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Robust-Control-Relevant Coprime Factor Identification: A Numerically Reliable Frequency Domain Approach

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Abstract— **In approximate identification, the goal of the model should be taken into account when evaluating model quality. The purpose of this paper is the development of a system identification procedure, resulting in model sets that are suitable for subsequent robust control design. Incorporation of control relevance in the procedure results in a closed-loop frequency response-based multivariable system identification procedure. The model is represented as a coprime factorization, enabling the usage of stable model perturbations. The main result is the direct estimation of control-relevant coprime factors, exploiting knowledge of a stabilizing controller during the identification experiment. A numerically reliable iterative algorithm is devised, which is illustrated by means of experimental results.**

I. INTRODUCTION

When evaluating model quality, the intended goal of the model should always be taken into account. In fact, any model is a simplification of reality and can only reproduce certain phenomena accurately. Whether a certain model is useful thus depends on its purpose.

If the goal of the approximate model is subsequent control design, then the model should only represent the phenomena that are relevant for control. In fact, low complexity models are highly desirable in control design since the complexity of the model commonly dictates the complexity of the resulting controller. In addition, a robust feedback control design can cope with large systematic modeling errors in certain frequency ranges [1], enabling the use of low complexity models.

High performance robust feedback control design requires a control-relevant model set, however, such model sets are not delivered by standard system identification procedures. Typical model sets for robust control consist of a nominal model with a certain perturbation model [2], [3]. In this paper, robust control-relevant model sets are investigated, where both the control performance and the size of the perturbation model are measured in the $\mathcal{H}_\infty$-norm.

In [4], [5], methods are presented that directly deliver nominal models with a guaranteed error bound in $\mathcal{H}_\infty$. The resulting models can be used directly in $\mathcal{H}_\infty$-optimal robust control. However, these methods deliver high order nominal models, especially in case of lightly damped systems [6]. To obtain a controller of restricted complexity, a low order nominal model is desirable.

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A model set of restricted complexity can be obtained if a low order nominal model is first estimated, followed by an error-modeling step, *e.g.*, [7], [8]. To ensure the model set is control-relevant, both the nominal model and model error should be control relevant. Control-relevant nominal models [9] typically involve iterative procedures, alternating between closed-loop identification and control design. If the nominal model is identified in closed-loop, then it is most natural to determine the modeling error under the same operating conditions, *i.e.*, from closed-loop data.

Determining the model error in closed-loop requires precautions regarding the structure of the perturbation model. Typical perturbation models, including $\mathcal{H}_\infty$-norm bounded perturbations, correspond to stable operators. If identification is performed in closed-loop, it is generally impossible to determine whether the open-loop system is stable, excluding the usage of standard model uncertainty structures [2], *e.g.*, additive perturbations models at the plant output. This fact has also been observed in [10], [11], [12], [13], where the plant is represented as a coprime factorization, and stable perturbations on stable factors are considered.

Identification of coprime factors followed by error modeling is a sensible approach to merge identification and robust control. In [14] and [15], coprime factors are identified in the time and frequency domain, respectively. In [16] and [17], [18] the coprime factor identification approach is further refined in the time and frequency domain, resulting in normalized coprime factors. Normalized coprime factors play a central role in certain robust control methodologies [10]. Indeed, estimation of normalized coprime factors corresponds to a general type of uncertainty, resulting in a small distance between the designed and the achieved loop [1]. However, the fact that a known controller has stabilized the system and has achieved a certain performance during the identification experiment, is not exploited in a normalized coprime factor domain. This controller is often 'close' to optimal, hence this knowledge should be exploited if a high performance control design is pursued.

The main contributions of this paper include a new definition of control relevant coprime factorizations (Section II and Section III). These coprime factorizations are not necessarily normalized and can be identified directly from data, *i.e.*, no iterations as in [16] are required, since only knowledge of the controller that is used during identification is required. In fact, a nonparametric frequency response representation of these coprime factors can be determined directly from closed-loop frequency response data (Section IV). In addition, a novel tailor-made parametrization for coprime fac-

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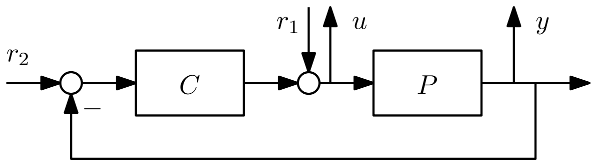

Fig. 1. Feedback configuration.

torizations is introduced (Section V), connecting requirements regarding coprimeness and control-relevance. This parametrization extends the results in [16], [14], where these conditions are separated. Indeed, separation of the conditions results in a cumbersome optimization problem. Finally, a numerically reliable procedure for frequency response estimation is presented (Section VI). Addressing the numerical properties is essential in frequency response-based identification of multivariable systems, since frequency response-based identification is notoriously ill-conditioned [19]. The resulting procedure is illustrated by experimental results (Section VII), followed by conclusions (Section VIII).

