theta-\theta_0)/\theta_\nu} & for\ \theta_1\ne0\\ r_0\left|\dfrac{\theta}{\theta_0}\right|^{A_1}\prod_{\nu=2}^{\infty}\left|\dfrac{\theta-\theta_\nu}{\theta_0-\theta_\nu}\right|^{A_\nu} e^{A_\nu(\theta-\theta_0)/\theta_\nu} & for\ \theta_1=0\end{cases} \tag{23}$$

This is the closed form solution of $r(\theta)$ of PPN law for $p>1$. Once the solution of $r(\theta)$ is obtained, $\dot{\theta}$ may be obtained from (10), or

$$H(\theta)=\frac{r\dot{\theta}}{V_T} \tag{24}$$

The complex analysis of (24) is the function (17) with simple and real zeros at $z_\nu=\theta_\nu$. Then, according to Weierstrass' factor theorem, $H(z)$ may be represented by a convergent infinite product. Considering the Rouche's Theorem [11],

$$\frac{d}{dz}\ln H(z)=\frac{H'(z)}{H(z)} \tag{25}$$

This form of the meromorphic function, with simple real poles at $z_\nu=\theta_\nu$, has residue = 1 [11, pp 148].

From the Mittag-Leffler expansion theorem for $\theta_1\ne0$:

$$\frac{H'(z)}{H(z)}=\frac{H'(0)}{H(0)}+\sum_{\nu=1}^{\infty}\left(\frac{1}{z-\theta_\nu}+\frac{1}{\theta_\nu}\right) \tag{26}$$

where $$\frac{H'(0)}{H(0)}=-\frac{pk(\cos\varphi_0+\varepsilon\sin\varphi_0)+1}{p\sin\varphi_0-p\varepsilon\cos\varphi_0} \tag{27}$$

For $\theta_1=0$, let $J(z)=H(z)/z$. Thus the logarithmic derivatives of $J(z)$ and $H(z)$ are same except at $\theta_1=0$.

Thus

$$\dot{\theta}=\begin{cases}\dfrac{V_T}{r_0}H(\theta_0)e^{[H'(0)/H(0)-F(0)](\theta-\theta_0)} \\ \qquad\times\displaystyle\prod_{\nu=1}^{\infty}\left|\frac{\theta-\theta_\nu}{\theta_0-\theta_\nu}\right|^{1-A_\nu} e^{(1-A_\nu)(\theta-\theta_0)/\theta_\nu} & for\,\theta_1\neq 0 \\ \dfrac{V_T}{r_0}H(\theta_0)\left|\dfrac{\theta}{\theta_0}\right|^{A_1}\displaystyle\prod_{\nu=2}^{\infty}\left|\frac{\theta-\theta_\nu}{\theta_0-\theta_\nu}\right|^{1-A_\nu} e^{(1-A_\nu)(\theta-\theta_0)/\theta_\nu} & for\,\theta_1=0\end{cases} \quad (28)$$

As in [1], it is observed that only factors of the product series corresponding to small $|\theta_\nu|$ and $|\theta_\nu-\theta_0|$ are significant.

IV. Simulation

The simulation results presented in [1] have been verified with the current derivations in our work. Fig 3 shows the comparison of Fig 2 (a) in [1] and our derivation of the same with $\varepsilon=0$, with the exact solution of (1), (2) and (3).

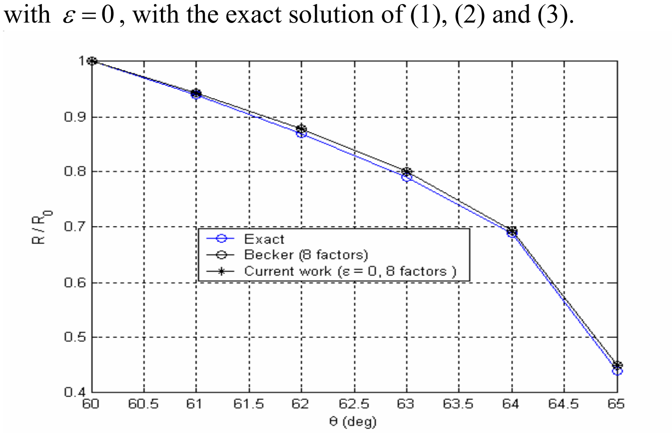

Fig 3. Normalized range as function of θ; $p=2, k=2,\ \theta_0=60^\circ,\ \gamma_0=15^\circ$

