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Formation Stability of Multi-Agent Systems with Limited Information

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Abstract

Formation control of multi-agent systems in the face of limitation on the number of communication channels is considered. The feedback controller switches the communication structure so that the agents share the information of all the other agents over certain period. We present stability analysis for two types of such limitations. For both cases, it is shown that stability of formation is maintained by appropriately constructing the switching sequence of the network structure. Furthermore, we provide stability conditions for the case where the communication channels collapse with known probability in the face of channel constraints placed on the overall system.

1. Introduction

The framework of networked multi-agent systems has variety of applications [1–4] and a whole range of approaches to control [5–8]. However, most of the approaches in the literature are aimed at providing full-state feedback controllers to achieve control objectives. The problem of information limitation appears to be fairly neglected, a problem which is very much likely to occur during the implementation of multi-agent systems.

In a recent paper [9], we developed a formation control framework for multiple agents with intermittent information exchange between the agents. Specifically, we derived the relationship between the magnitude of the feedback gain and the rate of information exchange. Even though emulation of a mass-spring-damper system is a trivial approach to stabilize the multi-agent systems [5,6] for the continuous-time case , it does not apply to the case of sampled-data setting [10]. In [9] we ended up obtaining a simple, low-dimensional criterion whereby formation stability of the (large number of) agents is shown to be guaranteed.

In this paper, we consider the case where the number of communication channels that can be used at each time instant is limited. Specifically, when there is not enough number of communication channels, information network of the multi-agent system cannot be connected and a group of agents may be isolated from the other agents at any time instant. Hence, a switching sequence of communication topologies should be appropriately constructed to stabilize the multi-agent system. In particular, we consider two cases; one where the channel constraint is placed on each agents and the other where the overall multi-agent system has constrains. Under these cases we analyze stability of the multi-agent system for various switching sequences. Furthermore, we provide stability conditions for the case where the communication channels collapse with known probability in the face of channel constraints placed on the overall system.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{N}_0$ denotes the set of non-negative integers. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose, $x_{i:j}$ for the sub-vector of $x \in \mathbb{R}^n$ having the ith to the jth elements of x, 0_n for the $n \times n$-dimensional zero matrix, I_n for the identity matrix of dimension n, $X_{(i,j)}$ for the (i,j)th element of matrix X, $\mathrm{row}_{i:j}(X)$ for the sub-matrix of X consisting of all the rows from i to j of X, $\otimes$ for the Kronecker product, $\mathrm{spec}(A)$ for the spectrum of the matrix A, 1_n for the vector $[1,\ldots,1]^{\mathrm{T}} \in \mathbb{R}^n$, $\|\cdot\|$ for the Euclidean vector norm, $\mathrm{mod}(a,b)$ for the remainder on division of a by b, and $|\mathcal{N}|$ for the cardinal number of the finite set $\mathcal{N}$.

2. Motivation and Problem Setting

In this section we begin by considering the problem of characterizing formation control laws for multi-agent systems. Specifically, we assume that each agent is a normalized point mass and is subject to the force input. For simplicity of exposition, we assume that the agents are collision-free and allowed to move in a one-dimensional space. The extension to multi-dimensional systems is straightforward. The control objective is to regulate each agent's position and velocity such that all the agents travel with asymptotic zero relative positions and common velocities.

To this end, consider the dynamics of n identical agents given by

$$\begin{aligned} \dot{q}_i(t) &= p_i(t), \quad q_i(0) = q_{i0}, \quad t \geq 0, \\ \dot{p}_i(t) &= u_i(t), \quad p_i(0) = p_{i0}, \end{aligned} \tag{1}$$

where $i = 1, \cdots, n$, $q_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$ are the position and the velocity, respectively, and $u_i \in \mathbb{R}$ is the force input of the ith agent. With this notation the control objective mentioned above can be written as

$$\lim_{t\to\infty}(q_i(t) - q_j(t)) = 0, \quad \lim_{t\to\infty}(p_i(t) - p_j(t)) = 0, \quad i,j = 1,\cdots,n, \quad i \neq j. \tag{2}$$

In the preceding work [5, 6], Tanner *et al.* considered an energy-based controller that emulates forces due to springs and dampers in the continuous-time setting. In particular,

