2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Sliding Mode Adaptive State Observation for Time-Delay Uncertain Nonlinear Systems

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Abstract— **In this paper a method to design robust adaptive sliding mode observers (ASMO) for a class of nonlinear time-delay systems with uncertainties, is proposed. The objective is to achieve insensitivity and robustness of the proposed sliding mode observer to matched disturbances. A novel systematic design method is synthesized to solve matching conditions and compute observer stabilizing gains. The Lyapunov-Krasovskii theorem is employed to prove the ultimate stability with arbitrary boundedness radius of the estimation error of the proposed filter. Finally, the ability of ASMO for fault reconstruction is studied.**

I. INTRODUCTION

The discontinuous approach of control design has attracted the attention of many researchers in recent years [1], [2]. Sliding mode control theory has been developed as a practical strategy to be implemented with uncertain systems (see [12] and references therein). The design of early sliding mode controllers assumed the availability of full state information which is not possible in many real systems. These facts motivate employing the sliding mode technique to design robust nonlinear state observers. Variable structure observer design has received significant attention in recent years [3]-[11]. By injecting a nonlinear discontinuous term, the sliding mode observer (SMO) forces the trajectories of the estimated error to remain on a sliding surface in the error space after a finite time. Therefore nonlinear sliding injection enables the robust observer to reject disturbances.

[3] designed a sliding observer based on the Lyapunov theorem to prove the stability. [4] proposed a simple observer with a discontinuous sliding term fed back through a suitable gain design. A sliding mode observer for nonlinear models with unbounded noise and measurement uncertainties was studied by [5]. [6] proposed a canonical form of sliding observer design in which a sufficient linear matrix inequality (LMI) was derived. Their method is based on some complex coordinate transformations. An LMI based sliding mode observer design method was proposed by [7] for a class of multivariable uncertain systems with matched uncertainties. The gain matrices of the sliding mode observer are characterized using the solution of the LMI existence condition which does not suffer from complexity. Sliding mode observer design for a class of nonlinear systems in which the nonlinear part satisfies the Lipschitz condition, whilst the uncertain part is bounded, was addressed by [8]. [9] designed a new systematic sliding mode observer for nonlinear systems subject to unknown inputs. An adaptive sliding mode observer with a boundary layer sliding term was suggested by [10]. Recently a second-order sliding mode observer based on a super-twisting algorithm was studied in order to design a robust state estimator for uncertain mechanical systems [11].

In this paper we propose a sliding mode observer design method to tackle matched disturbance for time delay Lipschitz systems. Moreover, most of the previous work deals with non-adaptive sliding mode observers necessitating knowing the upper bound of uncertainties. We will employ adaptive algorithms, since the matched disturbance is assumed to be unknown (but bounded).

Constant or time-varying delay is frequently encountered in engineering systems to be controlled or observed [13]-[15], and is commonly a source of instability. For uncertain dynamical systems with a time delay, no results for designing SMO have been reported in the control literature. We will consider time delay in our problem and prove the stability of the robust observation error.

Another contribution of this paper is to develop the use of observer information (state estimates) as an upper bound of matched uncertainties under a Lipschitz constraint. It should be noted that in all previous work the uncertainty or disturbance was assumed to be bounded by functions of only the output measurement of the system. To cope with this, a particular adaptive compensator will be constructed to guarantee the stability of the error system. Furthermore, a state transformation matrix based on the orthogonal complement concept is employed to analyze the error system in the sliding mode.

Finally a systematic approach using the orthogonal complements and generalized pseudo-inverse will be proposed to solve straightforwardly the matching conditions and compute the observer gain.

Section 2 provides preliminaries and the assumptions of the nonlinear system to be addressed. The design of the adaptive sliding mode observer and the analysis of the stability for the error dynamic system are given in Section 3. Synthesis of the error system in the sliding mode will be studied in Section 4. In Section 5 we propose a systematic design procedure. The ASMO based disturbance (fault) estimation will be studied in Section 6, and will be followed by some concluding remarks in Section 7.

