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Nonlinear Control of the Burgers PDE—Part I: Full-State Stabilization

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Abstract— **We consider the problem of stabilization of unstable "shock-like" equilibrium profiles of the viscous Burgers' equation with actuation at the boundaries. These equilibria are not stabilizable (even locally) using the standard "radiation boundary condition." Using a nonlinear spatially-scaled transformation (that employs three ingredients, of which one is the Hopf-Cole nonlinear integral transformation) and linear backstepping design, we design an explicit nonlinear full-state control law that achieves exponential stability, with a region of attraction for which we give an estimate. The region of attraction is not the entire state space since the Burgers equation is known not to be globally controllable, however, the stability result achieved is stronger than being infinitesimally local. In a companion paper we consider output feedback stabilization, for which we design a nonlinear observer with boundary sensing, and solve the problems of trajectory generation and tracking.**

I. INTRODUCTION

We study a nonlinear control problem for the viscous Burgers equation, which is considered a basic model of nonlinear convective phenomena such as those that arise in Navier-Stokes equations. While the Burgers model is not able to capture the complexity of turbulence, its nonlinearity makes it challenging and a starting point towards developments for the more realistic Navier-Stokes equations. We consider a family of stationary solutions called "shock profiles" [9] (or "shock-like" profiles) which are unstable and not stabilizable (even locally) by simple means such as the standard "radiation boundary conditions." We achieve exponential stabilization (in spatial L^2 norm) of the shock profiles using two control inputs (one at each boundary).

Early results with linear control of the Burgers equation were presented in [4], achieving local stabilization. Optimal control was considered in [7]. In [6] a linear static collocated output feedback (a.k.a. "radiation" boundary conditions) is proved to achieve local L^2 exponential stability. In [20] this result is improved to L^∞, but remains local. Global stabilization is achieved in [13], using nonlinear boundary conditions, and extended to KdV, Kuramoto-Sivashinsky, adaptive control, and other problems in [1], [2], [15], [16], [17], [18]. References [10] and [11] study the null controllability of the Burgers equation with one and two inputs respectively, deriving bounds on the minimal time of controllability. In [3] the stabilization problem is solved using nonlinear model reduction techniques, with in-domain actuation.

Our recent design for nonlinear parabolic PDEs [22], [23] (without convective nonlinearities) is based on a feedback linearizing nonlinear transformation in the form of a Volterra series and has opened the avenue for fully nonlinear designs for PDEs. In this paper we follow a conceptually similar strategy (though rather different in its execution) and find a nonlinear spatially-scaled transformation (based on three ingredients, one of which is the Hopf-Cole nonlinear transformation [8]), that transforms the system (with the help by one of the two boundary controls) into a linear reaction-diffusion PDE. This PDE is stabilized using the linear backstepping approach [19], yielding a control law that is nonlinear in the original state variable. We provide an estimate of the region of attraction for the closed-loop system. This estimate is finite because the Burgers system unfortunately is not globally controllable [10], [11].

We illustrate our theoretical results with numerical examples. It is first shown, via a study of eigenvalues and via simulations of the nonlinear system, that the "radiation feedback" is not stabilizing, even locally, for sufficiently large shock profiles. The stabilization properties of the feedback laws are illustrated in simulations.

In a companion paper [14] we pursue output feedback stabilization, for which we use a nonlinear observer that uses injection of the output estimation error at one of the boundaries. We also solve the problem of trajectory generation and tracking for the Burgers equation.

II. BURGERS EQUATION AND ITS SHOCK PROFILES

Consider the viscous Burgers' equation

$$u_t = u_{xx} - u_x u, \quad x \in [0,1] \tag{1}$$

where $u(x,t)$ is the state, with boundary conditions

$$u_x(0,t) = \omega_0(t), \quad u_x(1,t) = \omega_1(t), \tag{2}$$

where $\omega_0(t)$ and $\omega_1(t)$ are the control inputs. In the sequel we drop the arguments (x,t) whenever the context allows to do this without harming clarity.

