2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Nonlinear Control of the Burgers PDE—Part II: Observer Design, Trajectory Generation, and Tracking

Miroslav Krstic, Lionel Magnis, and Rafael Vazquez

***Abstract*— In a companion paper we have solved the problem of full-state stabilization of unstable "shock-like" equilibrium profiles of the viscous Burgers equation with actuation at the boundaries. In this paper we consider the problem of output feedback stabilization. We design a nonlinear observer for the Burgers equation that employs only boundary sensing. We employ its state estimates in an output feedback control law which we prove to be locally stabilizing. The main idea is to use a nonlinear spatially-scaled transformation (that employs three ingredients, of which one is the Hopf-Cole nonlinear integral transformation) and then employ the linear backstepping observer design method. The stabilization properties of the output feedback law are illustrated with numerical simulations of the closed-loop system. We also consider the problems of trajectory generation and tracking for the fully nonlinear Burgers equation. Our algorithm is applicable to a large class of functions of time as reference trajectories of the boundary output, though we focus in more detail on the special case of sinusoidal references. Since the Burgers equation is not globally controllable, the reference amplitudes cannot be arbitrarily large. We provide a sufficient condition that characterizes the allowable amplitudes and frequencies, under which the state trajectory is bounded and tracking is achieved.**

I. INTRODUCTION

In this paper we study various nonlinear control problems for the viscous Burgers equation, which is considered a basic model of nonlinear convective phenomena such as those that arise in Navier-Stokes equations. We consider a family of stationary solutions called "shock profiles" (or "shock-like" profiles) which are unstable and not stabilizable (even locally) by simple means such as the standard "radiation boundary conditions." In a companion paper [8] we have studied the problem stabilization of the shock profiles using two control inputs (one at each boundary) by full-state feedback. We have also provided an estimate of the region of attraction for the closed-loop system under the full-state feedback laws, which is finite because the Burgers system is not globally controllable [4], [5].

This work was supported by NSF.

M. Krstic is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA, krstic@ucsd.edu

L. Magnis is with the Ecole des Mines de Paris, 60 Boulevard Saint Michel, 75006 Paris, France lionel.magnis@ensmp.fr

R. Vazquez is with the Departmento de Ingeniería Aeroespacial, Universidad de Sevilla, Avda. de los Descubrimientos s.n., 41092 Sevilla, Spain rvazquez@ucsd.edu

In this paper, we consider boundary-output feedback stabilization and the problems of trajectory generation and tracking for the Burgers equation with two inputs and one reference output.

Early efforts on output feedback stabilization for the Burgers system where presented in [3], where a linear static collocated output feedback (a.k.a. "radiation" boundary conditions) is proved to achieve local L^2 exponential stability. In [12] this result is improved to L^∞, but remains local. Global stabilization is achieved in [7] (see also [1] and [9]) using nonlinear static output feedback.

Our output feedback design is based on a nonlinear spatially-scaled transformation (which is also used in the full-state design [8]) that transforms the system (with the help of one of the two boundary controls) into a linear reaction-diffusion PDE with nonlinear boundary conditions. For this system we design a nonlinear observer (with gains computed using the backstepping observer design method [11]) that uses injection of the output estimation error at one of the boundaries. We combine the observer with the full-state feedback design of [8]. The resulting output feedback is fully collocated and decentralized, as in the case with "radiation boundary conditions." However, while our feedback at one of the boundaries is static (like the "radiation" feedback), at the other boundary it is dynamic (and nonlinear). The stabilization properties of the observer-based feedback laws are illustrated in simulations.

Finally, we present results for trajectory generation and tracking for the Burgers equation, which are enabled by our transformation into the heat equation, for which the general trajectory generation problem has been solved in [6] and for which explicit solutions are shown here for a particular class of functions. While we do not achieve a global result due to the lack of global controllability mentioned above, for the case of tracking a sinusoid in time we give a bound that quantifies the trade-off between the maximum amplitudes and frequencies for which tracking is achieved.

II. FULL-STATE STABILIZATION OF THE SHOCK PROFILES OF THE BURGERS EQUATION

In this section we summarize the results of [8]. Consider the viscous Burgers equation

$$u_t = u_{xx} - u_x u, \tag{1}$$

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

where $u(x,t)$ is the state variable, for $x \in [0,1]$, with boundary conditions

$$u_x(0,t) = \omega_0(t), \quad u_x(1,t) = \omega_1(t), \tag{2}$$

where $\omega_0(t)$ and $\omega_1(t)$ are the control inputs. To save space, in the sequel we will drop the arguments (x,t) whenever the context allows.