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II. Preliminaries and Assumptions

Consider an uncertain nonlinear time-delay system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + \sum_{j=1}^{r} A_j x(t-\tau_j(t)) + f(x,t) + \phi(y) \\ \quad +B(u(t)+g(x,u,t)) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x \in R^n$ is the system state, $u \in R^m$ is the control input, $y \in R^p$ represents the measured system output and $t \in R^+$ Assuming that $n \geq p \geq m$ and $n \geq p \geq q$. $f(x,t): R^n \times R^+ \rightarrow R^n$ and $\phi(y): R^p \rightarrow R^n$ are the known nonlinear parts of the system. $g(x,u,t): R^n \times R^m \times R^+ \rightarrow R^m$ is matched uncertainty and disturbance. Furthermore (A,B,C,D) is the set of real constant known matrices of appropriate dimensions with B and C both being full rank. $\tau_j(t), j=1,..,r$ are known continuously differentiable time delays satisfying $\tau_j(t) \leq \tau$ and $\frac{d}{dt}\tau_j(t) \leq d_j \leq 1$ for all $t > 0$. Finally, we make the following assumptions:
p
(**A1**) ([6], [21]) Assume that

- rank(CB) = rank (B).
- The triple (A,B,C) is minimum phase or equivalently, the invariant zeros of (A,B,C) are in C_{-}.

(**A2**) The pair (A,C) is assumed to be observable so that there exists an observer gain $K \in R^{n\times p}$ such that $A_0 = A - KC$ is a strictly Hurwitz matrix.

(**A3**) The known nonlinearity $f(x,u,t)$ satisfies a Lipshitz condition

$$\|f(x_1,u,t) - f(x_2,u,t)\| \leq \gamma_f \|x_1 - x_2\| \tag{2}$$

where $x_1, x_2 \in R^n$ and $\gamma_f \in R^+$ is a known positive constant.

(**A4**) The following algebraic Riccati equation (ARE) equation has a positive solution $P = P^\top > 0 \in R^{n\times n}$ for a positive definite matrices $Q = Q^T > 0$, $\bar{P}_j = \bar{P}_j^\top > 0 \in R^{n\times n)}$, $\bar{P}_j = (1-d_j)P_j$ and $\varepsilon > 0$

$$A_0^\top P + PA_0 + \sum_{j=1}^{r}(P_j + PA_j\bar{P}_j^{-1}A_j^\top P) + \varepsilon I = -Q \tag{3}$$

(**A5**) *Matching Condition*: Assume that there exists an arbitrary matrix $F \in R^{m\times p}$ satisfying [3]

$$B^\top P = FC \tag{4}$$

(**A6**) The matched uncertainty $g(x,u,t)$ is bounded (but unknown) in the Euclidean norm such as

$$\|g(x,u,t)\| \leq \sum_{i=1}^{\bar{m}} \rho_i \alpha_i(x,y,u,t) \doteq \alpha^\top(x,y,u,t)\rho \leq l_\rho < \infty \tag{5}$$

where $\rho \in R^{\bar{m}}$ and $\alpha: R^n \times R^p \times R^m \times R^+ \rightarrow R^{\bar{m}}$ are respectively an unknown constant vector and an unknown vector function of the form

$$\alpha(x,y,u,t) = [\ \alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_{\bar{m}}]^\top,\ \alpha_i(x,y,u,t) \geq 0$$

$$\rho = [\ \rho_1 \quad \rho_2 \quad \ldots \quad \rho_{\bar{m}}]^\top\ ,\ \rho_i \geq 0$$

Without loss of generality it is assumed that $\alpha(x,y,u,t)$ is continuous uniformly with respect to time and locally uniform bounded with respect to $x(t), u(t)$ and Lipschitzian

$$\|\alpha(x_1,y,u,t) - \alpha(x_2,y,u,t)\| \leq \gamma_\alpha \|x_1 - x_2\| \tag{6}$$

where $x_1, x_2 \in R^n$ and $\gamma_\alpha \in R^+$ is an unknown positive Lipschitz constant.

Remark 2.1: It is well known that the matching condition (4) is satisfied if and only if Assumption (A1) holds [21]. Furthermore, Assumption (A1) presents existence condition of the stable sliding motion (see [22] for details).

III. Adaptive Sliding Mode Observer Design

In this section, we propose our new improved ASMO to reconstruct the states of the uncertain time-delay nonlinear system (1). Later, we will analyze its stability using Lyapunov theory. Consider the following sliding filter

