2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Instrumental variable identification of fading channel models from irregularly sampled noisy data

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***Abstract*— A continuous-time stochastic model is considered for describing fading channels from irregularly sampled data affected by measurement noise. The model parameters are estimated using an instrumental variable approach that gives consistent estimates as the number of data tends to infinity and the upper bound on the irregular sampling interval tends to zero. The proposed estimator is robust to measurement noise, computationally efficient, and easy to implement. Once the model parameters are estimated, the design of efficient algorithms for power control is possible.**

I. Introduction

Reliable models of fading channels are essential for power control and receiver design in modern cellular systems. In some situations, samples are only taken when there is no severe fading. This means that only irregularly sampled data are available and that there is a need for models that describe and estimation algorithms that work with such data. When only irregularly sampled data are available from a stochastic system, it can be advantageous to use a continuous-time stochastic model. This facilitates the design of a computationally efficient estimation algorithm for the model parameters. In this paper, a second order stochastic differential equation is considered and an instrumental variable (IV) estimator of its parameters is proposed.

II. Problem description

Following [1], assume that the Doppler power spectrum $P(\mathrm{i}\omega)$ of the channel can be factorized as $P(\mathrm{i}\omega) = |G(\mathrm{i}\omega)|^2$. As a consequence, the output can be represented as $y(t) = \int_0^t g(\tau)e(t-\tau)d\tau$, where $e(t)$ is continuous-time Gaussian white noise of unit incremental variance. Consider the stochastic differential equation (SDE)

$$y^{(2)}(t) + \theta_1 y^{(1)}(t) + \theta_2 y(t) = e(t), \tag{1}$$

where $y^{(i)}(t)$ denotes the ith derivative of $y(t)$ and where $y(0)$ and $y^{(1)}(0)$ are Gaussian random variables. Here, $y(t)$ represents the in-phase and quadrature components of the total received signal. Note that the SDE (1) realizes $G(\mathrm{i}\omega) = 1/(-\omega^2 + \mathrm{i}\theta_1\omega + \theta_2)$. Define the parameter vector $\boldsymbol{\theta} = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^T$, where, in the case of a time-varying channel, the parameters θ_1 and θ_2 are considered as dependent on t and τ.

The parameter vector $\boldsymbol{\theta}$ is to be estimated from the irregularly sampled measurements $x(t_k) = y(t_k) + v(t_k)$, where $k = 1, \dots, N$ and where $v(t_k)$ is zero mean discrete-time Gaussian white noise of variance σ^2, independent of $e(t)$. In order to get a well-defined problem, the following notations and assumptions regarding the irregular sampling scheme are made. **1**) Let $h_k = t_{k+1} - t_k$, and assume that $\underline{h} \leqslant h_k \leqslant \bar{h}$ for some $\underline{h} > 0$ and finite $\bar{h}$. **2**) The sequence of sampling intervals, $\{h_k\}$, is independent and identically distributed with an associated probability density function. Moreover, $\{h_k\}$ is independent of the process. **3**) Let the operator $\mathrm{E}\{\cdot\}$ denote expectation with respect to the process, and introduce the operator $\bar{\mathrm{E}}\{\cdot\} \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \mathrm{E}\{\cdot\}$.

III. Estimation

Consider the difference operator D_k^m such that

$$D_k^m f(t_k) = \sum_{\mu=\mu_1}^{\mu_2} c_{m,k,\mu} f(t_{k+\mu}) \tag{2}$$

with the property that for $f(t)$ smooth enough

$$D_k^m f(t_k) = p^{(m)} f(t_k) + \mathcal{O}(\bar{h}^p), \tag{3}$$

where $p^{(m)} = d^m/dt^m$. Here, it is assumed that $0 \leqslant \mu_1 \leqslant \mu_2$.

Result 1. *Relation (3) is fulfilled if the extended natural conditions*

$$\sum_{\mu=\mu_1}^{\mu_2} c_{m,k,\mu}\lambda_k^a(\mu) = \begin{cases} 0, & a = 0, \dots, m-1 \\ m!, & a = m \\ 0, & a = m+1, \dots, m+p-1 \end{cases} \tag{4}$$

hold, where $\lambda_k(\mu) = t_{k+\mu} - t_k$.