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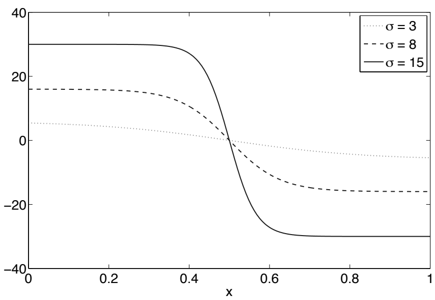

Fig. 1. Three example shock profiles for different values of σ.

We are interested in the family of "shock-like" stationary solutions determined in [5] as

$$U(x) = -2\sigma \tanh\left(\sigma\left(x - 1/2\right)\right), \tag{3}$$

which are parameterized by the variable $\sigma \geq 0$. The family of solutions given by (3) is called shock profiles since, the more positive σ is, the more "shock-like" $U(x)$ looks, as illustrated in Fig. 1.

Remark 2.1: We point out that all the stabilization and observer design results in this paper can be derived for any other continuous equilibrium profile $U(x)$. However, we concentrate on the symmetric profiles (3) as the most interesting class of equilibria, where the open-loop instability is the most pronounced. We also point out that, while the Burgers equation is often studied in the form $u_t = \varepsilon u_{xx} - u_x u$ for a non-unity viscosity parameter ε, to reduce the notational burden, in this paper we study the problem with $\varepsilon = 1$ only. All the results in the paper can be derived for any $\varepsilon \neq 1$ and the dependence on the ε is non-essential. The equilibrium profile $U(x) = -2\varepsilon\sigma \tanh\left(\sigma\left(x - 1/2\right)\right)$ scales with ε, which results in the open-loop eigenvalues simply scaling with ε. The primary effect of $\varepsilon \neq 1$ on the "nonlinear aspects" of the results is that it affects the actual size of the region of attraction but not the fundamental form of the estimate of its size.

Continuing with (3), we first observe that

$$U'(x) = -2\sigma^2\left(1 - \tanh^2\left(\sigma\left(x - 1/2\right)\right)\right), \tag{4}$$

hence

$$U'(0) = U'(1) = -2\sigma^2\left(1 - \tanh^2(\sigma/2)\right). \tag{5}$$

Then from (2) we obtain that

$$\omega_0 = \omega_1 = -2\sigma^2\left(1 - \tanh^2(\sigma/2)\right) \leq 0 \tag{6}$$

are the constant values of the open-loop control laws $\omega_0(t)$ and $\omega_1(t)$ that produce the equilibrium profile (3).

Let us denote the fluctuation variable around the shock profile as $\tilde{u}(x,t) = u(x,t) - U(x)$, where $U(x)$ is

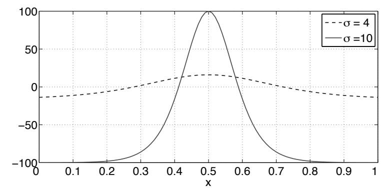

Fig. 2. Value of the reaction coefficient in (12). It is positive and large in the region around $x = 1/2$, potentially destabilizing the system.

defined as in (3). Then the Burgers equation written in the perturbation variable $\tilde{u}$ is

$$\tilde{u}_t = \tilde{u}_{xx} - U(x)\tilde{u}_x - U'(x)\tilde{u} - \tilde{u}_x\tilde{u}. \tag{7}$$

Let us denote $\tilde{\omega}_0(t) = \omega_0(t) - U'(0), \tilde{\omega}_1(t) = \omega_1(t) - U'(1)$. Then the boundary conditions for (7) are

$$\tilde{u}_x(0,t) = \tilde{\omega}_0(t), \quad \tilde{u}_x(1,t) = \tilde{\omega}_1(t). \tag{8}$$

III. Instability of Shock Profiles in Open Loop and Under "Radiation Boundary Feedback"

To study the stability of the origin of the open-loop system (7), we linearize it, obtaining

$$\theta_t = \theta_{xx} + 2\sigma\left(\tanh\left(\sigma\left(x - 1/2\right)\right)\theta\right)_x, \tag{9}$$
$$\theta_x(0) = \theta_x(1) = 0, \tag{10}$$

where $\theta(x,t)$ is the linearization of $\tilde{u}$. Equation (9) is an advection-reaction-diffusion PDE. To simplify the study of stability, we eliminate the advection term using the invertible transformation $\zeta(x,t) = G(x)\theta(x,t)$, where

$$G(x) = \frac{\cosh\left(\sigma(x - 1/2)\right)}{\cosh(\sigma/2)}, \tag{11}$$

which transforms (9)–(10) into

$$\zeta_t = \zeta_{xx} + \sigma^2\left[\frac{2}{\cosh^2\left(\sigma\left(x - 1/2\right)\right)} - 1\right]\zeta, \tag{12}$$
$$\zeta_x(0) = -\sigma\tanh(\sigma/2)\zeta(0), \tag{13}$$
$$\zeta_x(1) = \sigma\tanh(\sigma/2)\zeta(1). \tag{14}$$