Notation. The considered feedback configuration is depicted in Figure 1, where $C, P \in \mathcal{R}$ denote the controller and plant, respectively, and $\mathcal{R}$ is the set of finite dimensional real-rational transfer function matrices. Arguments are often omitted if these are clear from the context. It is assumed that both C and P are proper and the feedback system is well-posed [3]. The closed-loop transfer function matrix $T(P,C)$ is given by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} (I+CP)^{-1} \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} = T(P,C) \begin{bmatrix} r_2 \\ r_1 \end{bmatrix}, \quad (1)$$

where r_1, u and r_2, y are n_u and n_y dimensional signals, respectively. If $T(P,C) \in \mathcal{RH}_\infty$, then C internally stabilizes P. The pair $\{N, D\}$ denotes a Right Coprime Factorization (RCF) of P if D is invertible, $N, D \in \mathcal{RH}_\infty$, $P = ND^{-1}$, and $\exists X_r, Y_r \in \mathcal{RH}_\infty$ such that the Bézout identity $X_r D + Y_r N = I$ holds. The pair $\{\bar{N}, \bar{D}\}$ is said to be a Normalized RCF (NRCF) if it is an RCF and in addition $\bar{D}^* \bar{D} + \bar{N}^* \bar{N} = I$. Dual definitions hold for Left Coprime Factorizations (LCFs) and Normalized LCFs (NLCFs), see [20]. Throughout, N and D are used exclusively to denote rational coprime factorizations over $\mathcal{RH}_\infty$. Occasionally, P is represented by the polynomial Right Matrix Fraction Description (RMFD) $P = BA^{-1}$, $B, A \in \mathbb{R}[\xi]$, *i.e.*, polynomial matrices of appropriate sizes, see [21].

II. Problem formulation

The problem considered in this paper can be summarized as follows. Given the true plant P_o, stabilized by a controller C^{exp}, design a controller C^{opt} that achieves optimal performance $J(P_o, C)$. Firstly, the criterion $J(P_o, C)$ is defined in Section II-A, whereas the related identification problem is elaborated on in Section II-B.

A. Control goal

The considered criterion is a weighted $\mathcal{H}_\infty$-norm, given by

$$J(P_o, C) = \|WT(P_o, C)V\|_\infty, \quad (2)$$

where $T(P,C)$ is defined in (1) and $W = \mathrm{diag}(W_y, W_u)$ and $V = \mathrm{diag}(V_2, V_1)$ are bistable weighting filters of appropriate sizes. The reason for considering the $\mathcal{H}_\infty$-norm is twofold. Firstly, the $\mathcal{H}_\infty$-norm is an induced norm, enabling the incorporation of model uncertainty. Secondly, the $\mathcal{H}_\infty$-norm enables the usage of general performance specifications, including loopshaping design techniques [22], [10], [23]. In virtue of the discrete time nature of many system identification techniques and controller implementations, it has been motivated in [24] that discrete time systems should be considered in (2) for optimal sampled-data control. Hence, the norm in (2) should be interpreted as a discrete time system norm. The discrete time nature of systems is tacitly assumed throughout this paper, generalization to the continuous time case is conceptually straightforward.

B. Control-relevant identification of model sets

The criterion (2) depends on the true plant, hence knowledge of P_o is required to compute C^{opt}. This knowledge of P_o is reflected by a plant model $\hat{P}$. First, identification of a nominal model is considered, followed by a discussion on error modeling.

1) Control-relevant identification: Since $\hat{P}$ is an approximation of P_o, the goal of the model should be taken into account in the estimation criterion. In case identification is performed in closed-loop, a particular controller C^{exp} is known that is stabilizing and achieves reasonable performance. This knowledge can be exploited in the control-relevant estimation of a plant model as explained next.

The optimization of the criterion (2) is solved by employing a model $\hat{P}$ of P_o. Rewriting the criterion and applying the triangle inequality yields

$$\begin{aligned} &\|WT(P_o, C)V\|_\infty \leq \\ &\|WT(\hat{P}, C)V\|_\infty + \|W(T(P_o, C) - T(\hat{P}, C))V\|_\infty. \end{aligned} \quad (3)$$

The first term on the right hand side of (3) involves a model-based control design, whereas the second term amounts to a control-relevant identification problem of a nominal model. In view of (2), the control-relevant identification involves an optimization over both C and $\hat{P}$, *i.e.*, control-relevant estimation of $\hat{P}$ depends on C^{opt}. Approximating C^{opt} by C^{exp} yields the following control-relevant identification criterion, which is the main problem that is considered in this paper.

Definition 1 *The control-relevant identification criterion is given by*

$$\min_{\hat{P}} \|W(T(P_o, C^{\mathrm{exp}}) - T(\hat{P}, C^{\mathrm{exp}}))V\|_\infty. \quad (4)$$

This approximation is justified if C^{exp} is sufficiently close to C^{opt}. If the distance between C^{exp} and C^{opt} is large, (3) can be used as a basis for iterative identification and control [9].

2) Control-relevant error modeling: As motivated in Section I, the main reason to consider coprime factorizations is to enable uncertainty modeling. In the next section, (4) is replaced by an equivalent criterion over coprime factors.