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \sum_{j=1}^{r} A_j\hat{x}(t-\tau_j(t)) + f(\hat{x},u,t) + Bu(t) \\ \quad +\phi(y) + K(y - C\hat{x}) + M(t) \\ M(t) = S(\hat{x},y,u,\hat{\rho}(t)) + B\hat{\Xi}(t)F(y - C\hat{x}) \end{cases} \tag{7}$$

in which

$$S(t) = (\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t))^2 P^{-1}C^\top F^\top \frac{FCe}{\|FCe\|\,\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t) + \delta} \tag{8}$$

where $S: R^n \times R^m \times R^+ \rightarrow R^n$ is adaptive smooth sliding surface with continuous approximating factor $\delta \in R^+$ and $\delta \ll 1$. $\hat{\rho}(t) \in R^{\bar{m}}$ is the adaptive sliding estimate of $\rho \in R^{\bar{m}}$ respectively. We establish $\hat{\Xi}(t) \in R^+$ as an adaptive compensation gain correspond to the Lipschitzian Assumption (**A6**). The essential role of the term $\hat{\Xi}(t)$ will be studied later. The parameters $\hat{\rho}(t)$ and $\hat{\Xi}(t)$ are updated by the followings continuous adaptation algorithms

$$\begin{cases} \frac{d\hat{\rho}(t)}{dt} = \Gamma_\rho(-\eta\hat{\rho}(t) + \|FCe(t)\|\,\alpha(\hat{x},y,u,t)) \\ \frac{d\hat{\Xi}(t)}{dt} = \Gamma_\Xi(-\eta\hat{\Xi}(t) + \|FCe(t)\|^2) \end{cases} \tag{9}$$

where Γ_ρ is a positive definite matrix of appropriate dimension. Γ_Ξ is a positive scalar constant and $0 < \eta \ll 1$. Additionally $\rho(0)$ and $\Xi(0)$ are finite. Thus, we have the following theorem:

Theorem 1: Given the nonlinear uncertain time delay system (1) with the associated assumptions (**A1**)-(**A7**), the robust adaptive sliding mode observer (7)-(9) results in the uniformly ultimately bounded error of the state reconstruction.

Proof: Consider the Lyapunov-Krasovskii function

$$\begin{aligned} V(e(t),\tilde{\rho}(t),\tilde{\Xi}(t)) &= e^\top(t)Pe(t) + \sum_{j=1}^{r}\int_{t-\tau_j(t)}^{t} e^\top(t)P_je(t) \\ &+\tilde{\rho}^\top(t)\Gamma_\rho^{-1}\tilde{\rho}(t) + \Gamma_\Xi^{-1}\tilde{\Xi}^2(t) \end{aligned} \tag{10}$$

where P is the solution of the algebraic Riccati differential equation (3), $e(t)=x(t)-\hat{x}(t)$ is defined as the state estimation error and furthermore

$$\tilde{\rho}(t)=\rho-\hat{\rho}(t),\ \tilde{\Xi}(t)=\Xi-\hat{\Xi}(t)$$

Using (1) and (7) the evolution of the estimation error dynamics is

$$\begin{aligned}\dot{e}(t)&=A_0e(t)+\sum_{j=1}^{r}A_je(t-\tau_j(t))+(f(x,u,t)-f(\hat{x},u,t))\\&+Bg(x,u,t)-S(t)-B\hat{\Xi}(t)F(y-C\hat{x})\end{aligned} \tag{11}$$

The derivative of $V(e(t),\tilde{\rho}(t),\tilde{\Xi}(t))$ is evaluated along $e(t)$, $\hat{\rho}(t)$ and $\hat{\Xi}(t)$

$$\begin{aligned}\dot{V}=e_\tau^\top(t)&\begin{bmatrix} A_0^\top P+PA_0+\sum_{j=1}^{r}P_j+\varepsilon I & PA_1 & \cdots & PA_r\\ A_1^\top P & -\bar{P}_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_r^\top P & 0 & \cdots & -\bar{P}_r\end{bmatrix}e_\tau(t)\\&-e^\top(t)(\varepsilon I)e(t)+2e^\top(t)P(f(x,u,t)-f(\hat{x},u,t))\\&+2e^\top(t)P(Bg(x,u,t)-S(t))\\&-2e^\top(t)PB\hat{\Xi}(t)FCe(t)\\&+2\left(\tilde{\rho}^\top(t)\Gamma_\rho^{-1}\frac{d\tilde{\rho}(t)}{dt}+\Gamma_\Xi^{-1}\tilde{\Xi}(t)\frac{d\tilde{\Xi}(t)}{dt}\right)\end{aligned}$$

where

$$e_\tau(t)=\begin{bmatrix} e(t)\\ e(t-\tau_1(t))\\ \vdots\\ e(t-\tau_r(t))\end{bmatrix}$$