For $\sigma = 0$ the system is neutrally stable. For $\sigma > 0$, in addition to the boundary conditions being of destabilizing ("anti-radiation") type, the reaction term in (12) is also destabilizing in the vicinity of $x = 1/2$. This is shown in Fig. 2. The larger the value of $\sigma > 0$, the more positive the first eigenvalue of (12)–(14) becomes. For example, for $\sigma = 15$ the first eigenvalue is $+0.6$.

Remark 3.1: With "radiation boundary feedback,"

$$\tilde{\omega}_0(t) = k\tilde{u}(0,t), \quad \tilde{\omega}_1(t) = -k\tilde{u}(1,t), \quad k > 0, \tag{15}$$

motivated by [6], the system (12)–(14) changes only in its boundary conditions,

$$\zeta_x(0) = \left(k - \sigma\tanh(\sigma/2)\right)\zeta(0), \tag{16}$$
$$\zeta_x(1) = -\left(k - \sigma\tanh(\sigma/2)\right)\zeta(1). \tag{17}$$

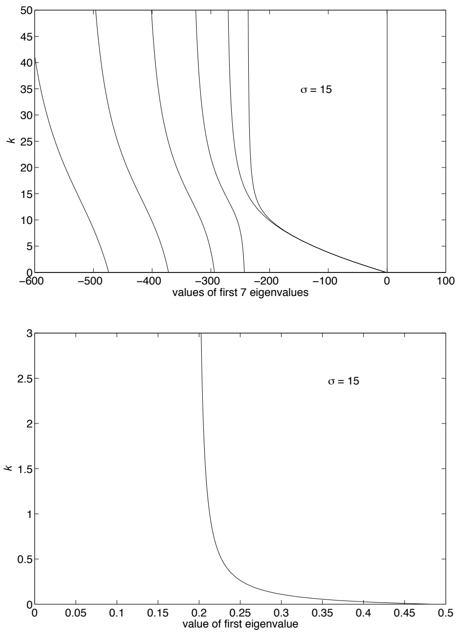

Fig. 3. Top: Values of the 7 rightmost eigenvalues of (12), (16), (17) for growing values of k and fixed $\sigma = 15$. Notice that all eigenvalues are real and move leftward as k increases. Bottom: Detail for the first eigenvalue, which remains positive for any value of k.

The stability properties of the system (12), (16), (17) improve as $k \to +\infty$. However, numerical eigenvalue calculation shows that for $\sigma > \sigma^*$, where $\sigma^* \approx 10$, the system always has exactly one unstable eigenvalue, so no value of k exists that stabilizes the system, see Fig. 3. The same negative result applies to the "nonlinear radiation boundary feedback" in [13]. So, a more sophisticated form of feedback, either full-state feedback or dynamic output-feedback, is needed to stabilize the shock-like equilibria (even locally).

IV. Full State Feedback Law

We perform a two-step feedback linearizing design. In the first step, we use a state transformation and a feedback law for $\tilde{\omega}_0$ to linearize the transformed PDE and its boundary condition at $x = 0$. In the second step we design a feedback law for $\tilde{\omega}_1$ to stabilize the resulting linear system using the backstepping method.

A. Linearizing transformation and the design of $\tilde{\omega}_0$

Define a new state variable $v(x,t)$ as follows:

$$v(x,t) = \tilde{u}(x,t)\mathrm{e}^{-\frac{1}{2}\int_0^x [\tilde{u}(y,t)+U(y)]\,dy}, \tag{18}$$

which can be written using (11) as

$$v(x,t) = G(x)\tilde{u}(x,t)\mathrm{e}^{-\frac{1}{2}\int_0^x \tilde{u}(y,t)\,dy}. \tag{19}$$

Remark 4.1: Transformation (19) is a composition of the Hopf-Cole transformation on $\tilde{u}(x,t)$ and the transformation ∂_x, scaled by $G(x)$. The former is used to study the solutions of Burgers' equation [8]. The transformation ∂_x was used in [22] to design a feedback control law for a semilinear parabolic system. The scaling $G(x)$ is the standard "gauge" transformation used to eliminate advection terms in parabolic PDEs.