Thus, the exact solution is closely approximated by the series presented in [1] and so derived in our study, with ε assigned to zero for comparison. To study the effect of error bound, ε is assigned values and the comparison results for the same at different values of ε are presented. From Fig 4, we see that for ε =0.1, the series solution with 8 factors give a close approximation of the exact solution with same error bound. It may be noted that the increase in the error bound in LOS rate produces a decrease in the rate of change of range-to-go. Similarly in Fig 5, for ε =0.1, the series solution with 8 factors give a close approximation of the exact solution with same error bound. Thus the results show that the product series rapidly converge to the exact solution. Also, it may be noted that a few factors are to be considered for close approximation to the exact solution.

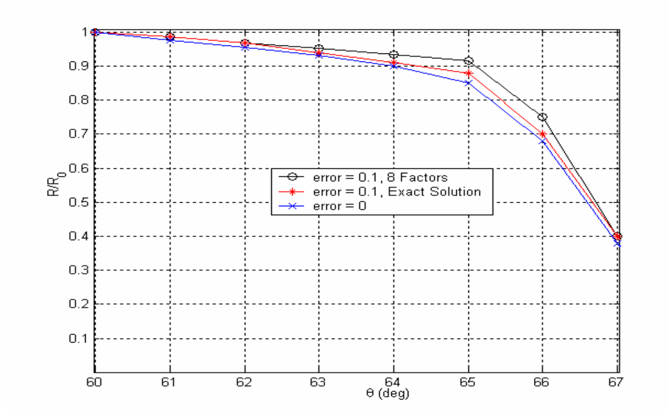

Fig 4. Normalized range as function of θ; $p=2, k=3,\ \theta_0=60^\circ,\ \gamma_0=15^\circ$

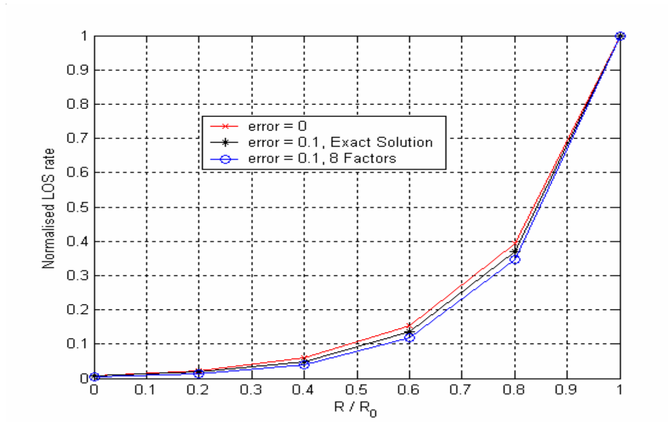

Fig 5. Normalized LOS rate as function of normalized range; $p=2, k=3,\ \theta_0=60^\circ,\ \gamma_0=15^\circ$

V. Conclusion

Our study extends the work in [1] for a estimator lag in the set of closed form solutions of the differential equations for the equations of motion of a missile pursuing a non-maneuvering target according to PPN guidance law. The complex analysis theory has been applied to obtain the uniform convergent series for $r(\theta)$ and $\dot{\theta}(\theta)$. The results reflect that the series so obtained closely approximate the exact solution of the differential equations. The work of generalizing the series for maneuvering target is going on. The analytical closed form solution of the homing loop is a significant contribution from the point of view of guidance law design and acts as the basic outline to do so.

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REFERENCES

[1] K. Becker, "Closed-Form Solution of Pure Proportional Navigation", *IEEE Trans. of Aerospace and Electronic Systems,* Vol. AES - 26, No.3, July 1990, pp 526-533.