Defining

$$-\underline{Q}=\begin{bmatrix} A_0^\top P+PA_0+\sum_{j=1}^{r}P_j+\varepsilon I & PA_1 & \cdots & PA_r\\ A_1^\top P & -\bar{P}_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_r^\top P & 0 & \cdots & -\bar{P}_r\end{bmatrix}$$

and taking into account Assumption (**A3**) we get

$$\begin{aligned}\dot{V}&\leq -e_\tau^\top(t)\underline{Q}e_\tau(t)-e^\top(t)(\varepsilon I)e(t)+2\gamma_f\lambda_M(P)\|e(t)\|^2\\&+2e^\top(t)P(Bg(x,u,t)-S(t))\\&-2e^\top(t)PB\hat{\Xi}(t)FCe(t)\\&+2\left(\tilde{\rho}^\top(t)\Gamma_\rho^{-1}\frac{d\tilde{\rho}(t)}{dt}+\Gamma_\Xi^{-1}\tilde{\Xi}(t)\frac{d\tilde{\Xi}(t)}{dt}\right)\end{aligned} \tag{12}$$

First we seek the condition to attain the negative definiteness of the matrix $-\underline{Q}$, i.e.

$$-\underline{Q}<0 \tag{13}$$

As indicated by [18], the following quadratic structure can be assumed

$$-\underline{Q}=T^\top NT \tag{14}$$

where

$$N=\begin{bmatrix}\Delta & 0 & \dots & 0\\ 0 & -\bar{P}_1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & -\bar{P}_r\end{bmatrix}$$

and

$$T=\begin{bmatrix} I_n & 0 & \dots & 0\\ -\bar{P}_1^{-1}A_1^\top P & I_n & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\bar{P}_r^{-1}A_1^\top P & 0 & \dots & I_n\end{bmatrix}$$

in which T is nonsingular and

$$\Delta=A_0^\top P+PA_0+\sum_{j=1}^{r}(P_j+PA_j\bar{P}_j^{-1}A_j^\top P)+\varepsilon I \tag{15}$$

If $\Delta<0$, then it can be shown that $N<0$ since $P_j>0, j=1,\dots,r$ Thus

$$T^\top NT<0 \tag{16}$$

and (13) is satisfied. Additionally, with regard to Assumptions (**A5**)-(**A6**) and (8), one can obtain

$$\begin{aligned}&2e^T(t)P(Bg(x,u,t)-S(\hat{x}(t),y(t),\hat{\rho}(t)))=\\&2e^T(t)PP^{-1}C^\top F^\top g(x,u,t)\\&\quad-2e^\top(t)P(\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t))^2P^{-1}C^\top F^\top\frac{FCe}{\|FCe\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)+\delta}\\&\quad\leq 2\|FCe(t)\|\sum_{i=1}^{\bar{m}}\rho_i(t)\alpha_i(x,y,u,t)\\&\quad\quad-2(\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t))^2\frac{\|FCe(t)\|^2}{\|FCe(t)\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)+\delta}\end{aligned}$$

Using $\rho=\tilde{\rho}(t)+\hat{\rho}(t)$ and by adding and subtracting the term $2\|FCe(t)\|\rho\alpha^\top(\hat{x},y,u,t)$

$$\begin{aligned}&2e^T(t)P(Bg(x,u,t)-S(\hat{x}(t),y(t),\hat{\rho}(t)))\leq\\&\quad 2\|FCe(t)\|\left\|\alpha^\top(x,y,u,t)-\alpha^\top(\hat{x},y,u,t)\right\|\|\rho\|\\&\quad+2\|FCe(t)\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)\\&\quad-2(\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t))^2\frac{\|FCe(t)\|^2}{\|FCe(t)\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)+\delta}\\&\quad+2\|FCe(t)\|\alpha^\top(\hat{x},y,u,t)\tilde{\rho}(t)\\&\quad=2\|FCe(t)\|\left\|\alpha^\top(x,y,u,t)-\alpha^\top(\hat{x},y,u,t)\right\|\|\rho\|\\&\quad+2\frac{\|FCe(t)\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)\delta}{\|FCe(t)\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)+\delta}\\&\quad+2\|FCe(t)\|\alpha^\top(\hat{x},y,u,t)\tilde{\rho}(t)\end{aligned}$$