Precisely, $P_o = N_o D_o^{-1}$ is approximated by $\hat{P} = \hat{N}\hat{D}^{-1}$, where $\{N_o, D_o\}$, $\{\hat{N}, \hat{D}\}$ are RCFs.

By considering a perturbation Δ on the RCFs $\{\hat{N}, \hat{D}\}$, there always exists a $\Delta \in \mathcal{RH}_\infty$ such that P_o is in the uncertain model set [1], [20]. In fact, coprime factor models can be further refined to guarantee that all models are stabilized by $C^{\exp}$. This refinement is known as the dual-Youla parameterization, see [25], and introduces candidate plant models most parsimoniously.

In this paper the freedom in the coprime factors is exploited to enable error modeling in the same domain as identification of $\{\hat{N}, \hat{D}\}$. This approach enables control-relevant error modeling in both the coprime factor and dual-Youla case. Due to space limitations, uncertainty modeling is not discussed further in this paper.

III. Control-relevant coprime factor identification

In this section, the control-relevant identification problem (4) is recast as a coprime factor identification problem. The pursued approach exploits the structure of the four-block problem (4) and results in an equivalent optimization problem over coprime factors. Besides the fact that the obtained coprime factors enable control-relevant model uncertainty representations, the resulting two-block problem reduces the complexity of the optimization problem by a factor $\frac{n_y+n_u}{n_u}$. Throughout, the superscript in $C^{\exp}$ is omitted if it is clear from the context.

The following proposition reveals that the four-block problem can be recast as a two-block problem.

Proposition 2 *Consider the control-relevant identification criterion* (4) *and let* $\{\bar{\tilde{N}}_e, \bar{\tilde{D}}_e\}$ *be an NLCF of* $V_1^{-1}CV_2$. *Then,* (4) *is equivalent to*

$$\min_{\hat{N},\hat{D}} \|W\big(\begin{bmatrix} N_o \\ D_o \end{bmatrix} - \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix}\big)\|_\infty, \tag{5}$$

where

$$\begin{bmatrix} N_o \\ D_o \end{bmatrix} = \begin{bmatrix} P_o \\ I \end{bmatrix} (\bar{\tilde{D}}_e V_1^{-1} + \bar{\tilde{N}}_e V_2^{-1} P_o)^{-1} \tag{6}$$

$$\begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \hat{P} \\ I \end{bmatrix} (\bar{\tilde{D}}_e V_1^{-1} + \bar{\tilde{N}}_e V_2^{-1} \hat{P})^{-1}. \tag{7}$$

Proof: Let $\{\tilde{N}_e, \tilde{D}_e\}$ be any LCF of $V_1^{-1}CV_2$ and observe that $WT(P,C)V = W\begin{bmatrix} P \\ I \end{bmatrix}(I + CV_2V_2^{-1}P)^{-1}V_1\tilde{D}_e^{-1}\begin{bmatrix}\tilde{N}_e & \tilde{D}_e\end{bmatrix}$. Next, multiplication of the latter expression to the right by a right inverse of $\begin{bmatrix}\tilde{N}_e & \tilde{D}_e\end{bmatrix}$ and rearranging yields $W\begin{bmatrix} P \\ I \end{bmatrix}(\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1}$. Employing $\|XY\|_\infty = \|X\|_\infty$ if Y is co-inner, *i.e.*, $YY^* = I$, and considering an NLCF $\{\bar{\tilde{N}}_e, \bar{\tilde{D}}_e\}$ of $V_1^{-1}CV_2$, *i.e.*, $\begin{bmatrix}\bar{\tilde{N}}_e & \bar{\tilde{D}}_e\end{bmatrix}\begin{bmatrix}\bar{\tilde{N}}_e & \bar{\tilde{D}}_e\end{bmatrix}^* = I$ establishes the equivalence between (4) and (5). ∎

The following proposition reveals that (6) and (7) correspond to coprime factorizations under a certain stability condition.

Proposition 3 *Let* $T(P,C) \in \mathcal{RH}_\infty$ *and let* $\{\tilde{N}_e, \tilde{D}_e\}$ *be a (not necessarily normalized) LCF of* $V_1^{-1}CV_2$. *Then,* $\{P(\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1}, (\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1}\}$ *is an RCF of* P.

Proof: Suppose $T(P,C) \in \mathcal{RH}_\infty$ and $V, V^{-1} \in \mathcal{RH}_\infty$. In addition, since $\{\tilde{N}_e, \tilde{D}_e\}$ is an LCF, $\exists Y_l, X_l \in \mathcal{RH}_\infty$ such that $\tilde{N}_eY_l + \tilde{D}_eX_l = I$. Hence, $T(P,C) \in \mathcal{RH}_\infty \Leftrightarrow T(P,C)V \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} P \\ I \end{bmatrix}(\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1}\begin{bmatrix}\tilde{N}_e & \tilde{D}_e\end{bmatrix} \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} P \\ I \end{bmatrix}(\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1}\begin{bmatrix}\tilde{N}_e & \tilde{D}_e\end{bmatrix}\begin{bmatrix} Y_l \\ X_l \end{bmatrix} \in \mathcal{RH}_\infty. \Leftrightarrow \begin{bmatrix} P \\ I \end{bmatrix}(\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1} \in \mathcal{RH}_\infty$, revealing that $\{P(\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1}, (\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1}\}$ is a stable factorization of P. The factorization is an RCF if $\exists X_r, Y_r$ such that the Bézout identity holds. Let $X_r = \tilde{N}_eV_2^{-1}$ and $Y_r = \tilde{D}_eV_1^{-1}$, hence $X_r, Y_r \in \mathcal{RH}_\infty$ by assumption. In addition, $\begin{bmatrix}X_r & Y_r\end{bmatrix}\begin{bmatrix} P \\ I \end{bmatrix}(\tilde{D}_eV_1^{-1} + \tilde{N}_eV_2^{-1}P)^{-1} = I$, completing the proof that the factorization indeed is an RCF. ∎