We consider the "shock-like" stationary solutions determined in [2] as

$$U(x) = -2\sigma \tanh\left(\sigma\left(x - 1/2\right)\right), \tag{3}$$

which are parameterized by the variable $\sigma \geq 0$.

From (2) and (3) we obtain that

$$\omega_0 = \omega_1 = -2\sigma^2 \left(1 - \tanh^2(\sigma/2)\right) \leq 0 \tag{4}$$

are the constant values of the open-loop control laws $\omega_0(t)$ and $\omega_1(t)$ that produce the equilibrium profile (3).

Let us denote the fluctuation variable around the shock profile as $\tilde{u}(x,t) = u(x,t) - U(x)$. Then the Burgers equation written in the perturbation variable $\tilde{u}$ is

$$\tilde{u}_t = \tilde{u}_{xx} - U(x)\tilde{u}_x - U'(x)\tilde{u} - \tilde{u}_x\tilde{u}. \tag{5}$$

Let us denote $\tilde{\omega}_0(t) = \omega_0(t) - U'(0), \tilde{\omega}_1(t) = \omega_1(t) - U'(1)$. Then the boundary conditions for (5) are

$$\tilde{u}_x(0,t) = \tilde{\omega}_0(t), \quad \tilde{u}_x(1,t) = \tilde{\omega}_1(t). \tag{6}$$

In [8] it is shown that the origin of the $\tilde{u}$ system (5) is unstable for $\sigma > 0$ (and neutrally stable for $\sigma = 0$), and the following full-state feedback laws are designed:

$$\begin{aligned} \tilde{\omega}_0 &= 2\sigma\tanh(\sigma/2)\tilde{u}(0) + \tilde{u}^2(0)/2, && (7) \\ \tilde{\omega}_1 &= \tilde{u}(1)^2/2 + \left(k(1,1) - 2\sigma\tanh\left(\sigma/2\right)\right)\tilde{u}(1) \\ &\quad + \int_0^1 \left(k_x(1,y) + \sigma\tanh\left(\sigma/2\right)k(1,y)\right) \\ &\quad \times G(y)\mathrm{e}^{\int_y^1 \tilde{u}(\xi)d\xi}\tilde{u}(y)dy, && (8) \end{aligned}$$

where $k(x,y)$ is given in [8] and

$$G(x) = \frac{\cosh\left(\sigma(x-1/2)\right)}{\cosh(\sigma/2)}, \tag{9}$$

III. Observer and Output Feedback Law

The control in Section II requires to know the state $\tilde{u}(x,t)$ for all $x \in [0,1]$. We now design output feedback laws for $\tilde{\omega}_0(t)$ and $\tilde{\omega}_1(t)$ using only boundary measurement. We will design fully collocated (decentralized) feedback laws, i.e., using only the measurement of $\tilde{u}(0,t)$ for the control $\tilde{\omega}_0(t)$, and only the measurement of $\tilde{u}(1,t)$ for $\tilde{\omega}_1(t)$.

Notice that the control law (7) is already an output feedback law requiring only the knowledge of $\tilde{u}(0)$. We saw in [8] that applying (7) and the mapping

$$v(x,t) = G(x)\tilde{u}(x,t)\mathrm{e}^{-\frac{1}{2}\int_0^x \tilde{u}(y,t)\,dy}, \tag{10}$$

which has the inverse

$$\tilde{u}(x) = \frac{v(x)/G(x)}{1 - \frac{1}{2}\int_0^x \frac{v(y)}{G(y)}\,dy}, \tag{11}$$

transforms the plant into the linear system

$$\begin{aligned} v_t &= v_{xx} + \sigma^2\left[\frac{2}{\cosh^2\left(\sigma\left(x-1/2\right)\right)} - 1\right]v, && (12) \\ v_x(0) &= \sigma\tanh\left(\sigma/2\right)v(0), && (13) \\ v_x(1) &= \sigma\tanh\left(\sigma/2\right)v(1) + \left(1 - \frac{1}{2}\int_0^1 \frac{v(y)}{G(y)}dy\right) \\ &\quad \times \left(\tilde{\omega}_1 - \frac{1}{2}\tilde{u}(1)^2\right). && (14) \end{aligned}$$

Hence the problem reduces to the design of an observer-based feedback controller for (12)–(14) using the measurement of $\tilde{u}(1,t)$.