The transformation (19) from $\tilde{u}$ to v is invertible:

$$\tilde{u}(x) = \frac{v(x)/G(x)}{1 - \frac{1}{2}\int_0^x \frac{v(y)}{G(y)}\,dy}. \tag{20}$$

Substituting the transformation (19) into the system (7)–(8) we obtain that v verifies the following equations:

$$\begin{aligned} v_t &= v_{xx} - \left(U'(x) + \sigma^2\right) v \\ &\quad +1/2\left(\tilde{\omega}_0 - U(0)\tilde{u}(0) - \tilde{u}^2(0)/2\right) v, \qquad (21)\\ v_x(0) &= \tilde{\omega}_0 - \frac{1}{2}\left(\tilde{u}(0) + U(0)\right)\tilde{u}(0), \qquad (22)\\ v_x(1) &= \left(\tilde{\omega}_1 - \frac{1}{2}\left(\tilde{u}(1) + U(1)\right)\tilde{u}(1)\right) \\ &\quad \times \mathrm{e}^{-\frac{1}{2}\int_0^1 [\tilde{u}(y,t)+U(y)]\,dy}. \qquad (23) \end{aligned}$$

Setting the feedback law

$$\begin{aligned} \tilde{\omega}_0 &= U(0)\tilde{u}(0) + \frac{\tilde{u}^2(0)}{2} \\ &= 2\sigma\tanh(\sigma/2)\tilde{u}(0) + \tilde{u}^2(0)/2, \qquad (24) \end{aligned}$$

we obtain the following system for v, in which we express all the coefficients as explicitly as possible:

$$v_t = v_{xx} + \sigma^2\left[\frac{2}{\cosh^2\left(\sigma\left(x - 1/2\right)\right)} - 1\right] v, \tag{25}$$

with boundary conditions

$$\begin{aligned} v_x(0) &= \sigma\tanh(\sigma/2)v(0), \qquad (26)\\ v_x(1) &= \sigma\tanh(\sigma/2)v(1) + \left(\tilde{\omega}_1 - \tilde{u}(1)^2/2\right) \\ &\quad \times\left(1 - \frac{1}{2}\int_0^1 \frac{\cosh(\sigma/2)v(y)}{\cosh\left(\sigma\left(y - 1/2\right)\right)}dy\right). \qquad (27) \end{aligned}$$

Notice that (25) is a *linear* reaction-diffusion parabolic equation with the same destabilizing reaction coefficient that the linearized system (12) had.

B. Design of $\tilde{\omega}_1$ using the backstepping method

To find a feedback law for $\tilde{\omega}_1$ that stabilizes (25)–(27), we use the backstepping method for 1-D parabolic equations [19]. We define a new state $w(x,t)$, which is obtained from v using the transformation

$$w(x,t) = v(x,t) - \int_0^x k(x,y)v(y,t)\,dy. \tag{28}$$

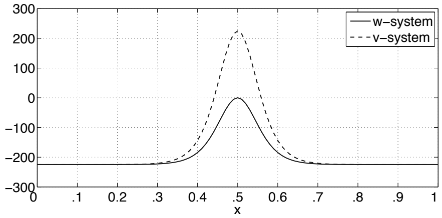

Fig. 4. Value of the target system reaction coefficient in (29), compared with the coefficient of the v plant in (25), for $\sigma = 15$. Notice that the coefficient only has been changed close to $x = 1/2$, where it was destabilizing. The new coefficient is negative everywhere.

The state w satisfies the following PDE:

$$
\begin{aligned}
w_t &= w_{xx} + \sigma^2 \left[\frac{1}{\cosh^2(\sigma(x-1/2))} - 1 \right] w \\
&\quad -cw, && (29) \\
w_x(0) &= \sigma \tanh(\sigma/2) w(0), && (30) \\
w_x(1) &= -\sigma \tanh(\sigma/2) w(1), && (31)
\end{aligned}
$$

where $c \geq 0$ is a parameter of the controller. The system (29)–(31) will be referred to as the target system.