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their control law has the form of

$$u_i(t) = [-1,-1] \sum_{j\in\mathcal{N}_i} \begin{bmatrix} q_i(t)-q_j(t) \\ p_i(t)-p_j(t) \end{bmatrix}, \quad i=1,\ldots,n, \tag{3}$$

where $\mathcal{N}_i \subset \{1,\ldots,n\}\backslash\{i\}$ represents the set of agents which the agent i can communicate with. It is also assumed that if the agent i knows its relative state with respect to agent j, then the agent j also knows its relative state with respect to agent i. Note that the control law (3) has the form of

$$u(t) = -Lq(t) - Lp(t), \tag{4}$$

where $q(t) \triangleq [q_1(t),\ldots,q_n(t)]^{\mathrm{T}}$, $p(t) \triangleq [p_1(t),\ldots, p_n(t)]^{\mathrm{T}}$, $u(t) \triangleq [u_1(t),\ldots,u_n(t)]^{\mathrm{T}}$, and L is the symmetric Laplacian matrix defined as

$$L_{(i,j)} \triangleq \begin{cases} |\mathcal{N}_i|, & i=j, \\ -1, & j\in\mathcal{N}_i, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

which in a way represents the communication architecture.

In this paper, we consider a more realistic case than the continuous-time setting presented above. In particular, we assume that each controller can simultaneously receive the information of relative positions and relative velocities of the agents prescribed by $\mathcal{N}_1,\ldots,\mathcal{N}_n$ with time interval T. This synchronized intermittent information exchange naturally leads to a formulation of sampled-data control.

For applying the control, we employ zero-order hold so that the control input between the sampling instants is given by

$$u_i(t) = u_i[k], \quad kT \le t < (k+1)T, \quad k\in\mathbb{N}_0, \quad i=1,\ldots,n, \tag{6}$$

where $u_i[k]$ denotes the input signal of the ith agent computed at the kth sampling instant $t=kT$. In this case, discretizing the equations of motion (1) with sampling period T, we obtain

$$\begin{aligned} q_i[k+1] &= q_i[k] + Tp_i[k] + \tfrac{1}{2}T^2u_i[k], \quad q_i[0]=q_{i0}, \\ p_i[k+1] &= p_i[k] + Tu_i[k], \quad p_i[0]=p_{i0}, \\ & k\in\mathbb{N}_0, \quad i=1,\ldots,n, \end{aligned} \tag{7}$$

where $q_i[k]$ represents (resp., $p_i[k]$) the position (resp., velocity) of the ith agent at the kth sampling instant.

In our earlier work [9] it was shown that if the Laplacian matrix L represents a connected topology, then the control input (6) with

$$u[k] = -Lq[k] - Lp[k], \tag{8}$$

which is inspired by (4) for the continuous-time case, can stabilize the discretized multi-agent system in the case where the sampling period T is sufficiently small. This control law (6), (8) is *time invariant* in the sense that the communication topology does not change over time. Conversely, when there is a limitation on the available number of channels so that a connected topology cannot be produced, the control objective (2) is not achievable with the constant feedback gain as used in (8). Hence, we need to construct a control framework that requires switching of the communication network.

In this paper, we consider the *time-varying* form of (8) which is given by

$$u[k] = -L_{\xi(k)}q[k] - L_{\xi(k)}p[k], \tag{9}$$

where $\xi\in\mathbb{N}$ is used to index the network topology satisfying the limitation on the allowable number of channels and L_ξ is the Laplacian matrix corresponding to the network topology indexed by ξ. Specifically, we derive an $r\in\mathbb{N}$ periodic switching sequence $L_{\xi(k)}$, where $\xi(k) = 1+\mathrm{mod}(k,r)$, such that the closed-loop system achieves formation stability given by (2). In the following sections we consider two types of such limitations; one where each agent is allowed only a single channel at each sampling instant and the other where the overall multi-agent system is allowed only a limited number of channels which is not sufficient to produce a connected topology.