A. Observer equations

We start with an observation that the boundary condition (14) contains a state nonlinearity given by the integral term in $v(y)$. Our observer is designed as a copy of the (nonlinear) plant with injection of the output error,

$$\begin{aligned} \hat{v}_t &= \hat{v}_{xx} + \sigma^2\left[\frac{2}{\cosh^2\left(\sigma\left(x-1/2\right)\right)} - 1\right]\hat{v} \\ &\quad + \left[\left(1 - \frac{1}{2}\int_0^1 \frac{\hat{v}(y)}{G(y)}dy\right)\tilde{u}(1) - \hat{v}(1)\right] \\ &\quad \times\rho(x), && (15) \\ \hat{v}_x(0) &= \sigma\tanh(\sigma/2)\hat{v}(0), && (16) \\ \hat{v}_x(1) &= \left(\sigma\tanh(\sigma/2) + \rho_1\right)\left[\left(1 - \frac{1}{2}\int_0^1 \frac{\hat{v}(y)}{G(y)}dy\right)\right. \\ &\quad \left.\times\tilde{u}(1) - \hat{v}(1)\right] + \left(1 - \frac{1}{2}\int_0^1 \frac{\hat{v}(y)}{G(y)}dy\right) \\ &\quad \times\left(\tilde{\omega}_1 - \frac{1}{2}\tilde{u}(1)^2 + \sigma\tanh(\sigma/2)\tilde{u}(1)\right), && (17) \end{aligned}$$

where $\hat{v}(x,t)$ denotes the estimate of the state $v(x,t)$. Notice that, using (11),

$$\left(1 - \frac{1}{2}\int_0^1 \frac{v(y)}{G(y)}dy\right)\tilde{u}(1) = v(1)/G(1) = v(1), \tag{18}$$

since $G(1) = 1$. Hence the term

$$\left(1 - \frac{1}{2}\int_0^1 \frac{\hat{v}(y)}{G(y)}dy\right)\tilde{u}(1), \tag{19}$$

appearing in (15)–(17), is an estimate of $v(1)$ and it is used for output injection. The gains $\rho(x)$ and ρ_1 are determined to ensure convergence of $\hat{v}$ to v.

B. Design of ouput injection gains using backstepping

To design $\rho(x)$ and ρ_1 we use the backstepping method for observer design [11]. First, we denote the observer error as $e(x,t) = v(x,t) - \hat{v}(x,t)$. Subtracting (15)–(17) from (12)–(14) we get equations for e, which linearized around the origin are

$$\begin{aligned} e_t &= e_{xx} + \sigma^2 \left[\frac{2}{\cosh^2(\sigma(x-1/2))} - 1 \right] e \\ &\quad -\rho(x)e(1), && (20) \\ e_x(0) &= \sigma \tanh(\sigma/2) e(0), && (21) \\ e_x(1) &= -\left(\sigma \tanh(\sigma/2) + \rho_1\right) e(1). && (22) \end{aligned}$$

We need to design the gains $\rho(x)$ and ρ_1 so that the system (20)–(22) is exponentially stable. The plant (20)–(22) is a linear 1-D reaction-diffusion PDE with Robin boundary conditions, so the backstepping observer design method in [11] can be applied. We map $e(x,t)$ into a new variable $\eta(x,t)$ using

$$e(x) = \eta(x) - \int_x^1 p(x,y)\eta(y)dy, \qquad (23)$$

with η verifying the (exp. stable) error target system

$$\begin{aligned} \eta_t &= \eta_{xx} - \left[\sigma^2 \tanh^2(\sigma(x-1/2))\right] \eta \\ &\quad -c\eta, && (24) \\ \eta_x(0) &= \sigma \tanh(\sigma/2)\eta(0), && (25) \\ \eta_x(1) &= -\sigma \tanh(\sigma/2)\eta(1). && (26) \end{aligned}$$

Following the method in [11], we find the kernel $p(x,y)$ appearing in (23) from

$$\begin{aligned} p_{xx} - p_{yy} &= -\sigma^2 \left[1 - 2\tanh^2(\sigma(x-1/2)) \right. \\ &\quad \left. +\tanh^2(\sigma(y-1/2))\right] p - cp, && (27) \\ p(x,x) &= -\frac{1}{2}\left[\sigma \tanh(\sigma(x-1/2)) \right. \\ &\quad \left. +\sigma\tanh(\sigma/2) + cx\right], && (28) \\ p_x(0,y) &= \sigma \tanh(\sigma/2) p(0,y). && (29) \end{aligned}$$

Once $p(x,y)$ is computed, it is used to compute the output injection gains as follows:

$$\begin{aligned} \rho(x) &= -\left(p_y(x,1) + \sigma\tanh(\sigma/2)p(x,1)\right), && (30) \\ \rho_1 &= -p(1,1). && (31) \end{aligned}$$

Comparing (27)–(29) with the kernel equation in [8], we deduce that $p(x,y) = k(y,x)$. Hence it is not necessary to solve (27)–(29) and we can use the solution for k in (30)–(31), obtaining

$$\begin{aligned} \rho(x) &= -\left(k_x(1,x) + \sigma\tanh(\sigma/2)k(1,x)\right), && (32) \\ \rho_1 &= -k(1,1) = \sigma\tanh(\sigma/2) + \frac{c}{2}. && (33) \end{aligned}$$

Note that $\rho(x)$ in (32) is the same function as the control gain in (8), just changing the sign. Thus, the output injection gains can be computed from the control kernel.