In Propositions 2 and 3, there is still freedom in the choice of $\bar{\tilde{D}}_e$. The impact of this freedom on the factorizations (6) and thus (7) is investigated next and is given by the following proposition.

Proposition 4 *The coprime factors* $\{N_o, D_o\}$ *and* $\{\hat{N}, \hat{D}\}$ *in* (5) *are unique up to a right multiplication by a constant unitary matrix* Q.

Proof: The proof is provided for $\{N_o, D_o\}$ and follows along similar lines for $\{\hat{N}, \hat{D}\}$. Suppose that $\{\bar{\tilde{N}}_e, \bar{\tilde{D}}_e\}$ is an NLCF of $V_1^{-1}CV_2$, resulting in the coprime factors $\{N_o, D_o\}$ as in (6). Let $Q, Q^{-1} \in \mathcal{RH}_\infty$. Then, it directly follows from the Bézout identity that $\{Q^{-1}\bar{\tilde{N}}_e, Q^{-1}\bar{\tilde{D}}_e\}$ generates all LCFs of $V_1^{-1}CV_2$ resulting in all RCFs $\{N_oQ, D_oQ\}$ of P_o. Next, observing that $\begin{bmatrix}Q^{-1}\bar{\tilde{N}}_e & Q^{-1}\bar{\tilde{D}}_e\end{bmatrix}$ is co-inner iff Q is a constant unitary matrix, *i.e.*, it satisfies $Q^*Q = I$, directly yields that $\{N_o, D_o\}$ is unique up to right multiplication by a constant unitary matrix Q. ∎

Proposition 4 implies that if the factorization $\{N_o, D_o\}$ as defined in (6) is not normalized, then there does not exist a normalized coprime factorization of P_o that is control-relevant in virtue of Proposition 2.

Finally, it is shown that in contrast to estimating normalized coprime factorizations, *e.g.*, [16], the suggested coprime factorizations in Propositions 2 and 3 exploit the freedom in the coprime factors to obtain a control-relevant coprime factorization in the sense of (4). Thereto, a connection between the coprime factorizations in Propositions 2 and 3 and previous results in a prediction error framework [16] is established. First, suppose that $W = I$ and $V = I$. Next,

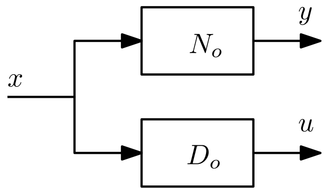

Fig. 2. Intermediate signal x.

observe that the coprime factorizations $\{N_o, D_o\}$ can be interpreted as the mapping from an intermediate signal x to y, u in Figure 2. In the prediction error framework, x is obtained by filtering $\begin{bmatrix} r_2 & r_1 \end{bmatrix}^T$ by $F \begin{bmatrix} C & I \end{bmatrix}$. In [16], [26] it is shown that $\{N_o, D_o\}$ in Figure 2 is an RCF of P_o iff $F = W\tilde{D}_c$, where $\{\tilde{N}_c, \tilde{D}_c\}$ is an LCF of C and $W, W^{-1} \in \mathcal{RH}_\infty$. If only r_1 is considered, then r_1 is filtered by $W\tilde{D}_c$. In [16], [26], the freedom in the filter F is exploited to iteratively obtain an NRCF of P_o. In contrast, the suggested choice in this paper is based on the controller C^{exp} and is given by $V_1 \bar{\tilde{D}}_e^{-1}$.

IV. Frequency response-based control-relevant coprime factor identification

Identification using (5) cannot be performed directly due to the use of the $\mathcal{H}_\infty$-norm. In this section, the frequency domain interpretation of the $\mathcal{H}_\infty$-norm is exploited to formulate a solvable identification problem. Note that identification in practice is always performed over a finite time interval, hence a discrete frequency grid Ω is used in frequency response-based identification. Introducing an appropriate parametrization $\begin{bmatrix} \hat{N}^T(\theta) & \hat{D}^T(\theta) \end{bmatrix}^T$ for (7), see Section V, gives the following result.