Remark 4.2: The reaction term in (29) is non-positive, so the w-system is exponentially stable. Note that we have not completely eliminated the reaction term from the original system (25) but only lowered it to eliminate its positive part, without wasting control effort on changing its negative part, see Fig. 4.

Remark 4.3: The coefficient c in (29) can be set to zero whenever $\sigma > 0$. When $\sigma = 0$, we need $c > 0$ because the resulting system $w_t = w_{xx} - cw, w_x(0,t) = w_x(1,t) = 0$ would be only neutrally stable for $c = 0$.

We need to determine the kernel $k(x,y)$ in (28) so that w verifies (29)–(31). Following [19], we find that the kernel k has to verify the following equation:

$$
\begin{aligned}
k_{xx} &= k_{yy} + \sigma^2 \left[1 - 2\tanh^2\left(\sigma(y-1/2)\right) \right. \\
&\quad \left. + \tanh^2\left(\sigma(x-1/2)\right) \right] k + ck, && (32) \\
k(x,x) &= -\sigma/2 \left[\tanh\left(\sigma\left(x-1/2\right)\right) \right. \\
&\quad \left. + \tanh\left(\sigma/2\right) \right] - cx/2, && (33) \\
k_y(x,0) &= \sigma \tanh\left(\sigma/2\right) k(x,0), && (34)
\end{aligned}
$$

which is a linear hyperbolic PDE in the domain $\mathcal{T} = \{(x,y) : 0 \leq y \leq x \leq 1\}$. In [19] it is shown that (32)–(34) is well-posed and that $k \in \mathcal{C}^2(\mathcal{T})$. The kernel k can be computed numerically or symbolically [19].

From (31), (28) and (27) we find the control law

$$
\begin{aligned}
\tilde{\omega}_1(t) &= \frac{\tilde{u}(1,t)^2}{2} + \left(k(1,1) - 2\sigma\tanh\left(\sigma/2\right)\right)\tilde{u}(1,t) \\
&\quad + \int_0^1 \left(k_x(1,y) + \sigma\tanh\left(\sigma/2\right)k(1,y)\right) \\
&\quad \times G(y) \mathrm{e}^{\int_y^1 \tilde{u}(\xi,t)d\xi} \tilde{u}(y,t)dy. && (35)
\end{aligned}
$$

V. Closed-Loop Stability Under Full-State Control Law

Denote the initial condition as $\tilde{u}_0(x) = \tilde{u}(x,0)$, and define a class $\mathcal{K}_\infty$ [12] function

$$
g(r) = r/2\mathrm{e}^{r/2}. \qquad (36)
$$

Theorem 1: Assume that $\tilde{u}_0 \in H^2$ is such that

$$
\|\tilde{u}_0\|_{L^2} < g^{-1}\left(1/m\sqrt{\sigma/\sinh\sigma}\right) \qquad (37)
$$

and that it verifies the compatibility conditions

$$
\begin{aligned}
\tilde{u}_0'(0) &= 2\sigma\tanh(\sigma/2)\tilde{u}_0(0) + \tilde{u}_0^2(0)/2, && (38) \\
\tilde{u}_0'(1) &= \tilde{u}_0(1)^2/2 + \left(k(1,1) - 2\sigma\tanh(\sigma/2)\right)\tilde{u}_0(1) \\
&\quad + \int_0^1 \left(k_x(1,y) + \sigma\tanh\left(\sigma/2\right)k(1,y)\right) \\
&\quad \times G(y)\mathrm{e}^{\int_y^1 \tilde{u}_0(\xi)d\xi}\tilde{u}_0(y)dy. && (39)
\end{aligned}
$$

Then the equilibrium profile $\tilde{u} \equiv 0$ of system (7)–(8) with feedback laws (24) and (35) is exponentially stable in the L^2 norm, i.e., for all $t > 0$,

$$
\|\tilde{u}(t)\| \leq \frac{\sqrt{2\sigma\coth(\sigma/2)}g\left(\|\tilde{u}_0\|_{L^2}\right)}{\frac{1}{m}\sqrt{\sigma/\sinh\sigma} - g\left(\|\tilde{u}_0\|_{L^2}\right)}\mathrm{e}^{-\alpha t}, \qquad (40)
$$

where $m, \alpha > 0$.