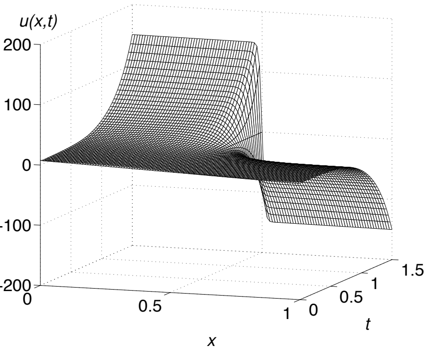

Fig. 1. Finite time blow-up of open-loop system (with constant inputs (4)) for $\sigma = 3$ and $u_0(x) = U(0) + 2 + (U(1) - U(0) - 4)x$.

C. Output feedback laws

Using control law (7) and the estimate $\hat{v}$ [which is transformed into an estimate of $\tilde{u}$ using (11)] in control law (8), we obtain our nonlinear output feedback control laws, which are defined as follows

$$\begin{aligned} \tilde{\omega}_0(t) &= 2\sigma\tanh(\sigma/2)\tilde{u}(0,t) + \frac{\tilde{u}^2(0,t)}{2}, && (34) \\ \tilde{\omega}_1(t) &= \frac{1}{2}\tilde{u}(1,t)^2 + \left(-2\sigma\tanh(\sigma/2) + k(1,1)\right)\tilde{u}(1,t) \\ &\quad + \left(\int_0^1 \left[k_x(1,y) + \sigma\tanh(\sigma/2)k(1,y)\right] \right. \\ &\quad \left. \times \hat{v}(y,t)dy \right) \left(1 - \frac{1}{2}\int_0^1 \frac{\hat{v}(y,t)}{G(y)}dy\right)^{-1}. && (35) \end{aligned}$$

Thus we have obtained a *diagonal* TITO compensator

$$\begin{bmatrix} \tilde{u}(0,t) \\ \tilde{u}(1,t) \end{bmatrix} \mapsto \begin{bmatrix} \tilde{\omega}_0(t) \\ \tilde{\omega}_1(t) \end{bmatrix}. \qquad (36)$$

IV. Simulations

The open-loop system (1)–(2) and (4) is unstable, as shown in [8]. A numerical study of the linearized system around the shock profile shows the presence of one positive (though possibly small) eigenvalue for any $\sigma > 0$. In Fig. 1 one can see a finite time blow-up of the open-loop system for $\sigma = 3$.

For the same initial conditions, we show in Figure 2 the numerical evolution of the system with the linear *observer-based* backstepping controller of Section III.

V. Trajectory Generation and Tracking

Given the system (1)–(2), with the two inputs $\omega_0(t)$ and $\omega_1(t)$, we consider a trajectory generation problem with $u(0,t)$ as the system's single output. Then, the problem of trajectory generation consists on finding open loop control input functions $\omega_0^r(t)$ and $\omega_1^r(t)$ to

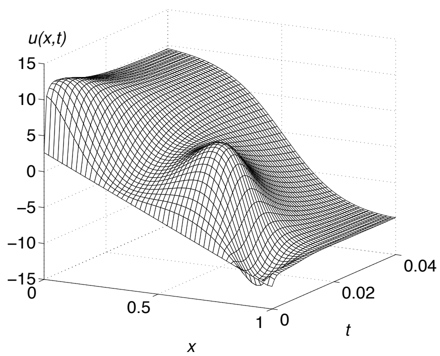

Fig. 2. Convergence of the closed-loop system under the linear output-feedback controller for $\sigma = 5$.

make $u(0,t)$ evolve *exactly* according to a given reference signal $u^r(0,t)$. While the open-loop system might be stable around the trajectory found in the trajectory generation problem, usually this is not the case. Then the problem of trajectory tracking, i.e., of determining feedback laws to stabilize the trajectory, has to be solved.