Proposition 5 *A lower bound for* (5) *is given by*

$$\min_{\theta} \max_{\omega_i \in \Omega} \bar{\sigma}\big(W\big(\begin{bmatrix} N_o(\omega_i) \\ D_o(\omega_i) \end{bmatrix} - \begin{bmatrix} \hat{N}(\theta, \omega_i) \\ \hat{D}(\theta, \omega_i) \end{bmatrix}\big)\big) \qquad (8)$$
$$\textit{subject to } T(\hat{P}, C) \in \mathcal{RH}_\infty.$$

Proof: Follows directly by considering the $\mathcal{H}_\infty$-norm. ■

Throughout, criterion (8) is used as an approximation for the control-relevant identification criterion (4).

In Proposition 5, $\{N_o(\omega_i), D_o(\omega_i)\}$ can be determined directly from data if $T(P_o, C^{\text{exp}})$ has been identified for $\omega_i \in \Omega$, which is in fact an open-loop identification problem. In particular, $\{N_o(\omega_i), D_o(\omega_i)\}$ is obtained if $T(P_o(\omega_i), C^{\text{exp}}(\omega_i))$ is appended with weighting filters and multiplied to the right by $\begin{bmatrix} \bar{\tilde{N}}_e(\omega_i) & \bar{\tilde{D}}_e(\omega_i) \end{bmatrix}^*$.

V. Model parametrization

In the previous section, the control-relevant estimation of P_o has been recast as an equivalent optimization problem of coprime factor frequency response functions. To estimate parametric coprime factors of limited complexity, a suitable parametrization should be selected. The parametrization $\hat{P} = \hat{N}\hat{D}^{-1}$ should satisfy: 1) $T(\hat{P}, C) \in \mathcal{RH}_\infty$, see (8); 2) $\{\hat{N}, \hat{D}\}$ stable and of low order; 3) efficient optimization algorithms can be employed. To address the latter point, *i.e.*, the optimization algorithm, it is most convenient to parameterize the coprime factors by means of (fractions of) polynomial matrices of suitable dimensions. To satisfy the above requirements, a tailor-made parametrization is suggested. In particular, let $\hat{P}(\theta) = B(\theta)A(\theta)^{-1}$, where $A(\theta) \in \mathbb{R}[\xi]^{n_u \times n_u}$, $B(\theta) \in \mathbb{R}[\xi]^{n_y \times n_u}$ are parameterized as a canonical MFD [21] that is linear in the parameters, *e.g.*, as in [27], and $\theta \in \mathbb{R}$, hence

$$\begin{bmatrix} \hat{N}(\theta) \\ \hat{D}(\theta) \end{bmatrix} = \begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (\tilde{D}_e V_1^{-1} A(\theta) + \tilde{N}_e V_2^{-1} B(\theta))^{-1}. \qquad (9)$$

Throughout, it is assumed that A and B are coprime polynomial matrices. In this case, $\hat{P}$ is of McMillan degree $\deg\det A$. Note that the McMillan degree of the coprime factorization $\{\hat{N}, \hat{D}\}$ is of McMillan degree $\geq \deg\det A$, since C, V_1, V_2 also contribute to the McMillan degree of the coprime factors. By the particular choice of the *normalized* LCF of $V_1^{-1} C V_2$ in Section IV, however, the McMillan degree is not unnecessarily high. In addition, the structural indices of B, A can be based only on the structural properties of the true plant P_o. It is not straightforward in the suggested parametrization to enforce both 1) $T(\hat{P}, C) \in \mathcal{RH}_\infty$, 2) $\hat{N}, \hat{D} \in \mathcal{RH}_\infty$. However, these conditions can be verified by one test, as formalized in the following proposition.

Proposition 6 *Consider the factorization* (9). *Then, the following statements are equivalent:*

1) $T(\hat{P}(\theta), C) \in \mathcal{RH}_\infty$.
2) $\begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (\tilde{D}_e V_1^{-1} A(\theta) + \tilde{N}_e V_2^{-1} B(\theta))^{-1} \in \mathcal{RH}_\infty$
3) $(\tilde{D}_e V_1^{-1} A(\theta) + \tilde{N}_e V_2^{-1} B(\theta))^{-1} \in \mathcal{RH}_\infty$

Proof: Let $\{\tilde{N}_e, \tilde{D}_e\}$ denote an LCF of $V_1^{-1} C V_2$ with Bézout factors $Y_l, X_l \in \mathcal{RH}_\infty$. Suppose that C internally stabilizes $\hat{P}(\theta)$, *i.e.*, $T(\hat{P}(\theta), C) \in \mathcal{RH}_\infty$. Then, $T(\hat{P}(\theta), C) \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (A(\theta) + CB(\theta))^{-1} \begin{bmatrix} CV_2 & V_1 \end{bmatrix} \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (\tilde{D}_e V_1^{-1} A(\theta) + \tilde{N}_e V_2^{-1} B(\theta))^{-1} \begin{bmatrix} \tilde{N}_e & \tilde{D}_e \end{bmatrix} \begin{bmatrix} Y_l \\ X_l \end{bmatrix} \in \mathcal{RH}_\infty \Leftrightarrow \begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix} (\tilde{D}_e V_1^{-1} A(\theta) + \tilde{N}_e V_2^{-1} B(\theta))^{-1} \in \mathcal{RH}_\infty$, proving 1) $\Leftrightarrow$ 2). Next, observing that B, A in the latter expression are polynomial matrices directly proves 1) $\Leftrightarrow$ 3). ■

In Section VI, an optimization algorithm will be presented that enables optimization of (5) using the parametrization (9).