Proof: The result follows directly from a Taylor series expansion of $f(t_{k+\mu})$ around $f(t_k)$. ■

In this paper, Result 1 will be considered for $i = \{1, 2\}$ and $p = 2$. Assume that the sampled data $\{x(t_k)\}_{k=1}^N$ are available. Rewrite (1) by using the operator (2), where the weights fulfill (4), as $w(t_k) = \boldsymbol{\varphi}^T(t_k)\boldsymbol{\theta} + \varepsilon(t_k)$, where $w(t_k) = D_k^2 x(t_k)$, $\boldsymbol{\varphi}(t_k) = \begin{bmatrix}-D_k x(t_k) & -x(t_k)\end{bmatrix}^T$, and $\varepsilon(t_k)$ is an equation error.

Proposition 1. *Define*

$$\mathbf{z}(t_k) = \begin{bmatrix}x(t_{k-\alpha}) & x(t_{k-\alpha-1})\end{bmatrix}^T, \tag{5}$$

where $\alpha \geqslant 1$. *The solution to*

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{z}(t_k)\boldsymbol{\varphi}^T(t_k)\hat{\boldsymbol{\theta}} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{z}(t_k) w(t_k) \tag{6}$$

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978-1-4244-2079-7/08/$25.00 ©2008 AACC.

gives an IV estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$.

Note that the estimator can easily be implemented recursively for tracking time-varying parameters.

Result 2. *Equation (6) can be written as*

$$\bar{\mathrm{E}}\{\mathbf{z}(t_k)\boldsymbol{\varphi}^T(t_k)\}\hat{\boldsymbol{\theta}} = \bar{\mathrm{E}}\{\mathbf{z}(t_k)w(t_k)\} \tag{7}$$

when $N \to \infty$.

Proof: The result follows from the proof of Result 2 in [2]. ■

The main result of the paper can now be given.

Result 3. *Choose the weights for the operator (2) so that (4) is fulfilled. Then it holds that*

$$\lim_{N\to\infty} \hat{\boldsymbol{\theta}} = \boldsymbol{\theta} + \mathcal{O}(\bar{h}^2), \tag{8}$$

where $\hat{\boldsymbol{\theta}}$ is given as the solution to (6).

Proof: The proof is inspired by calculations in [3], [4]. From Result 2 and by rewriting (7), we get

$$\sum_{i=0}^{2} \hat{\theta}_{2-i}\bar{\mathrm{E}}\{D_k^i x(t_k)\cdot x(t_{k-\alpha-\beta})\} = 0, \tag{9}$$

or equivalently

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{i=0}^{2}\hat{\theta}_{2-i}\,\mathrm{E}\{D_k^i x(t_k)\cdot x(t_{k-\alpha-\beta})\} = 0, \tag{10}$$

for $\beta = 0, 1$, where $\hat{\theta}_{2-i}$ denotes element $2-i$ of $\hat{\boldsymbol{\theta}}$, with $\hat{\theta}_0 \equiv 1$. Consider the element $\mathrm{E}\{D_k^i x(t_k)\cdot x(t_{k-\alpha-\beta})\}$ in (10) and define $r_x(|t_a - t_b|) = \mathrm{E}\{x(t_a)x(t_b)\}$ and $r_y(|t_a - t_b|) = \mathrm{E}\{y(t_a)y(t_b)\}$. For $i = 0$, it holds that

$$\begin{aligned}&\mathrm{E}\{x(t_k)x(t_{k-\alpha-\beta})\} = r_x(|t_k - t_{k-\alpha-\beta}|)\\ &= r_y(|t_k - t_{k-\alpha-\beta}|) + \sigma^2\delta_{0,\alpha+\beta} = r_y(|t_k - t_{k-\alpha-\beta}|),\end{aligned} \tag{11}$$

where the last equality follows from the fact that $\alpha+\beta \geqslant 1$. From the result