Using the well-known inequalities

$$\begin{aligned}0\leq\frac{ab}{a+b}\leq b,&\qquad\forall a,b\in R^+\\ 2ab\leq\varepsilon^{-1}a^2+\varepsilon b^2,&\qquad\forall a,b\in R,\ \forall\varepsilon\in R^+\end{aligned}$$

because of inequality (6), after some manipulation we obtain

$$
\begin{aligned}
&2e^T(t)P(Bg(x,u,t)-S(\hat{x}(t),y(t),\hat{\rho}(t)))\leq\\
&\;2\left\|FCe(t)\right\|\left\|\alpha^\top(x,y,u,t)-\alpha^\top(\hat{x},y,u,t)\right\|\left\|\rho\right\|\\
&\;+2\left\|FCe(t)\right\|\alpha^\top(\hat{x},y,u,t)\tilde{\rho}(t)+2\tfrac{\left\|FCe(t)\right\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)\delta}{\left\|FCe(t)\right\|\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t)+\delta}\\
&\quad\leq 2\left\|FCe(t)\right\|\left\|\rho\right\|\gamma_\alpha\left\|e(t)\right\|+2\delta\\
&+2\left\|FCe(t)\right\|\alpha^\top(\hat{x},y,u,t)\tilde{\rho}(t)\\
&\quad\leq\varepsilon^{-1}(\rho\gamma_\alpha)^2\left\|FCe(t)\right\|^2+e^\top(t)(\varepsilon I)e(t)+2\delta\\
&\quad\;+2\left\|FCe(t)\right\|\alpha^\top(\hat{x},y,u,t)\tilde{\rho}(t)
\end{aligned}
\tag{17}
$$

By substituting (17) into (12)

$$
\begin{aligned}
\dot{V}\leq&-e_\tau^\top(t)\underline{Q}e_\tau(t)+2\gamma_f\lambda_M(P)\left\|e(t)\right\|^2\\
&+\varepsilon^{-1}(\left\|\rho\right\|\gamma_\alpha)^2\left\|FCe(t)\right\|^2+2\delta\\
&+2\left\|FCe(t)\right\|\alpha^\top(\hat{x},y,u,t)\tilde{\rho}(t)\\
&-2e^T(t)PB\hat{\Xi}(t)FCe(t)+2(\tilde{\rho}^\top(t)\Gamma_\rho^{-1}\tfrac{d\tilde{\rho}(t)}{dt}\\
&+\Gamma_\Xi^{-1}\tilde{\Xi}(t)\tfrac{d\tilde{\Xi}(t)}{dt})
\end{aligned}
$$

Let

$$
\Xi=\frac{\varepsilon^{-1}(\left\|\rho\right\|\gamma_\alpha)^2}{2}
$$

Thus

$$
\begin{aligned}
\dot{V}\leq&-e_\tau^\top(t)\underline{Q}e_\tau(t)+2\gamma_f\lambda_M(P)\left\|e(t)\right\|^2\\
&+2\Xi\left\|FCe(t)\right\|^2+2\delta+2\left\|FCe(t)\right\|\alpha^\top(\hat{x},y,u,t)\tilde{\rho}(t)\\
&-2\hat{\Xi}(t)\left\|FCe(t)\right\|^2+2(\tilde{\rho}^\top(t)\Gamma_\rho^{-1}\tfrac{d\tilde{\rho}(t)}{dt}\\
&+\Gamma_\Xi^{-1}\tilde{\Xi}(t)\tfrac{d\tilde{\Xi}(t)}{dt})
\end{aligned}
$$

Using the adaptation laws (9) yields

$$
\begin{aligned}
\dot{V}\leq&-e_\tau^\top(t)\underline{Q}e_\tau(t)+(2\gamma_f\lambda_M(P)+1)\left\|e(t)\right\|^2\\
&+4\delta+2\eta(\tilde{\rho}(t)^\top\hat{\rho}(t)+\tilde{\Xi}(t)\hat{\Xi}(t))
\end{aligned}
$$

Taking into account the satisfied condition (13), or equivalently

$$
A_0^\top P+PA_0+\sum_{j=1}^{r}(P_j+PA_j\bar{P}_j^{-1}A_j^\top P)+\varepsilon I=-Q
$$

and using the inequality $\left\|e_\tau(t)\right\|\geq\left\|e(t)\right\|$, one can simplify the above inequality

$$
\begin{aligned}
\dot{V}\leq&-\lambda_m(\underline{Q})\left\|e_\tau(t)\right\|^2+(2\gamma_f\lambda_M(P)+1)\left\|e_\tau(t)\right\|^2\\
&-2\eta\{\sum_{i=1}^{\bar{m}}(\tfrac{1}{2}\rho_i-\hat{\rho}_i(t))^2+(\tfrac{1}{2}\Xi-\hat{\Xi}(t))^2\}\\
&+\tfrac{1}{2}\eta(\sum_{i=1}^{\bar{m}}\rho_i^2+\Xi^2)+4\delta\\
&\leq-(\lambda_m(\underline{Q})-(2\gamma_f\lambda_M(P)+1))\left\|e_\tau(t)\right\|^2\\
&\;+\tfrac{1}{2}\eta(\sum_{i=1}^{\bar{m}}\rho_i^2+\Xi^2)+4\delta\\
&\;\doteq-(\lambda_m(\underline{Q})-(2\gamma_f\lambda_M(P)+1))\left\|e_\tau(t)\right\|^2+\varpi
\end{aligned}
\tag{18}
$$

Thus, if

$$
\lambda_m(\underline{Q})\geq 2\gamma_f\lambda_M(P)+1 \tag{19}
$$

then the uniformly ultimate stability of the error system is guaranteed. Thus the proof is complete.