[2] M Guelman, "The Closed-Form Solution of True Proportional Navigation", *IEEE Trans. of Aerospace and Electronic Systems,* Vol. AES-12, No.4, July 1976, pp 472-482.

[3] C.D. Yang, F.B. Yeh, and J.H. Chen, "The Closed-Form solution of Generalized Proportional Navigation", *Journal of Guidance, Control and Dynamics,* Vol-10, March – April 1987, pp 216-218.

[4] J. Wang, and X. Xie, " A Derivation of Pure Proportional Navigation", *Proceedings of the American Control Conf.*, San Diego, California, June 1999, pp 3758 – 3759.

[5] A.Chakravarthy and D. Ghose, "Capturability of Realistic Generalized True Proportional Navigation", *IEEE Trans. of Aerospace and Electronic Systems,* Vol. AES-32, No.1, July 1996, pp 407-418.

[6] M R Ananthasayanam, A K Sarkar, P Vohra, A Bhattacharya, R Srivastava, *Estimation of LOS Rates and Angles from Noisy Seeker Measurements using EKF*, IEEE SPCOM, Dec. 2004.

[7] A Bhattacharya, R N Bhattacharya, *Seeker based Optimal Guidance Law, a few issues,* AIAA GNC Conf, 11-14 August 2003, Austin, Texas.

[8] Y. Bar-Shalom, X.-Rong Li, T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*, John Wiley and Sons, Inc., 2001.

[9] C K Chui, G Chen, *Kalman filtering with real-time applications*, Springer,
Third edition, 1999.

[10] Lars V. Ahlfors, *Complex Analysis*, Third Edition, McGraw-Hill International, 1984.

[11] John M. Howie, *Complex Analysis*, Springer-Verlag, London, 2003,
ISBN 1852337338.

[12] P. Zarchan, *Tactical and Strategic Missile Guidance*, Forth edition,
AIAA Inc., Virginia, 2002.

APPENDIX I

The following proof implies that the zeros of the function $H(z) = p[\sin(\varphi_0 - kz) - \varepsilon\cos(\varphi_0 - kz)] - \sin z$ are all real and simple. It is assumed that $k > 1$ and $p > 1$. Fig 6 shows the plot of $H(z)$.

For the complex plane $z = \theta + i\,y$, Euler's formula is

$$e^{z} = \cos z + i \sin z \tag{29}$$

Then,

$$\begin{aligned} H(z) = {} & p\{\sin(\varphi_0 - k\theta)\cosh(ky) - \varepsilon\cos(\varphi_0 - k\theta)\cosh(ky)\} - \\ & \sin\theta\cosh(y) - i[p\{\cos(\varphi_0 - k\theta)\sinh(ky) + \\ & \varepsilon\sin(\varphi_0 - k\theta)\sinh(ky)\} + \cos\theta\sinh(y)] \end{aligned} \tag{30}$$

For $H(z)$ to vanish, both real and imaginary parts have to be zero. For the above $H(z)$, if $y = 0$, we obtain the real roots θ_v of $H(\theta)$. If $y \neq 0$, then $H(z) = 0$, if

$$\begin{aligned} & p\{\sin(\varphi_0 - k\theta)\cosh(ky) - \\ & \quad \varepsilon\cos(\varphi_0 - k\theta)\cosh(ky)\} - \sin\theta\cosh(y) \end{aligned} = 0 \tag{31}$$

$$\begin{aligned} & p\{\cos(\varphi_0 - k\theta)\sinh(ky) + \\ & \quad \varepsilon\sin(\varphi_0 - k\theta)\sinh(ky)\} + \cos\theta\sinh(y) \end{aligned} = 0 \tag{32}$$

are satisfied together.