$$\begin{aligned}&\mathrm{E}\{D_k^i y(t_k)\cdot y(t_{k-\alpha-\beta})\} = D_k^i r_y(|t_k - t_{k-\alpha-\beta}|)\\ &= p^{(i)} r_y(|t_k - t_{k-\alpha-\beta}|) + \mathcal{O}(\bar{h}^2)\end{aligned} \tag{12}$$

for $i = 1, 2$, it follows that

$$\begin{aligned}&\mathrm{E}\{D_k^i x(t_k)\cdot x(t_{k-\alpha-\beta})\}\\ &\quad = p^{(i)} r_x(|t_k - t_{k-\alpha-\beta}|) + \mathcal{O}(\bar{h}^2)\\ &\quad = p^{(i)} r_y(|t_k - t_{k-\alpha-\beta}|) + \sigma^2\delta_{0,\alpha+\beta} + \mathcal{O}(\bar{h}^2)\\ &\quad = p^{(i)} r_y(|t_k - t_{k-\alpha-\beta}|) + \mathcal{O}(\bar{h}^2)\end{aligned} \tag{13}$$

for $i = 1, 2$, where, again, it is used that $\alpha + \beta \geqslant 1$. Using (11) and (13) in (10) gives

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{i=0}^{2}\hat{\theta}_{2-i}\,\mathrm{E}\{D_k^i x(t_k)\cdot x(t_{k-\alpha-\beta})\}\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{i=0}^{2}\hat{\theta}_{2-i}\big(p^{(i)} r_y(|t_k - t_{k-\alpha-\beta}|)\\ &\quad + \mathcal{O}(\bar{h}^2)\big) = 0.\end{aligned} \tag{14}$$

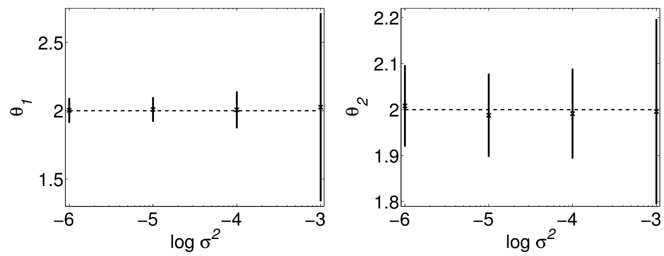

Fig. 1. The mean values plus/minus the standard deviations for the estimates of θ_1 (left) and θ_2 (right) as functions of the measurement noise variance σ^2. The true parameter values are indicated by dashed lines.

The covariance function $r_y(\tau)$ satisfies the Yule-Walker equation

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{i=0}^{2}\theta_{2-i}p^{(i)} r_y(|t_k - t_{k-\alpha-\beta}|) = 0 \tag{15}$$

for $\tau \geqslant 0$, see [3], [4]. By comparing (14) and (15), (8) follows. ■

IV. Numerical Example

The properties of the IV estimator in Proposition 1 is investigated in a Monte Carlo simulation with 200 realizations. The estimator is implemented with $\alpha = 1$ in (5). In each realization, $N = 10000$ data points are generated from the process $y(t_k)$ defined by $\theta_1 = \theta_2 = 2$, and measurements $x(t_k)$ are affected by noise σ^2 in the interval $[10^{-6}, 10^{-3}]$.

As an example of an irregular sampling scheme, we take $t_k = kT + \sum_{l=1}^{k}\delta_l$, $k = 1,\ldots,N$, where δ_l is uniformly distributed between $-\tilde{\delta}$ and $\tilde{\delta}$. Moreover, δ_l is independent of $e(t)$ and $v(t_k)$ for all l, t, and k, and δ_l is independent of δ_j for all $j \neq l$. Here, the choice $\tilde{\delta} = T/5$ is made, where the average sampling interval $T = 0.1$.

The results are shown in Fig. 1, where the mean values plus/minus the standard deviations for the estimates of θ_1 and θ_2, respectively, are plotted as functions of the measurement noise variance σ^2.

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