The suggested parametrization has certain advantages over a direct polynomial parametrization of coprime factors. Indeed, the factorization (7) may suggest a parametrization of the form [16]

$$\begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} B_N \\ B_D \end{bmatrix} A_F^{-1}, \qquad (10)$$

where A_F, B_N, B_D are polynomial matrices of appropriate sizes. In general, $\deg(B_N) \geq \deg(B)$ and $\deg(B_D) \geq \deg(A)$. Since common dynamics of B_N, B_D may not cancel exactly, in general $\hat{P} = \hat{N}\hat{D}^{-1} = B_N B_D^{-1}$ has a higher McMillan degree than $\deg\det(A)$. In addition, compared to the parametrization (9), stability of the nominal model $\hat{P}$ in (10) depends on B_N, B_D, whereas stability of the factorization $\{\hat{N}, \hat{D}\}$ depends on A_N. Hence, two separate conditions are required if the parametrization (10) is used, as opposed to the parametrization in Proposition 6.

VI. A NUMERICALLY RELIABLE ITERATIVE PROCEDURE

In this section, the control-relevant estimation of coprime factors in (8) using the parametrization (9) is performed via a numerically reliable procedure. In Section VI-A, the pursued approach is briefly described, followed by numerical aspects in Section VI-B.

A. ℓ_∞ Approximation using Lawson's algorithm

The optimization (8) involves an ℓ_∞-type criterion, resulting in a nonsmooth optimization problem, *i.e.*, efficient gradient-based optimization techniques cannot be used directly. In addition, the parametrization (9) is nonlinear in the parameters θ, resulting in a generally non-convex optimization problem.

To solve (8), Lawson's algorithm [28], [27] is employed as follows.

Algorithm 7 *Set $\theta^{<o>} = 0$ and $w_i^{<0>} = \frac{1}{n_\omega}$, n_ω denoting the number of frequencies in Ω. Iterate over k until convergence:*

$$\theta^{<k>} = \arg\min_\theta \sum_i w_i^{<k>} \|\varepsilon_i(\theta)\|_F^2 \tag{11}$$

$$\textit{where } w_i^{<k>} = \frac{\bar{\sigma}(\varepsilon_i(\theta^{<k>}))w_i^{<k-1>}}{\sum_i(\bar{\sigma}(\varepsilon_i(\theta^{<k>}))w_i^{<k-1>})}. \tag{12}$$

Algorithm 7 iteratively solves the nonlinear least squares problem (11), which is defined precisely in Section VI-B. The weighting function $w_i^{<k>}$ is employed to minimize (8). To anticipate on the results in the next section, convergence of Algorithm 7 cannot be guaranteed since the nonlinear least squares problem can result in local minima. However, numerical experience indicates that the algorithm often converges and the constraint $T(\hat{P}, C) \in \mathcal{RH}_\infty$ in (8) is commonly satisfied.

B. A numerically reliable approach for solving nonlinear least squares problems

The nonlinear least squares problem in (11) is equivalent to

$$\sum_i \|W_h^{<k>} \circ \left(W\left(\begin{bmatrix} N_o \\ D_o \end{bmatrix} - \begin{bmatrix} \hat{N}(\theta) \\ \hat{D}(\theta) \end{bmatrix}\right)\right)\|_F^2, \tag{13}$$

where the elements of $W_h^{<k>}$ are equal to $\sqrt{w_i^{<k>}}$ and $\circ$ denotes the Hadamard product, which is introduced to separate the weighting of Lawson's algorithm and the nonlinear least squares problem. Rearranging and using the following facts from Kronecker algebra for matrices of suitable dimensions, see, *e.g.*, [29]:

- $\|A\|_F = \|\mathrm{vec}(A)\|_2$
- $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$
- $\mathrm{vec}(A \circ B) = \mathrm{diag}(\mathrm{vec}(A))\mathrm{vec}(B)$,

reveals that (13) can be written as

$$\sum_i \|W_{\mathrm{lsq},i}^{<k>}(\theta)\mathrm{vec}\left(\begin{bmatrix} B(\theta) \\ A(\theta) \end{bmatrix}\right)\|_2^2, \tag{14}$$

$$W_{\mathrm{lsq},i}^{<k>}(\theta) = \mathrm{diag}(\mathrm{vec}(W_h^{<k>}))\Big(\Big[(\bar{\tilde{D}}_e V_1^{-1} A(\theta) + \bar{\tilde{N}}_e V_2^{-1} B(\theta))^{-1}\Big]^T \otimes \Big[W\Big(\begin{bmatrix} N_o \bar{\tilde{N}}_e V_2^{-1} & N_o \bar{\tilde{D}}_e V_1^{-1} \\ D_o \bar{\tilde{N}}_e V_2^{-1} & D_o \bar{\tilde{D}}_e V_1^{-1} \end{bmatrix} - I\Big)\Big]\Big), \tag{15}$$

If both $A(\theta)$ and $B(\theta)$ are parameterized in a suitable canonical polynomial basis, then (14) can be solved by iteratively solving the linear least squares problem

$$\sum_i \|W_{\mathrm{lsq},i}^{<k>}(\theta^{<f-1>})\mathrm{vec}\left(\begin{bmatrix} B(\theta^{<f>}) \\ A(\theta^{<f>}) \end{bmatrix}\right)\|_2^2, \tag{16}$$

for f, since (16) is linear in the parameter vector $\theta^{<f>}$.