Now, defining $x[k] \triangleq \left[x_1^{\mathrm{T}}[k],\ldots,x_n^{\mathrm{T}}[k]\right]^{\mathrm{T}}$, where $x_i[k] = [q_i[k],p_i[k]]^{\mathrm{T}}$, the closed-loop system (1), (6), (9) becomes

$$x[k+1] = \hat{A}_{\xi(k)}(T)x[k], \quad x[0]=x_0, \quad k\in\mathbb{N}_0, \tag{10}$$

where

$$\begin{aligned} \hat{A}_\xi(T) &\triangleq I_n\otimes A(T) + L_\xi\otimes(B(T)K), \\ A(T) &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, B(T) = \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}, K=[-1,-1]. \end{aligned}$$

Note that (10) is viewed as a jump linear system.

3. Equivalence of the Control Objective

In this section, we consider the closed-loop system given by (10) and derive an equivalent statement of the control objective (2) for this system.

Since L_i, $i=1,\ldots,r$, are symmetric and nonnegative definite, it follows from the Schur decomposition that there exists an orthogonal matrix S such that $S^{\mathrm{T}}L_1S = \mathrm{diag}[\lambda_1,\ldots,\lambda_n] \triangleq U$ and $\mathrm{col}_1(S) = \frac{1}{\sqrt{n}}1_n$, implying $\lambda_1=0$. Now, consider the coordinate transformation

$$\tilde{x}[k] = (S\otimes I_2)^{\mathrm{T}}x[k]. \tag{11}$$

Using Lemma 4.1 in [9], (10) becomes

$$\tilde{x}[k+1] = \Lambda_{\xi(k)}(T)\tilde{x}[k], \;\; \tilde{x}[0] = (S\otimes I_2)^{\mathrm{T}}x_0, \;\; k\in\mathbb{N}_0, \tag{12}$$

where

$$\begin{aligned} \Lambda_\xi &\triangleq (S\otimes I_2)^{\mathrm{T}}\left(I_n\otimes A + L_\xi\otimes(BK)\right)(S\otimes I_2) \\ &= I_n\otimes A + (S^{\mathrm{T}}L_\xi S)\otimes(BK) \\ &= \left[\begin{array}{c|cccc} A & 0_2 & \cdots & 0_2 \\ \hline 0_2 & A & & \\ \vdots & & \ddots & \\ 0_2 & & & A \end{array}\right] + \left[\begin{array}{c|c} 0_2 & 0_2 \;\; \cdots \;\; 0_2 \\ \hline 0_2 & \\ \vdots & \tilde{U}_\xi\otimes(BK) \\ 0_2 & \end{array}\right] \end{aligned}$$

$$= \left[\begin{array}{c:ccc} A & 0_2 & \cdots & 0_2 \\ \hdashline 0_2 & & & \\ \vdots & & \hat{\Lambda}_\xi & \\ 0_2 & & & \end{array}\right], \tag{13}$$

$$\hat{\Lambda}_\xi \triangleq I_{n-1} \otimes A + \tilde{U}_\xi \otimes (BK), \tag{14}$$

and $\tilde{U}_\xi \in \mathbb{R}^{(n-1)\times(n-1)}$ is a symmetric matrix such that

$$S^{\mathrm{T}} L_\xi S = \left[\begin{array}{c:ccc} 0 & 0 & \cdots & 0 \\ \hdashline 0 & & & \\ \vdots & & \tilde{U}_\xi & \\ 0 & & & \end{array}\right]. \tag{15}$$

The block-diagonal structure of (13) implies that the closed-loop dynamics of the multi-agent system given by (12) can be decoupled into

$$\tilde{x}_{1:2}[k+1] = A(T)\tilde{x}_{1:2}[k], \tag{16}$$

$$\tilde{x}_{3:2n}[k+1] = \hat{\Lambda}_{\xi(k)}(T)\tilde{x}_{3:2n}[k]. \tag{17}$$

Here note that

$$\begin{aligned} \tilde{x}_{3:2n}[k] &= (\mathrm{row}_{3:2n}(S^{\mathrm{T}} \otimes I_2))x[k] \\ &= (\mathrm{row}_{2:n}(S^{\mathrm{T}}) \otimes I_2)x[k] \\ &= (\mathrm{row}_{2:n}(S^{\mathrm{T}}) \otimes I_2) \\ &\quad \cdot (q[k] \otimes [1,0]^{\mathrm{T}} + p[k] \otimes [0,1]^{\mathrm{T}}) \\ &= (\mathrm{row}_{2:n}(S^{\mathrm{T}}))q[k] \otimes [1,0]^{\mathrm{T}} \\ &\quad + (\mathrm{row}_{2:n}(S^{\mathrm{T}}))p[k] \otimes [0,1]^{\mathrm{T}}. \end{aligned} \tag{18}$$