A. Trajectory generation

We use the invertible transformation

$$v(x,t) = u(x,t)\mathrm{e}^{-\frac{1}{2}\int_0^x u(y,t)dy}, \tag{37}$$

$$u(x,t) = \frac{v(x,t)}{1-\frac{1}{2}\int_0^x v(y,t)dy}, \tag{38}$$

which converts the Burgers system (1) into the form

$$v_t(x,t) = v_{xx}(x,t) + v(x,t) \times \frac{1}{2}\left(\omega_0(t) - \frac{1}{2}v(0,t)^2\right), \tag{39}$$

$$v_x(0,t) = \omega_0(t) - \frac{1}{2}v(0,t)^2, \tag{40}$$

$$v_x(1,t) = \left(1-\frac{1}{2}\int_0^1 v(y,t)dy\right)\omega_1(t) - \frac{1}{2}\frac{v(1,t)^2}{1-\frac{1}{2}\int_0^1 v(y,t)dy}. \tag{41}$$

From (37) we obtain $v(0,t) = u(0,t)$, hence we get that $v^r(0,t) = u^r(0,t)$. Thus the trajectory generation for the u-system (1) can be approached as a trajectory generation problem for the v-system (39). Then we are looking for functions $v^r(x,t)$, $\omega_0^r(t)$, and $\omega_1^r(t)$ that satisfy (39)–(41) and $v^r(0,t) = u^r(0,t)$. We choose

$$\omega_0^r(t) = \frac{1}{2}u^r(0,t)^2, \tag{42}$$

which, substituted in (39), simplifies the nonlinear trajectory generation problem to the trajectory generation problem for the linear heat equation (with nonlinear boundary conditions)

$$v_t^r(x,t) = v_{xx}^r(x,t), \quad v_x^r(0,t) = 0, \tag{43}$$

$$v_x^r(1,t) = \left(1-\frac{1}{2}\int_0^1 v^r(y,t)dy\right)\omega_1^r(t) - \frac{1}{2}\frac{v^r(1,t)^2}{1-\frac{1}{2}\int_0^1 v^r(y,t)dy}. \tag{44}$$

A general infinite-series solution for $v^r(x,t)$ in (43) exists for a very broad class of functions of time $u^r(0,t)$ (the Gevrey class), which has been developed in the framework of differential flatness [6]. Furthermore, an explicit solution can be derived for any function $u^r(0,t)$ that can be written as an output of a linear exosystem. For example, if

$$u^r(0,t) = b + a\sin\omega t, \tag{45}$$

i.e., we want to track a sinusoid with a bias, then the explicit solution for the reference state is

$$v^r(x,t) = b + a\mathrm{Im}\left\{\cosh\left(\sqrt{j\omega}x\right)\mathrm{e}^{j\omega t}\right\}. \tag{46}$$

Once $v^r(x,t)$ is found, the input reference $\omega_1^r(t)$ is computed from (44) as

$$\omega_1^r(t) = \frac{\frac{1}{2}v^r(1,t)^2 + \left(1-\frac{1}{2}\int_0^1 v^r(y,t)dy\right)v_x^r(1,t)}{\left(1-\frac{1}{2}\int_0^1 v^r(y,t)dy\right)^2}. \tag{47}$$

The formula (47) requires that both the derivative and the integral of the state trajectory $v^r(x,t)$ be known. In the case of the biased sinusoidal output reference (45) they are easily obtainable as

$$v_x^r(x,t) = a\mathrm{Im}\left\{\sqrt{j\omega}\sinh\left(\sqrt{j\omega}x\right)\mathrm{e}^{j\omega t}\right\} \tag{48}$$

and

$$\int_0^x v_x^r(y,t)dy = a\mathrm{Im}\left\{\frac{\sinh\left(\sqrt{j\omega}x\right)\mathrm{e}^{j\omega t}}{\sqrt{j\omega}}\right\} + bx. \tag{49}$$

It can be shown that following result holds for the particular case of the output given by (45).

Theorem 1: The following functions

$$u^r(x,t) = \frac{b + a\mathrm{Im}\left\{\cosh\left(\sqrt{j\omega}x\right)\mathrm{e}^{j\omega t}\right\}}{1-\frac{bx}{2}-\frac{a}{2}\mathrm{Im}\left\{\frac{\sinh\left(\sqrt{j\omega}x\right)\mathrm{e}^{j\omega t}}{\sqrt{j\omega}}\right\}}, \tag{50}$$

$$\omega_1^r(t) = \left(b + \frac{a}{2}\sin\omega t\right)^2, \tag{51}$$

$$\omega_1^r(t) = \frac{1}{\left(1-\frac{b}{2}-\frac{a}{2}\mathrm{Im}\left\{\frac{\sinh\left(\sqrt{j\omega}\right)}{\sqrt{j\omega}}\mathrm{e}^{j\omega t}\right\}\right)^2}$$