Convergence of the iterative procedure (16) hinges on the accurate computation of the solution at each iteration step. Hence, a necessary condition for convergence of the iterative procedure is the computation of an accurate solution to the linear least squares problem (16). If the normal equations corresponding to the least squares problem (16) are poorly conditioned, then the solution $\theta^{<f>}$ is prone to numerical errors and may be inaccurate.

Employing a monomial basis in continuous time system identification leads to extremely ill-conditioned normal equations for moderate degrees of the vector polynomials [19]. Though the discrete time monomial basis is orthonormal with respect to a continuous inner product, the weighted evaluation on a discrete frequency grid Ω may result in a poor numerical conditioning.

In this paper, a high performance numerical algorithm is employed to construct a polynomial basis with respect to the weighted discrete inner product

$$< f, g >_{W_{\mathrm{lsq}}} = \sum_i f(\omega_i)^* W_{\mathrm{lsq},i}^{<k>}(\theta^{<f-1>})^* W_{\mathrm{lsq},i}^{<k>}(\theta^{<f-1>}) g(\omega_i), \tag{17}$$

resulting in optimally conditioned normal equations for the linear least squares problem (16), *i.e.*, a condition number $\kappa = 1$. The construction of these orthonormal vector polynomials (OVPs) is performed by a modification of the results in [30], see also [19], details of which are omitted due to space limitations.

The iterative procedure (16) typically converges closely to the global minimum. This is a remarkable property of such iterations and is enabled by the fact that the iterative procedure does not search the error surface as is the case in common optimization algorithms. However, in case of undermodeling and noise, the stationary point of the iterations (16) need not converge exactly to a minimum of (13) [31]. Thereto, the solution resulting from the iterations (16) is further refined using a Gauss-Newton optimization, possibly at the expense of an increasing condition number κ.

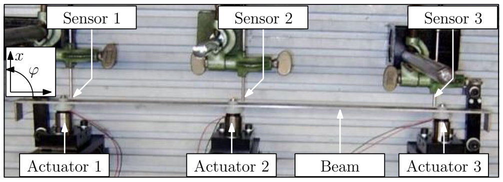

Fig. 3. Flexible beam experimental setup.

VII. Experimental Example

A. Experimental setup

A flexible beam system, see Figure 3 is considered, which is a prototype for high performance flexible positioning systems. The setup consists of a steel beam ($500\times20\times2$ mm) that is fixed to the environment by means of five leaf springs such that four DOFs are fixed and the x and φ DOFs are free.

The system has three inputs and three outputs for control purposes. The inputs consist of current-driven voice-coil actuators, whereas the outputs are contactless fiberoptic sensors with a positioning accuracy of 1 μm. To illustrate the results of the previous sections, only the x direction is considered using input 2 and output 2.

Due to the leaf springs, the remaining DOFs in x and ϕ direction are suspended by soft springs, resulting in an open-loop stable system. Due to weakly damped resonance phenomena, closed-loop identification is necessary. The control system is implemented in a rapid-prototyping environment in conjunction with Matlab/Simulink. The controller $C^{\exp}$ is given by

$$C^{\exp} = \frac{6.540 - 13.02z^{-1} + 7.248z^{-2} + 2.087z^{-3} - 2.570z^{-4}}{1 - 2.130z^{-1} + 1.805z^{-2} - 0.684z^{-3} + 0.095z^{-4}}, \tag{18}$$

and is implemented with a sampling frequency of 2 kHz. The controller is a proportional-derivative type of controller with a notch filter at 185 [Hz] and roll-off, achieving a bandwidth of approximately 15 Hz. Note that any stabilizing controller can be used, however, in view of the approximation in Section II-A, it is desirable that $C^{\exp}$ is close to C^{opt}.

B. Procedure

The closed-loop transfer function matrix $T(P_o, C^{\exp})$ is identified at $\omega_i \in \Omega$ using a multisine excitation signal consisting of 250 logarithmically spaced frequency components with equal amplitude and crest-factor optimized phases [32].

The weighting filters W and V are parametrized as suggested in [23], [24] as follows:

$$W_y = 13.97 \quad W_u = 1$$
$$V_1 = \frac{0.5031 - 0.4906z^{-1}}{1 - 0.9875z^{-1}} \quad V_2 = \frac{0.1136 - 0.05928z^{-1}}{1 + 0.5171z^{-1}}, \tag{19}$$

where the gain of the filters is chosen to scale the entries of $T(P, C)$ such that these are approximately equal to one around the target bandwidth (40 [Hz]), where the nonparametric information from Figure 5 is employed. In addition, V_1 and V_2 are used to enforce integral action and controller roll-off, respectively.