Therefore, $\lim_{k\to\infty} \tilde{x}_{3:2n}[k] = 0$ if and only if

$$\begin{aligned} \lim_{k\to\infty} (\mathrm{row}_{2:n}(S^{\mathrm{T}}))q[k] &= 0 \text{ and} \\ \lim_{k\to\infty} (\mathrm{row}_{2:n}(S^{\mathrm{T}}))p[k] &= 0. \end{aligned} \tag{19}$$

Meanwhile, since S is orthogonal and $\mathrm{col}_1(S) = \frac{1}{\sqrt{n}}1_n$, it follows that $\mathrm{row}_{2:n}(S^{\mathrm{T}})1_n = 0$. Hence, (19) holds if and only if

$$\begin{aligned} \lim_{k\to\infty} q[k] &\in \{m_1 1_n : m_1 \in \mathbb{R}\} \text{ and} \\ \lim_{k\to\infty} p[k] &\in \{m_2 1_n : m_2 \in \mathbb{R}\}, \end{aligned} \tag{20}$$

i.e.,

$$\lim_{k\to\infty} (q_i[k] - q_j[k]) = 0 \quad \text{and} \quad \lim_{k\to\infty} (p_i[k] - p_j[k]) = 0, \quad i,j = 1,\ldots,n, \quad i \neq j. \tag{21}$$

Thus, it follows that the agents travel with asymptotic zero relative positions and common velocities if and only if (17) is asymptotically stable for the jump linear system given by (10).

4. Stability Conditions with Limited Information

4.1. Agent-wise Limited Information

In this section, we consider the case where only a single communication channel is allowed for each agent at every sampling instant. This consideration is due to the fact that if more than one channel is allowed per agent, then a connected topology can be created at every sampling instant, which simplifies the process of achieving the control objective. Furthermore, we assume that only the topologies which use the maximum number of allowable channels in total are used in the switching sequence, i.e., the total number of channels are always $\frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd. Thus, a multi-agent system consisting of n agents may take $\frac{n!}{2^{\frac{n}{2}}\frac{n}{2}!}$ possible combinations of topologies if n is even and $\frac{(n+1)!}{2^{\frac{n+1}{2}}\frac{n+1}{2}!}$ if n is odd. Here we let $\mathcal{C}_n$ represent this possible number of combinations.

Now, consider an r-periodic switching sequence out of the above topologies. In this case, (17) yields

$$\tilde{y}[\kappa+1] = \tilde{\Lambda}\tilde{y}[\kappa], \quad \kappa \in \mathbb{N}_0, \tag{22}$$

where $\tilde{y}[\kappa] = \tilde{x}_{3:2n}[\kappa r]$ and $\tilde{\Lambda} \triangleq \hat{\Lambda}_r \cdots \hat{\Lambda}_1$. Therefore, it follows that the agents travel with asymptotic zero relative positions and common velocities if and only if $\tilde{\Lambda}$ is asymptotically stable, i.e., the moduli of the eigenvalues of $\tilde{\Lambda}$ are all less than one.

4.2. System-wise Limited Information

In this section, we consider the case where at any sampling instant the overall multi-agent system is allowed only a limited number of channels which are not sufficient to produce a connected network topology. Again, we assume that only the topologies which consist of the maximum number of allowable channels are used in the switching sequence, and hence the possible number of combinations $\mathcal{C}_n$ of the network topologies is given by $\frac{n_{\mathrm{a}}!}{(n_{\mathrm{a}}-n_{\mathrm{c}})!n_{\mathrm{c}}!}$, where $n_{\mathrm{a}} = \frac{n(n-1)}{2}$ and $n_{\mathrm{c}}(< n_{\mathrm{a}})$ is the total number of communication channels allowed at the sampling instants. Note that even though a similar stability analysis as in Section 4.1 can be carried out, we take a different approach, which can be extended to the case of stochastic communication channel collapse with system-wise channel constraint, as discussed in Section 5.