Remark 5.1: Theorem 1 is a statement of regional (less than global, more than infinitesimally local) stability in the L^2 space.

VI. Simulations

A. Open-loop system

The open-loop system (1)–(2), (6), is unstable, as mentioned in Section III. A numerical study of the linearized system around the shock profile shows the presence of one positive (though possibly small) eigenvalue for any $\sigma > 0$. In Fig. 5 one can see a finite time blow up of the open-loop system for $\sigma = 3$.

B. Radiation boundary feedback

In Remark 3.1 we explained that the shock profiles are not stabilizable (even locally) by "radiation feedback" for $\sigma > \sigma^*$, where $\sigma^* \approx 10$ (obtained numerically). The lack of stabilizability by "radiation feedback" is consistent with its objective being merely the regulation of the scalar outputs $\tilde{u}(0,t)$ and $\tilde{u}(1,t)$ to zero, rather than stabilization of the entire state $\tilde{u}(x,t)$. For small σ, the latter may be achieved as a bonus, a consequence of stability of the underlying "zero dynamics." However, for large σ the zero dynamics become unstable, and thus regulation of the output is not accompanied by closed-loop stability of the state.

Despite the instability of the linearized Burgers equation with the "radiation feedback" at the shock profiles, things are not so bad in fully nonlinear simulations.

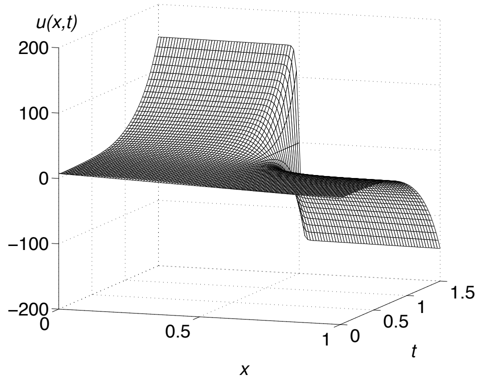

Fig. 5. Finite time blow-up of open-loop system (with constant inputs (6)) for $\sigma = 3$ and $u_0(x) = U(0) + 2 + (U(1) - U(0) - 4)x$.

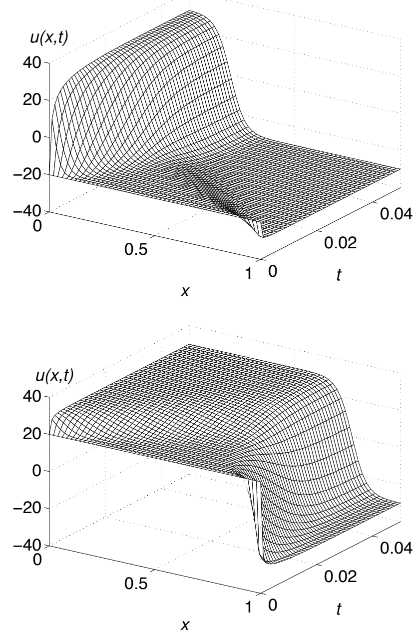

Fig. 6. Behavior for radiation feedback with negative (top) and positive (bottom) constant initial condition for $\sigma = 15$ and $k = 100$.

The nonlinearities prevent finite time blow-up (rather than causing it) and the closed-loop solutions merely converge to a different equilibrium (rather than the desired "shock profile" equilibrium). In Fig. 6 one can see the convergence of u to two different non-symmetric equilibria, which seem to (nearly) match the desired shock profile at the boundaries but not elsewhere.

In simuations we find that for odd initial conditions around $x = 1/2$ radiation feedback (15) achieves convergence to $U(x)$. However, for non-symmetric initial conditions, the system goes to other stationary solutions.