Remark: From (18) we have

$$
\varpi=\frac{1}{2}\eta(\sum_{i=1}^{\bar{m}}\rho_i^2+\Xi^2)+4\delta
$$

We can set optionally the parameters $\delta=\eta=0$. Thus

$$
\varpi=0
$$

and

$$
\dot{V}\leq-(\lambda_m(\underline{Q})-(2\gamma_f\lambda_M(P)+1))\left\|e_\tau(t)\right\|^2\leq 0
$$

Integrating the above inequality on the interval $[0,t]$ leads to

$$
V(t)\leq V(0)-\int_0^t(\lambda_m(\underline{Q})-(2\gamma_f\lambda_M(P)+1))\left\|e_\tau(s)\right\|^2ds
$$

Thus

$$
\begin{aligned}
V(0)&\geq V(t)+\int_0^t(\lambda_m(\underline{Q})-(2\gamma_f\lambda_M(P)+1))\left\|e_\tau(s)\right\|^2ds\\
&\geq\int_0^t(\lambda_m(\underline{Q})-(2\gamma_f\lambda_M(P)+1))\left\|e_\tau(s)\right\|^2ds
\end{aligned}
$$

Taking the limit as $t\rightarrow\infty$ on both sides, yields

$$
\infty>V(0)\geq(\lambda_m(\underline{Q})-(2\gamma_f\lambda_M(P)+1))\lim_{t\rightarrow\infty}\int_0^t\left\|e_\tau(s)\right\|^2ds
$$

From the well-known Barbalat Lemma [16] we obtain the global asymptotic convergence of the error

$$
\lim_{t\rightarrow\infty}\left\|e_\tau(t)\right\|=0 \tag{20}
$$

IV. Synthesis of the Error System in the Sliding Mode

In this section a sliding mode insensitivity synthesis procedure (similar to [7]) is set up for the system (1). The method is based on a state transformation matrix. From the structure of the sliding feedback injection gains (8) we can conclude that the hyperplane in the error space is $\mathscr{S}=\{e(t)\in R^n:FCe=0\}$. The matrix $F\in R^{m\times p}$ is scaling design parameter and therefore by choice can be chosen to be full row rank. In the case $m=p$, one can easily conclude that

$$
\mathscr{N}(F)=\emptyset
$$

So the sliding hyperplane is reduced to classical sliding motion $\mathscr{S}=\left\{e(t)\in R^n:e_y=Ce=0\right\}$. This implies that the observer necessarily tracks the system outputs. The case $m<p$ declares that the null space of scaling matrix F is nonempty, hence the ASMO tracks a necessary subspace of the system output to estimate robustly the system states in the presence of matched uncertainties. Consider the error system (11). By fulfilling the conditions (4) and (19) and setting $\varpi=0$, the asymptotic stability of the reduced order system in the sliding mode is attained. Assume that T is the nonsingular transformation matrix

$$
T=\left[\begin{array}{c}\left(C^{\top\perp}\right)^\top P\\ FC\end{array}\right]\doteq\left[\begin{array}{c}\Omega^\top P\\ FC\end{array}\right] \tag{21}
$$

where $(C^{\top\perp}) = \Omega$ is any permissible full rank matrix whose columns from the basis of the null space of the matrix C, i.e., Ω is an orthogonal complement of the matrix $C^{\top}$. Thus

$$T^{-1} = \begin{bmatrix} \Omega(\Omega^{\top}P\Omega)^{-1} & P^{-1}C^{\top}F^{\top}(FCP^{-1}C^{\top}F^{\top})^{-1} \end{bmatrix}$$

Let

$$\begin{bmatrix} e_1 \\ e_y \end{bmatrix} = Te$$

then

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_y \end{bmatrix} = TA_0T^{-1}\begin{bmatrix} e_1 \\ e_y \end{bmatrix} + \sum_{j=1}^{r}\left(TA_jT^{-1}\begin{bmatrix} e_1(t-\tau_j(t)) \\ e_y(t-\tau_j(t)) \end{bmatrix}\right)$$
$$+T(f(x,u,t)-f(\hat{x},u,t)) + TBg(x,u,t)$$
$$-T(S(t)+B\hat{\Xi}(t)F(y-C\hat{x}))$$