For the statement of the following results, we use $p_{\mathrm{cm}}(t) \in \mathbb{R}$ to represent the velocity of the center of mass of the n agents and define $p_{\mathrm{cm}}(t)$ as

$$p_{\mathrm{cm}}(t) \triangleq \frac{1}{n}\sum_{i=1}^{n} p_i(t). \tag{23}$$

Theorem 4.1. Consider the multi-agent system of n agents given by (1) where the sampling period T is predetermined, and the control law given by (6) and (9). If we switch the communication topology with an r-periodic switching sequence $L_{\xi(k)}$, then the agents travel with asymptotic zero relative positions and common velocities (i.e., (2) is satisfied) if and only if $\hat{\Lambda}$ is asymptotically stable (Schur), where

$$\hat{\Lambda} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \hat{\Lambda}_r \\ \hat{\Lambda}_1 & 0 & \cdots & 0 & 0 \\ 0 & \hat{\Lambda}_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{\Lambda}_{r-1} & 0 \end{bmatrix}, \quad (24)$$

and $\hat{\Lambda}_i$, $i = 1, \ldots, r$, are given by (14). Furthermore, $\dot{p}_{\mathrm{cm}}(t) = 0$ for all $t \geq 0$.

Proof. First, note that the closed-loop system of the multi-agent system (1) with the control law given by (6) and (9) takes the form of (12). As stated in Section 3, the agents travel with asymptotic zero relative positions and common velocities if and only if (17) is asymptotically stable. Since (17) falls under the category of a Markov jump linear system with transition probability one as shown in Figure 4.1, it follows from [11] that (17) is asymptotically stable if and only if for all $R_i > 0$, $R_i \in \mathbb{R}^{(2n-2)\times(2n-2)}$, $i = 1, \ldots, r$, there exist $P_i > 0$, $P_i \in \mathbb{R}^{(2n-2)\times(2n-2)}$, $i = 1, \ldots, r$, such that the coupled Lyapunov equations

$$\begin{aligned} P_1 &= \hat{\Lambda}_1^{\mathrm{T}} P_2 \hat{\Lambda}_1 + R_1, \\ P_2 &= \hat{\Lambda}_2^{\mathrm{T}} P_3 \hat{\Lambda}_2 + R_2, \\ &\vdots \\ P_r &= \hat{\Lambda}_r^{\mathrm{T}} P_1 \hat{\Lambda}_r + R_r, \end{aligned}$$

or, equivalently,

$$P = \hat{\Lambda}^{\mathrm{T}} P \hat{\Lambda} + R, \quad (25)$$

are satisfied, where $P \triangleq$ block-diag$[P_1, \ldots, P_r]$ and $R \triangleq$ block-diag$[R_1, \ldots, R_r] > 0$. Finally, (25) is equivalent to the fact that $\hat{\Lambda}$ is asymptotically stable.

To prove $\dot{p}_{\mathrm{cm}}(t) = 0$, using (1), (6), (9), (23), and the fact that $L_\xi 1_n = 0$, $\xi = 1, \ldots, r$, it follows that

$$\begin{aligned} \dot{p}_{\mathrm{cm}}(t) &= \frac{1}{n}\sum_{i=1}^{n} \dot{p}_i(t) = \frac{1}{n}\sum_{i=1}^{n} u_i(t) \\ &= \frac{1}{n}\sum_{i=1}^{n} u_i[k] \\ &= \frac{1}{n} 1_n^{\mathrm{T}}(-L_{\xi(k)} q[k] - L_{\xi(k)} p[k]) \\ &= 0, \quad kT \leq t < (k+1)T, \quad k \in \mathbb{N}_0. \quad (26) \end{aligned}$$

□

Note that a similar expression as in (24) is derived in [12, 13] by expressing the discrete-time periodic switching system (17) as a time-invarient system of higher dimension.