C. Backstepping feedback

Before showing the results with our nonlinear design, we show the results with a linearized backstepping controller

$$\tilde{\omega}_0(t) = 2\sigma \tanh(\sigma/2)\, \tilde{u}(0,t) \qquad (41)$$

$$\begin{aligned} \tilde{\omega}_1(t) &= -\left(3\sigma \tanh(\sigma/2) + c/2\right) \tilde{u}(1,t) \\ &\quad - \int_0^1 \rho(y) G(y) \tilde{u}(y,t) dy\,. \end{aligned} \qquad (42)$$

The kernel $-\rho(x)$ and a closed-loop solution with the controller (42), (43) are shown in Figure 7. The radiation feedback does not stabilize the desired equilibrium profile in this case, as shown in Figure 6.

In Fig. 8 we show solutions from various initial conditions under the *nonlinear* backstepping controller.

VII. Conclusions

We have solved the problem of full-state stabilization for the fully nonlinear viscous Burgers equation (see a companion paper [14] where we solve the problems of observer design, output feedback stabilization, trajectory generation, and tracking) . Our results are based on a nonlinear feedback linearizing transformation, which allows us to use the linear backstepping control design

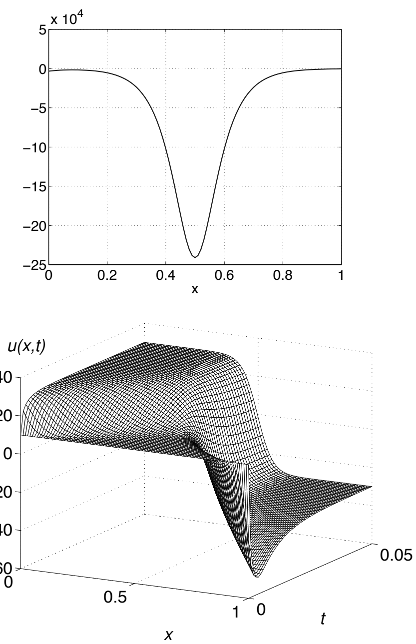

Fig. 7. The backstepping kernel $-\rho(x)$ (top) and the closed-loop solution under the linear backstepping feedback (bottom) for $\sigma = 15$.

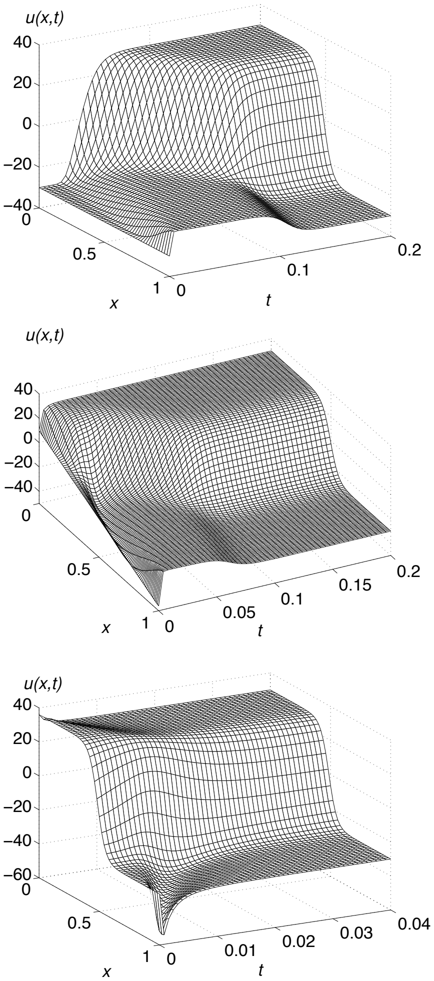

Fig. 8. Convergence of the closed-loop system under the nonlinear full-state feedback law for various initial conditions and $\sigma = 15$.

method [19] . Due to the explicit nature of the transformation and our methods, we were able to derive formulas for the feedback laws. Since our transformation is not globally invertible, our results are not global, which is consistent with lack of global controllability shown in [10], [11], however we derive a non-infinitesimal region of attraction, using the largest level set of our Lyapunov function within the regions of feasibility of the control law and invertibility of the transformation.

Particularly interesting problems for future research include a possible design using only one control input, $\tilde{\omega}_1$ (with $\tilde{\omega}_0$ set to zero), an extension to more general nonlinear parabolic PDEs with convective nonlinearities and to convective nonlinearities of more general form, to Burgers equations in higher dimensions, and to more challenging PDEs with convective nonlinearities such as Kuramoto-Sivashinsky and Navier-Stokes.

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