$$\times \left[\frac{1}{2}\left(b + a\mathrm{Im}\left\{\cosh\left(\sqrt{j\omega}\right)\mathrm{e}^{j\omega t}\right\}\right)^2 + \left(1 - \frac{b}{2} - \frac{a}{2}\mathrm{Im}\left\{\frac{\sinh\left(\sqrt{j\omega}\right)}{\sqrt{j\omega}}\mathrm{e}^{j\omega t}\right\}\right) \times a\mathrm{Im}\left\{\sqrt{j\omega}\sinh\left(\sqrt{j\omega}\right)\mathrm{e}^{j\omega t}\right\}\right], \quad (52)$$

satisfy the nonlinear PDE

$$u_t^r(x,t) = u_{xx}^r(x,t) - u_x^r(x,t)u^r(x,t), \quad (53)$$
$$u_x^r(0,t) = \omega_0^r(t), \quad u_x^r(1,t) = \omega_1^r(t), \quad (54)$$

and, in particular, $u^r(0,t) = b + a\sin\omega t$.

Remark 5.1: The functions (50)–(52) that solve the trajectory generation problem do not scale linearly with the amplitude a or the bias b. Furthermore, for values of a or b sufficiently large, the possibility exists of these functions taking infinite values for some (x,t) pairs.

B. Trajectory Tracking

Once the trajectory generation problem is solved, we need to find a feedback law that stabilizes the trajectory $u^r(x,t)$ from any initial condition $u(x,0)$ (rather than just generating the desired motion from the special initial condition $u^r(x,0)$).

We introduce the state tracking error $\tilde{v}(x,t) = v(x,t) - v^r(x,t)$. Note that occasionally, particularly for systems that are stable and linear, it suffices to apply the open-loop inputs, and the system's solution will converge to the desired state trajectory. However, in our case, the *linearization* of (39)–(41) around the reference trajectory $v^r(x,t)$ is

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) - \frac{1}{2}u^r(0,t)v^r(x,t)\tilde{v}(0,t), \quad (55)$$
$$\tilde{v}_x(0,t) = -u^r(0,t)\tilde{v}(0,t), \quad (56)$$
$$\tilde{v}_x(1,t) = -\frac{v^r(1,t)}{1-\frac{1}{2}\int_0^1 v^r(y,t)dy}\tilde{v}(1,t) - \frac{\frac{1}{2}\int_0^1 \tilde{v}(y,t)dy}{\left(1-\frac{1}{2}\int_0^1 v^r(y,t)dy\right)^2}\left[-\frac{1}{2}v^r(1,t)^2 + \left(1 - \frac{1}{2}\int_0^1 v^r(y,t)dy\right)v_x^r(1,t)\right]. \quad (57)$$

This complicated linear time-varying PDE system in general will not be exponentially stable, thus we need to develop feedback control laws to stabilize the equilibrium $\tilde{v}(x) \equiv 0$ to zero.

The PDE governing the tracking error $\tilde{v}(x,t)$ is

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) + v(x,t) \times \left(\omega_0(t) - u(0,t)^2/2\right)/2, \quad (58)$$
$$\tilde{v}_x(0,t) = \omega_0(t) - u(0,t)^2/2, \quad (59)$$
$$\tilde{v}_x(1,t) = \left(1 - \frac{1}{2}\int_0^1 v(y,t)dy\right) \times \left(\omega_1(t) - u(1,t)^2/2\right) - v_x^r(1,t). \quad (60)$$

First we choose the control $\omega_0(t)$ as the feedback law

$$\omega_0(t) = 1/2u(0,t)^2, \quad (61)$$

which changes (58)–(59) into

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t), \quad (62)$$
$$\tilde{v}_x(0,t) = 0, \quad (63)$$

while (60) is unchanged. Next, we choose

$$\omega_1(t) = c_1u(1,t) + \frac{1}{2}u(1,t)^2 + \mathrm{e}^{\frac{1}{2}\int_0^1 u(y,t)dy} \times \left(v_x^r(1,t) + c_1v^r(1,t)\right), \quad (64)$$

where c_1 is a positive gain. Using (37) and (38), it is found that this control law transforms (60) into

$$\tilde{v}_x(1,t) = -c_1\tilde{v}(1,t). \quad (65)$$

Therefore the closed-loop $\tilde{v}$-system is turned into a heat equation with one Neumann and one stabilizing Robin boundary condition:

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t), \quad (66)$$
$$\tilde{v}_x(0,t) = 0, \quad \tilde{v}_x(1,t) = -c_1\tilde{v}(1,t). \quad (67)$$

Using a Lyapunov estimate as in [8], we find that the closed-loop $\tilde{v}$ satisfies the following bound

$$\|\tilde{v}(t)\|_{L^2} \le \|\tilde{v}(0)\|_{L^2}\mathrm{e}^{-\tilde{c}t}, \quad (68)$$

for some $\tilde{c} > 0$ (whose exact value is not important).