5. Switching Controller with Stochastic Communication Channel Collapse

During the implementation of multi-agent systems, due to various physical reasons, the communication channels between the agents may collapse, preventing the state information from being shared between the agents. In this section,

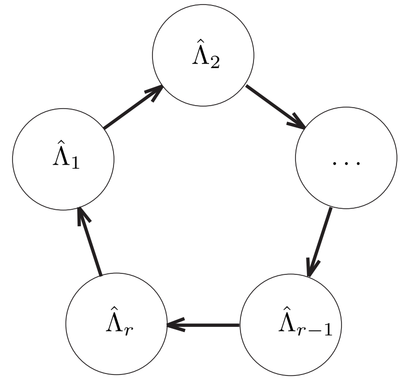

Figure 4.1: Periodic switching controller

we take the multi-agent system from the previous section which has a limitation on the number of channels and attains consensus by switching the network topologies. Specifically, assume that the communication channel between the agents i and j of this system collapses at the sampling instants with a known probability $e_{ij} (= e_{ji})$, $j \in \mathcal{N}_i$, $i \in \{1, \ldots, n\}$.

It is also assumed that if the controller of the agent i detects that the communication channel with the agent $j \in \mathcal{N}_i$ is missing, i.e., if the controller of the agent i cannot sense its relative state with the agent j, then it excludes the state of agent j in calculating the control input u_i, i.e., $j \notin \mathcal{N}_i$ in (3).

Since we assume that only the topologies which employ the maximum number n_{c} of channels are used in the switching sequence, there are $2^{n_{\mathrm{c}}}$ possible combinations of network topologies at every sampling instant k due to the stochastic communication channel collapse. Here, we use $\theta_\xi \in \{1, \ldots, 2^{n_{\mathrm{c}}}\}$, $\xi = 1, \ldots, r$, to index the above combinations of network topologies arising from L_ξ, the network topology when there is no communication channel collapse. Now, let $L_{\xi(k),\theta_{\xi(k)}}$ denote the network topology at the sampling instant k, where $\xi(k) = \mathrm{mod}(k, r) + 1$. Furthermore, let $p(\xi, \theta_\xi)$, $\xi \in \{1, \ldots, r\}$, $\theta_\xi \in \{1, \ldots, 2^{n_{\mathrm{c}}}\}$, represent the probability distribution of L_{ξ,θ_ξ} with respect to the random variable θ_ξ, which can be readily calculated using the given e_{ij}'s. Note that $\sum_{\theta_\xi=1}^{2^{n_{\mathrm{c}}}} p(\xi, \theta_\xi) = 1$, $\xi \in \{1, \ldots, r\}$.

Now, using the fact that the above multi-agent system with the switching controller and the stochastic communication channel collapses can be seen as a Markov jump linear system shown in Figure 5.1, we state the theorem on the stability of this system.

Theorem 5.1. Consider the multi-agent system of n agents given by (1) where the sampling period T is predetermined, and the control law given by (6) and (9). Suppose that the communication channels between the agents collapse at the sampling instants with known probability, i.e, the structure of the interconnection L_{ξ,θ_ξ} is a random variable with a known probability mass function $p(\xi, \theta_\xi)$. Then the relative states of the multi-agent system is mean-square stable (see Appendix A) if and only if for all $R \triangleq$ block-diag$[R_1, \ldots, R_r] > 0$ there exists $P \triangleq$ block-diag$[P_1, \ldots, P_r] > 0$ such that

$$P = r\mathbb{E}\left[\Delta_{\xi,\theta_\xi}^{\mathrm{T}} P \Delta_{\xi,\theta_\xi}\right] + R, \quad (27)$$

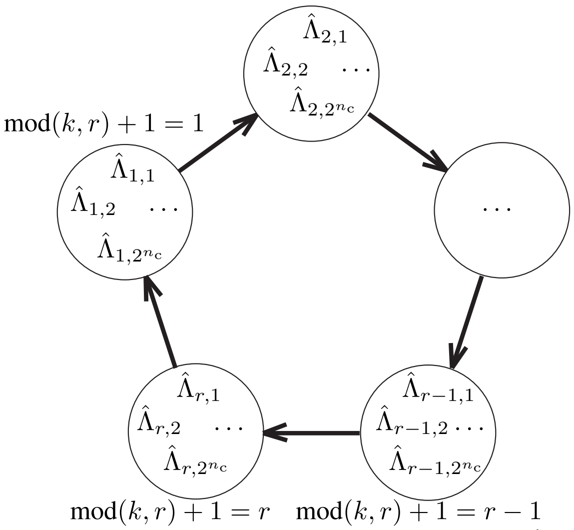

Figure 5.1: Periodic switching controller with stochastic communication channel collapse