Using (23) one can easily obtain

$$TB = \begin{bmatrix} 0 \\ FCP^{-1}C^{\top}F^{\top} \end{bmatrix}$$

Considering (9) and (10) we can rewrite $M(t) = P^{-1}C^{\top}N(t)$ thus the regular form is

$$\begin{aligned} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_y \end{bmatrix} &= \begin{bmatrix} A_{0_{11}} & A_{0_{12}} \\ A_{0_{21}} & A_{0_{22}} \end{bmatrix}\begin{bmatrix} e_1 \\ e_y \end{bmatrix} \\ &+ \sum_{j=1}^{r}\begin{bmatrix} A_{j_{11}} & A_{j_{12}} \\ A_{j_{21}} & A_{j_{22}} \end{bmatrix}\begin{bmatrix} e_1(t-\tau_j(t)) \\ e_y(t-\tau_j(t)) \end{bmatrix} \\ &+ \begin{bmatrix} \Omega^{\top}P \\ FC \end{bmatrix}(f(x,u,t)-f(\hat{x},u,t)) \\ &+ \begin{bmatrix} 0 \\ FCP^{-1}C^{\top} \end{bmatrix}(F^{\top}g(x,u,t)-N(t)) \end{aligned} \tag{22}$$

When the sliding mode arises we have $\forall t \in R^{+} : \dot{e}_y = e_y = 0$. Consequently the reduced order system in the sliding mode is governed by

$$\dot{e}_1 = A_{0_{11}}e_1 + \Omega^{\top}P(f(x,u,t)-f(\hat{x},u,t)) + \sum_{j=1}^{r}A_{j_{11}}e_1(t-\tau_j(t))$$

where

$$\begin{cases} A_{0_{11}} = \Omega^{\top}PA_0\Omega(\Omega^{\top}P\Omega)^{-1} = \Omega^{\top}PA\Omega(\Omega^{\top}P\Omega)^{-1} \\ A_{j_{11}} = \Omega^{\top}PA_j\Omega(\Omega^{\top}P\Omega)^{-1} \end{cases}$$

which the above reduced order system is clearly not sensitive to any matched disturbance. In addition, from the structure of $A_{0_{11}}$ one can deduce that the reduced order system characteristic is independent of the linear observer gain K.

Remark 4.1: In practice $\varpi \neq 0$. Therefore, for $0 < \varpi \ll 1$ the above discussion is valid in practice because the eventual tolerance error in the state estimation will be very small.

V. Recommended Design Procedure

To design robust adaptive sliding mode observers we need to determine the solution for the ARE (3) with the structure constrained by the matching condition (4). Consequently the designer has to find proper P, Q, F matrices in order to find a stabilizing observer gain K. In this section we look for the solutions for matching condition (4) and simultaneously assigning the stabilizing gain K.

A. Solution of the Matching Conditions

As we are dealing with both matched and mismatched disturbances in our problem, the simultaneous fulfillment of the classical matching conditions (4) is the essential step in the design of the proposed sliding mode observer. With regard to Assumption (**A1**), (4) can be straightforwardly solved exactly [7], if we select

$$\begin{cases} P = B^{\perp}X_1B^{\perp\top} + C^{\top}X_2C \\ B^{\perp}X_1B^{\perp\top} + C^{\top}X_2C > 0 \\ F = B^{\top}C^{\top}X_2 \end{cases} \tag{23}$$

where X_1 and X_2 are arbitrary weighting symmetric matrices with appropriate dimensions. Moreover, $B^{\perp}$ is any permissable full rank matrix whose columns form the basis of the null space of the matrix $B^{\top}$. Therefore, $B^{\perp}$ is a permissable orthogonal complement of the matrix B.

B. Observer Gain Design

In this subsection we investigate a design technique to assign the observer gain K with regard to the particular solution P in (24). Considering equation (3) we can rewrite

$$\Lambda = A^{\top}P + PA + \sum_{j=1}^{r}(P_j + PA_j\bar{P}_j^{-1}A_j^{\top}P) + \varepsilon I + Q = C^{\top}K^{\top}P + PKC \tag{24}$$