By rearranging,

$$p\{\sin(\varphi_0 - k\theta) - \varepsilon\cos(\varphi_0 - k\theta)\} = \sin\theta \frac{\cosh(y)}{\cosh(ky)} \tag{33}$$

$$p\{\cos(\varphi_0 - k\theta) + \varepsilon\sin(\varphi_0 - k\theta)\} = \cos\theta \frac{\sinh(y)}{\sinh(ky)} \tag{34}$$

By squaring and adding both sides,

$$\begin{aligned} p^2(1+\varepsilon^2) &= \sin^2\theta \frac{\cosh^2(y)}{\cosh^2(ky)} + \cos^2\theta \frac{\sinh^2(y)}{\sinh^2(ky)} \\ &\leq \sin^2\theta + \cos^2\theta = 1 \end{aligned} \tag{35}$$

This condition is clearly a contradiction if $p > 1$. Hence all zeros of $H(z)$ are real zeros θ_v of $H(\theta)$. Fig 6 shows the plot of $H(\theta)$. The x-axis is of θ_v and y-axis is of $H(\theta)$ function.

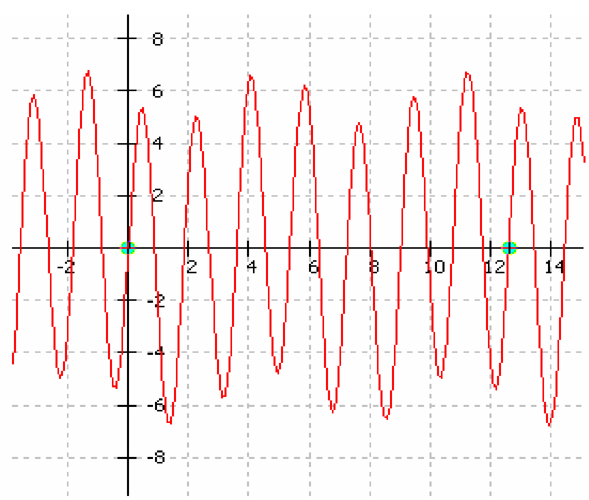

Fig 6. Plot of $H(\theta)$, $p = 2$, $k = 3$, $\theta_0 = 60^{\circ}$, $\gamma_0 = 15^{\circ}$, $\varepsilon = 0.1$

Next proof is that the zeros are simple as the derivative

$$\frac{dH}{d\theta} = -kp\,[\,\cos(\varphi_0 - k\theta) + \varepsilon\sin(\varphi_0 - k\theta)] - \cos\theta \qquad (36)$$

is not zero at all θ_v.

The proof is as follows:
If $\theta = \theta_v$,

$$p^2[\sin(\varphi_0 - k\theta_v) - \varepsilon\cos(\varphi_0 - k\theta_v)]^2 = \sin^2\theta_v \qquad (37)$$

Expanding

$$\begin{aligned} &p^2[1 - \cos^2(\varphi_0 - k\theta_v) - \\ &\quad 2\,\varepsilon\sin(\varphi_0 - k\theta_v)\cos(\varphi_0 - k\theta_v)] = 1 - \cos^2\theta_v \end{aligned} \qquad (38)$$

Assuming ε to be small

$$\begin{aligned} &\cos(\varphi_0 - k\theta_v)[\cos(\varphi_0 - k\theta_v) + \\ &\quad 2\,\varepsilon\sin(\varphi_0 - k\theta_v)] = 1 - \frac{1}{p^2} + \frac{1}{p^2}\cos^2\theta_v \end{aligned} \qquad (39)$$

This is a monotonically increasing function of p.
Hence for $p > 1$

$$\left|\{\cos(\varphi_0 - k\theta_v) + \varepsilon\sin(\varphi_0 - k\theta_v)\}\right| \geq \left|\cos\theta_v\right| \qquad (40)$$

Hence,

$$\frac{d}{d\theta}H(\theta) \neq 0 \quad \textit{at all } \theta_v. \qquad (41)$$

APPENDIX II

Considering the contour C_n in (19), it can be proved that the defined contour does not pass through any poles of $F(z)$ and $|F(z)| \leq M$.
The proof is as follows:
The parts of C_n which intersect the real axis are considered now. They intersect the real axis at

$$\theta_n = \frac{\varphi_0}{k} \pm \left(2n + \frac{1}{2}\right)\frac{\pi}{k} \qquad (42)$$