The solution to the plant state $u(x,t)$ is

$$u(x,t) = \frac{v^r(x,t) + \tilde{v}(x,t)}{1 - \frac{1}{2}\int_0^x v^r(y,t)dy - \frac{1}{2}\int_0^x \tilde{v}(y,t)dy}, \quad (69)$$

where $\tilde{v}(x,t)$ is the solution of (66)–(67) with initial condition $\tilde{v}_0 = u(x,0)\mathrm{e}^{-\frac{1}{2}\int_0^x u(y,0)dy} - v^r(x,0)$. Since $\lim_{t\to\infty}\tilde{v}(x,t) \equiv 0$, we have that $u(x,t)$ converges to

$$u^r(x,t) = \frac{v^r(x,t)}{1 - \frac{1}{2}\int_0^x v^r(y,t)dy}. \quad (70)$$

However, the tracking result fails to be global (i.e., to hold for all initial conditions) because the solution (69) is only valid if the condition

$$\int_0^x v^r(y,t)dy + \int_0^x \tilde{v}(y,t)dy < 2, \quad (71)$$

is verified for all x and t. This condition holds when $u^r(0,t) \equiv 0$, namely, when $v^r(x,t) \equiv 0$ (which is a basic result on stabilizing u around the origin, which is global), however, it does not necessarily hold in the presence of a nonzero trajectory $v^r(x,t)$.

The following theorem describes the behavior of the closed-loop system with our tracking controller.

Theorem 2: Consider the system (1)–(2) with control laws (61) and (64), where $v^r(x,t)$ is a solution of the motion planning problem of Section V-A. Let the reference trajectory $v^r(x,t)$ be bounded for all $x \in [0,1], t \geq 0$, and let $v^r_x(1,t)$ be bounded for all $t \geq 0$. If the following conditions hold

$$\|v^r(t)\|^2_{L^2} \leq 1/4\,, \qquad \forall t \geq 0, \tag{72}$$
$$\|u_0\|^2_{L^2} \leq h^{-1}(1/4)\,, \tag{73}$$

where $h(r) = r\mathrm{e}^{\sqrt{r}}$, then the solution $u(x,t)$ is bounded for all $x \in [0,1]$ and $t \geq 0$, the control inputs $\omega_0(t)$ and $\omega_1(t)$ are bounded for all $t \geq 0$, and the function $\tilde{u}(x,t) = u(x,t) - u^r(x,t)$ converges to zero exponentially in t, for all $x \in [0,1]$.

We apply now Theorem 2 to our example with a sinusoidal output (45) with $b = 0$ (no bias).

Theorem 3: Consider the closed-loop Burgers system (1)–(2) with the controls

$$\omega_0(t) = u(0,t)^2/2 \tag{74}$$
$$\begin{aligned}\omega_1(t) = {} & -c_1 u(1,t) + u(1,t)^2/2 \\ & + \mathrm{e}^{\int_0^1 u(y,t)/2dy} a\mathrm{Im}\left\{ \left(\sqrt{j\omega}\sinh\left(\sqrt{j\omega}x\right) \right.\right. \\ & \left.\left. + c_1\cosh\left(\sqrt{j\omega}x\right)\right)\mathrm{e}^{j\omega t}\right\}\end{aligned} \tag{75}$$

If $\|u_0\|^2_{L^2} \leq h^{-1}(1/4)$ and

$$a \leq a_{\max}(\omega) = \frac{1}{8}\sqrt{\frac{2\omega}{\cosh\sqrt{2\omega} - \cos\sqrt{2\omega}}} \tag{76}$$

where $a_{\max}(\omega)$ is a positive, decreasing function with $a_{\max}(0) = 1/8$, then the state and the control inputs remain bounded and the state $u(x,t)$ converges to

$$u^r(x,t) = \frac{a\mathrm{Im}\left\{\cosh\left(\sqrt{j\omega}x\right)\mathrm{e}^{j\omega t}\right\}}{1 - \frac{1}{2}a\mathrm{Im}\left\{\frac{1}{\sqrt{j\omega}}\sinh\left(\sqrt{j\omega}x\right)\mathrm{e}^{j\omega t}\right\}}, \tag{77}$$

which means, in particular, that $u(0,t)$ converges to $u^r(0,t) = a\sin\omega t$.