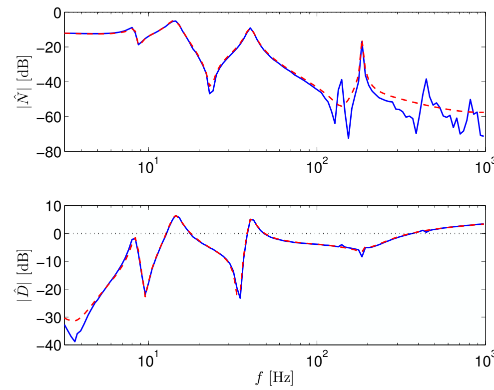

Fig. 4. Bode magnitude diagram of identified $\{N_o, D_o\}$ (solid) and parametric coprime factors (9) (dashed) at $\omega_i \in \Omega$.

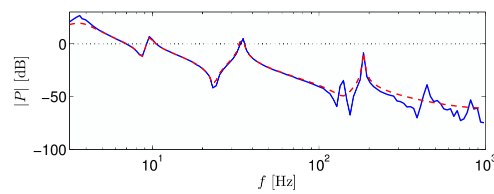

Fig. 5. Bode magnitude diagram of identified P_o (solid) and $\hat{P} = BA^{-1}$ (dashed) at $\omega_i \in \Omega$.

C. Results

The optimization procedure in Section VI has been invoked. After several iterations, Lawson's algorithm converges to a stationary point. After convergence, Condition 3 in Proposition 6 turns out to hold. Hence, the nominal model $\hat{P}$ is indeed stabilized by $C^{\exp}$ and $\{\hat{N}, \hat{D}\}$ are coprime factors. Bode diagrams of $\{N_o, D_o\}$ and $\{\hat{N}, \hat{D}\}$ are depicted in Figure 4. The open-loop plants are shown in Figure 5. The coprime factors are of order 14, whereas $\hat{P}$ is of order 8.

The estimated model accurately describes the resonance phenomena at 10, 35, and 185 Hz. The former two resonance phenomena are located in the cross-over region. The latter resonance is indeed control-relevant, since a notch filter was required in (18) to attenuate it. The resonance at approximately 3 Hz, which is the largest one in the open-loop situation, see Figure 5, is inaccurately modeled. Clearly, it has a small contribution in control-relevant criterion (4). Concluding, the suggested approach delivers a control-relevant nominal model.

In Figure 6, $N^*N + D^*D$ is depicted. Clearly, since $N^*N + D^*D$ is unequal to one, the coprime factors $\{N_o, D_o\}$ and $\{\hat{N}, \hat{D}\}$ are not normalized. Finally, in Figure 7, the condition number κ during the first series of Gauss-Newton iterations is depicted, *i.e.*, $k = 0$ in Algorithm 7. Clearly, the condition number is close to one, indicating that the numerical algorithm is reliable. The condition number during subsequent iterations of Lawson's algorithm is similar.

VIII. Conclusions

Control-relevant identification of limited order coprime factorizations has been considered. The contributions in-

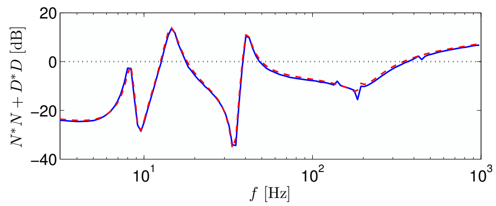

Fig. 6. Test of normalization condition, based on $\{N_o, D_o\}$ (solid) and parametric coprime factors (dashed) $\omega_i \in \Omega$.

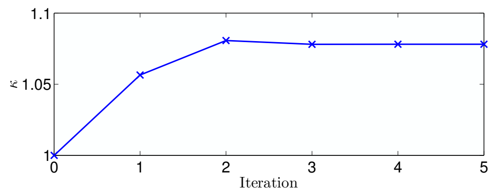

Fig. 7. Condition number during Gauss-Newton iterations.

clude the new definition of control relevant coprime factorizations (Section III), the computation of nonparametric control-relevant coprime factor frequency response functions (Section IV), the specific coprime factor parametrization (Section V), and a numerically reliable iterative optimization algorithm (Section VI). Experimental results illustrate the convergence of the algorithm for realistic measurements and the capability of the algorithm to deliver a model suitable for subsequent error modeling and robust control design. In fact, the control-relevant coprime factor domain is directly useful for control-relevant uncertainty modeling. Compared to estimating normalized coprime factors, the adopted approach does not require iterating over coprime factors.

Compared to normalized coprime factorizations, the McMillan degree of the identified coprime factors depends on the McMillan degree of the controller C^{exp} and weighting filters. Thus, the presented approach is especially useful if a low complexity controller is used. In this perspective, it is remarked that the normalized coprime factorizations in [10], [11] are factorizations of the shaped plant, *i.e.*, the nominal plant with weighting filters.

Further research includes convergence analysis of the iterative algorithm. Extensive simulations and experiments reveal promising results regarding convergence, in addition, the stability constraint in (8) is often satisfied for adequate model orders.

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