Since $p > 1$

$$\sin(\varphi_0 - k\theta_n) = \pm 1 \qquad (43)$$

$$\cos(\varphi_0 - k\theta_n) = 0 \qquad (44)$$

Hence C_n does not pass through any poles of $F(z)$. The numerator of $F(z)$ may be expressed as:

$$\begin{aligned} G(z) &= \cos z - p[\cos(\varphi_0 - kz) + \varepsilon\sin(\varphi_0 - kz)] \\ &= [\cos\theta\cosh(y) - p\{\cos(\varphi_0 - k\theta)\cosh(ky) + \\ &\quad \varepsilon\sin(\varphi_0 - k\theta)\cosh(ky)\}] - i[\sin\theta\sinh(y) + \\ &\quad p\{\sin(\varphi_0 - k\theta)\sinh(ky) - \varepsilon\cos(\varphi_0 - k\theta)\sinh(ky)\}] \end{aligned} \qquad (45)$$

and the denominator is as (30).

On the lines $\eta_n = \theta_n + iy$ parallel to the imaginary axis,

$$\begin{aligned} |G(\eta_n)|^2 &= [\cos\theta\cosh(y) \pm p\varepsilon\cosh(ky)]^2 \\ &\quad + [\sin\theta\sinh(y) \pm p\sinh(ky)]^2 \\ &\leq (1 + p\varepsilon)^2\cosh^2(ky) + (1 + p)^2\sinh(ky)^2 \end{aligned} \qquad (46)$$

Similarly,

$$\begin{aligned} |H(\eta_n)|^2 &= [\pm p\cosh(ky) - \sin\theta\cosh(y)]^2 \\ &\quad + [p\varepsilon\sinh(ky) + \cos\theta\sinh(y)]^2 \\ &\geq (p - 1)^2\cosh^2(ky) + (p\varepsilon + 1)^2\sinh^2(ky) \end{aligned} \qquad (47)$$

Thus, from (46), (47) and (16), we obtain

$$|F(\eta_n)|^2 = \frac{|G(\eta_n)|^2}{|H(\eta_n)|^2} \leq \frac{(p+1)^2 + (p\varepsilon + 1)^2}{(p-1)^2 + (p\varepsilon + 1)^2} = M_1 \qquad (48)$$

Next we consider the parts of C_n which intersect the imaginary axis. Let $\varsigma = \varphi_0 - k\theta$, then by Euler's formula

$$F(z) = i\frac{e^{i\theta}e^{-y} + e^{-i\theta}e^{y} - p(e^{i\varsigma}e^{ky} + e^{-i\varsigma}e^{-ky}) + ip\varepsilon(pe^{i\varsigma}e^{ky} - pe^{-i\varsigma}e^{-ky})}{p(e^{i\varsigma}e^{ky} + e^{-i\varsigma}e^{-ky}) - e^{i\theta}e^{-y} + e^{-i\theta}e^{y} - ip\varepsilon(pe^{i\varsigma}e^{ky} + pe^{-i\varsigma}e^{-ky})} \qquad (49)$$

Multiplying numerator and denominator by $e^{\mp ky}$ and at $y \to \pm\infty$

$$\lim_{y \to \pm\infty} F(z) = \begin{cases} \mp i & \textit{for } k \neq 1 \\ \pm i\dfrac{e^{\mp i\theta} - pe^{\pm i\varsigma} + ip\varepsilon e^{\pm i\varsigma}}{pe^{\pm i\varsigma} - e^{\mp i\theta} - ip\varepsilon e^{\pm i\varsigma}} & \textit{for } k = 1 \end{cases} \qquad (50)$$

Hence, for large n and small ε,

$$|F(z)| \leq 2\frac{p+1}{p-1} = M_2 \qquad (51)$$

for all C_n which intersect the imaginary axis.
Hence for all M greater than the greater of M_1 and M_2, we obtain

$$|F(z)| \leq M \qquad (52)$$

on C_n, where M is independent of n.