where $\Delta_{\xi,\theta_\xi} \triangleq \Xi(\xi) \otimes \hat{\Lambda}_{\xi,\theta_\xi}$, $\Xi \in \mathbb{R}^{r\times r}$,

$$\Xi_{(i,j)}(\xi) \triangleq \begin{cases} 1, & i = \mathrm{mod}(\xi+1, r) \text{ and } j = \xi, \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

and $\mathbb{E}[\cdot]$ is the expected value with respect to the probability mass function $\frac{p(\xi,\theta_\xi)}{r}$. Furthermore, $\dot{p}_{\mathrm{cm}}(t) = 0$ for all $t \geq 0$.

Proof. First, note that the closed-loop system of the multi-agent system (1) with the control law given by (6) and (9) and the added assumption of stochastic communication channel collapse becomes a Markov jump linear system described by

$$x[k+1] = \hat{A}_{\xi(k),\theta_{\xi(k)}} x[k], \tag{29}$$

where $\hat{A}_{\xi,\theta_\xi} \triangleq I_n \otimes A(T) + L_{\xi,\theta_\xi} \otimes (B(T)K)$. Using the same techniques as in Section 3 and decoupling off the relative state dynamics from (29), it follows that

$$\tilde{x}[k+1] = \hat{\Lambda}_{\xi(k),\theta_{\xi(k)}} \tilde{x}[k]. \tag{30}$$

Now, since (30) can be seen as a Markov jump linear system given by Figure 5.1 and using the fact that the probability of the current network topology does not depend on the probability of the previous network topology it follows that the the relative states of the multi-agent system is mean-square stable if and only if for all $R_i > 0$, $i = 1, \ldots, r$, there exist $P_i > 0$, $i = 1, \ldots, r$, such that the coupled Lyapunov-like equations

$$\begin{aligned} P_1 &= \mathbb{E}_1\left[\hat{\Lambda}^{\mathrm{T}}_{1,\theta_1} P_2 \hat{\Lambda}_{1,\theta_1}\right] + R_1, \\ P_2 &= \mathbb{E}_2\left[\hat{\Lambda}^{\mathrm{T}}_{2,\theta_2} P_3 \hat{\Lambda}_{2,\theta_2}\right] + R_2, \\ &\vdots \\ P_r &= \mathbb{E}_r\left[\hat{\Lambda}^{\mathrm{T}}_{r,\theta_r} P_1 \hat{\Lambda}_{r,\theta_r}\right] + R_r, \end{aligned}$$

are satisfied, where $\mathbb{E}_i[\cdot]$ denotes the expected value with respect to the probability mass function $p(i,\theta_i)$. Now, putting together the above coupled Lyapunov-like equations gives (27).

Furthermore, the proof of $\dot{p}_{\mathrm{cm}}(t) = 0$ is similar to the proof of Theorem 4.1 by replacing L_ξ with L_{ξ,θ_ξ}. □

Here, note that the condition for mean-square stability in the Theorem 5.1 is equivalent to a feasible solution $P > 0$ existing for the linear matrix inequalities

$$0 > r\mathbb{E}\left[\Delta^{\mathrm{T}}_{\xi,\theta_\xi} P \Delta_{\xi,\theta_\xi}\right] - P. \tag{31}$$

Furthermore, note that the mean-square stability of the relative states implies almost sureness of the control objective given by (2).

6. Illustrative Numerical Example

Consider the multi-agent system (1) where the number of agents is $n = 4$ and the maximum number of communication channels allowed at each sampling instant is $n_{\mathrm{c}} = 2$. Since a connected topology cannot be created between the 4 agents using only 2 communication channels, we employ the 2-periodic switching controller given by (9) with

$$L_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

and

$$L_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{33}$$

Furthermore, assume that the communication channels collapse with an independent but equal probability p_{c}. In this case, it can be easily deduced that there are $2^2 = 4$ possible combinations of network topologies at each sampling instant and the probability of a certain network topology occurring can be calculated using p_{c}. It follows from Theorem 5.1 that the stability of the multi-agent system under the above settings can be determined by solving the linear matrix inequalities (31) and $P > 0$ and showing whether a feasible solution P exists.