Assume that $C^{\dagger}$ is the generalized pseudo-inverse of C. Regarding (24) the condition

$$(I - C^{\dagger}C)\Lambda(I - C^{\dagger}C) = 0 \tag{25}$$

is both necessary and sufficient for the assignability of the desired P [5]. The necessity can be basically proven using the structure of Λ in (24) and the properties of pseudo-inverse [5]. Using (25) we obtain

$$\Lambda - \Lambda C^{\dagger}C - C^{\dagger}C\Lambda + C^{\dagger}C\Lambda C^{\dagger}C = 0$$

Using the well known Moore-Penrose pseudo-inverse $C^{\dagger}C = (C^{\dagger}C)^{\top}$ and the fact that $\Lambda = \Lambda^{\top}$

$$\Lambda = \Lambda C^{\dagger}C + C^{\dagger}C\Lambda - C^{\dagger}C\Lambda C^{\dagger}C \overset{\triangle}{=} N + N^{\top} - M \tag{26}$$

Assume that matrix $\bar{S}$ is skew-symmetric and satisfies

$$\bar{S}(I - C^{\dagger}C) = 0$$

If we choose

$$PKC = -\frac{1}{2}M + N + \bar{S} = \left[(I - \frac{1}{2}C^{\dagger}C)\Lambda + \bar{S}\right]C^{\dagger}C \tag{27}$$

then

$$PKC + C^\top K^\top P = \Lambda$$

which obviously guarantees that there exists a gain K that satisfies (24). Consequently

$$K = P^{-1}\left[(I - \frac{1}{2}C^\dagger C)\Lambda + \bar{S}\right]C^\dagger \tag{28}$$

which proves the sufficiency of (25) with the construction of the necessary gain matrix K. Finally if $A - KC$ is Hurwitz then the assigned K is acceptable. Note that the positive definite matrices $Q = Q^\top$ and P_j should be selected properly for the sake of getting appropriate observer stabilizing gain K by setting $\sum_{j=1}^{r}(P_j + PA_j\bar{P}_j^{-1}A_j^\top P) + \varepsilon I + Q > 0$.

VI. ASMO Based Disturbance Estimator

There is a need for fault and disturbance estimation (see for example [19], [20] and the references therein). Disturbance observers, which also are known as unknown input observers (UIO), can be constructed for fault detection and isolation (FDI). Regarding the excellent robustness of sliding mode observers, we discuss and prove the ability of using our proposed ASMO as a fault estimator for the matched uncertainties. We consider again the state transformation matrix (21). Assume that $\varpi = 0 (\Rightarrow \delta = 0)$. The regular form (22) yields

$$\begin{aligned}\dot{e}_y = &\begin{bmatrix} A_{0_{21}} & A_{0_{22}} \end{bmatrix}\begin{bmatrix} e_1 \\ e_y \end{bmatrix} \\ &+ \sum_{j=1}^{r}\left(\begin{bmatrix} A_{j_{21}} & A_{j_{22}} \end{bmatrix}\begin{bmatrix} e_1(t-\tau_j(t)) \\ e_y(t-\tau_j(t)) \end{bmatrix}\right) \\ &+FC(f(x,u,t) - f(\hat{x},u,t)) \\ &+FCP^{-1}C^\top g(x,u,t) - FCM(t)\end{aligned}$$

In the sliding mode $e_y = 0$ and $\dot{e}_y = 0$. Therefore

$$\begin{aligned}0 = A_{0_{21}}e_1 + \sum_{j=1}^{r} A_{j_{21}}e_1(t-\tau_j(t)) + FC(f(x,u,t) - f(\hat{x},u,t)) \\ +FCP^{-1}C^\top F^\top g(x,u,t) - FCM(t)\end{aligned} \tag{29}$$

The analysis in Section 3 proved that $\lim_{t\to\infty}\|e_\tau(t)\| = 0$. Consequently $f(\hat{x},u,t) \to f(x,u,t)$. Thus, from (29)

$$CP^{-1}C^\top F^\top g(x,u,t) \to CM(t) \tag{30}$$

According to the concept of approximated equivalent output error injection [20], assuming equivalently $\delta \neq 0$, then using the structure of $M(t)$ in (7), the matched disturbance is reconstructed via

$$g(x,u,t) \approx (\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t))^2 \frac{FCe(t)}{\|FCe(t)\|\,\alpha^\top(\hat{x},y,u,t)\hat{\rho}(t) + \delta} \tag{31}$$

VII. Conclusion

This paper deals with the problem of nonlinear robust adaptive sliding mode observer (ASMO) design for a class of continuous-time nonlinear systems with time-varying state delay and matched uncertainties. A full-order ASMO structure is used. An improved adaptation method in conjunction with a continuous approximated sliding injection feedback is introduced to cope with the disturbances. A systematic approach to compute the observer gain is employed.

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