C. Example

We illustrate the solution to the trajectory tracking problem for the output reference $u^r(0,t) = b + a\sin\omega t$ from Theorem 1. The explicit state trajectory is

$$\begin{aligned}u^r(x,t) = {} & \left\{ b + \frac{a}{2}\left[\mathrm{e}^{\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t + \sqrt{\frac{\omega}{2}}x\right)\right.\right. \\ & \left.\left. + \mathrm{e}^{-\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t - \sqrt{\frac{\omega}{2}}x\right)\right]\right\} \\ & \times\left\{1 - \frac{b}{2}x - \frac{a}{4\sqrt{\omega}}\right. \\ & \times\left[\mathrm{e}^{\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t - \frac{\pi}{4} + \sqrt{\frac{\omega}{2}}x\right)\right. \\ & \left.\left. + \mathrm{e}^{-\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t - \frac{\pi}{4} - \sqrt{\frac{\omega}{2}}x\right)\right]\right\}^{-1}.\end{aligned} \tag{78}$$

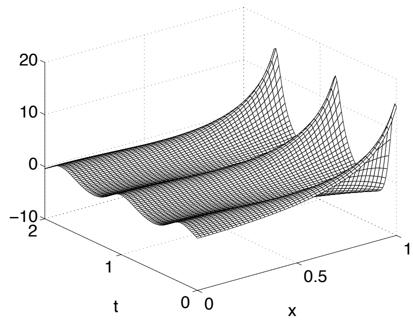

Fig. 3. Solution to the nonlinear trajectory generation problem for a sinusoidal reference.

The presence of the b term in the denominator of (78) will increase the possibility of a blow-up of the trajectory when $b > 0$, however $b < 0$ will help to keep the denominator away from 0. Figure 3 shows the plot of the solution to the nonlinear trajectory generation problem. We don't show a simulation for tracking as it simply represents convergence to the trajectory in Figure 3.

REFERENCES

[1] A. Balogh and M. Krstic, "Burgers' equation with nonlinear boundary feedback: H^1 stability, well posedness, and simulation," *Math. Probl. Eng.*, vol. 6, pp. 189-200, 2000.

[2] J. A. Burns, A. Balogh, D. S. Gilliam, V. I. Shubov, "Numerical Stationary Solutions for a Viscous Burgers' Equation," *Journal of Mathematical Systems, Estimation, and Control*, Vol. 8(2), 1998.

[3] C. I. Byrnes, D. S. Gilliam, V. I. Shubov, "On the global dynamics of a controlled viscous Burgers equation," *J. Dynam. Control Systems*, vol. 4 (4), pp. 457–519, 1998.

[4] E. Fernandez-Cara and S. Guerrero, "Null controllability of the Burgers' equation with distributed controls," *Syst. Contr. Lett.*, vol. 56, pp. 366–372, 2007.

[5] S. Guerrero and O. Y. Imanuvilov, "Remarks on global controllability for the Burgers' equation with two control forces," preprint, 2006.

[6] B. Laroche, P. Martin and P. Rouchon, "Motion planning for the heat equation," *Int. J. Rob. Nonl. Contr.*, vol. 10, pp. 629–643, 2000.

[7] M. Krstic, "On global stabilization of Burgers' equation by boundary control," *Syst. Contr. Lett.*, vol. 37, pp. 123–141, 1999.

[8] M. Krstic, L. Magnis, and R. Vazquez, "Nonlinear control of the Burgers PDE— Part I: Full-state stabilization," *2008 ACC*, 2008.

[9] W.-J. Liu and M. Krstic, "Backstepping boundary control of Burgers' equation with actuator dynamics," *Syst. Contr. Lett.*, vol. 41, pp. 291-303, 2000.

[10] A. Smyshlyaev and M. Krstic, "Closed form boundary state feedbacks for a class of partial integro-differential equations," *IEEE Tran. Aut. Contr.*, vol. 49, pp. 2185–2202, 2004.

[11] A. Smyshlyaev and M. Krstic, "Backstepping observers for parabolic PDEs," *Syst. Contr. Lett.*, vol. 54, pp. 1953–1971, 2005.

[12] H. Van Ly, K.D. Mease and E.S. Titi, "Distributed and boundary control of the viscous Burgers equation," *Numer. Funct. Anal. Optim.*, vol. 18, pp. 143–188, 1997.