With the initial condition $x_0 = [1, 1, 0, 0, 0, 0, 0, 0]^{\mathrm{T}}$ and $T = 0.1$, Figures 6.1 and 6.2 show the simulated results of the above multi-agent system with different channel collapse probabilities p_{c}. In Figure 6.1, where the communication channel collapse probability is low ($p_{\mathrm{c}} = 0.2$), the control objective is achieved. In this case, a feasible solution $P > 0$ exists for (31). On the other hand, when the communication channel collapse probability is high ($p_{\mathrm{c}} = 0.8$), the control objective is not achieved as shown in Figure 6.2. Indeed, a feasible solution $P > 0$ does not exist for (31).

7. Conclusion

In this paper we considered a sampled-data control framework for formation control of multi-agent systems in the face of the limitation on the number of communication channels available for the agents. Specifically, we constructed periodic switching controllers that change the network topology

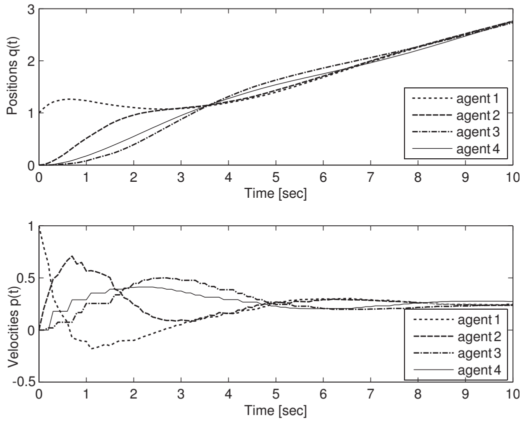

Figure 6.1: State trajectories versus time for $p_c = 0.2$

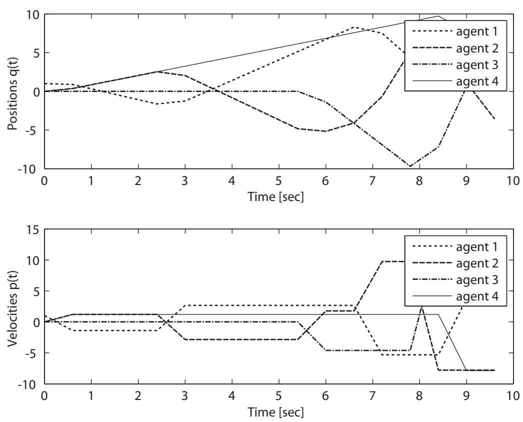

Figure 6.2: State trajectories versus time for $p_c = 0.8$

at every sampling instant. For the case where there is a limitation on the total number of communication channels, it was shown that formation stability can be determined through coupled Lyapunov equations. In addition, we derived the stability conditions for the case where the communication channels collapse with a known probability mass function.

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Appendix

A. Stability Notions in Stochastic Systems

Consider the Markov jump linear systems given by

$$x[k+1] = \Lambda_{\theta(k)} x[k], \quad x[0] = x_0, \quad k \in \mathbb{N}_0, \tag{34}$$

where the matrix Λ_θ is a function of θ, which takes values from a finite set $\{1, \ldots, \Theta\}$ with probability $p(\theta)$. In this section we first define stochastic stability, mean-square stability and exponential mean-square stability with respect to the system given by (34). And then show that these three notions of stability are equivalent in the case of a Markov jump linear systems given by (34).

Definition. For the above system given by (34), the equilibrium point $x = 0$ is

i) *stochastically stable* if for every initial state $x[0]$,

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \|x[k]\|^2\right] < \infty. \tag{35}$$

ii) *mean-square stable* if for every initial state $x[0]$,

$$\lim_{k\to\infty} \mathbb{E}\left[\|x[k]\|^2\right] = 0. \tag{36}$$

Note that here $\mathbb{E}[\,\cdot\,]$ denotes the expected value with respect to the probability distribution $p : \{1, \ldots, \Theta\} \times \mathbb{R}^n \to [0, 1]$, $\sum_{\theta=1}^{\Theta} p(\theta) = 1$.

Theorem A.1. For the Markov jump linear systems given by (34), it follows that stochastic stability and mean-square stability are equivalent.

Proof. see [14].