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Geometric Output Regulation for a Class of Nonlinear Distributed Parameter Systems

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***Abstract*— We consider the output regulation problem for a special class of nonlinear distributed parameter systems (NLDPS). The main goal of this work is to show that the geometric theory of nonlinear output regulation, which has been extensively developed for lumped nonlinear systems, can be extended in a local setting to this class of NLDPS. Our approach is geometric, based on the center manifold theorem. Even for local problems, however, one must surmount technical issues that inevitably arise in the infinite dimensional setting. In this paper, we describe a particular class of nonlinear systems and exogenous systems for which center manifold methods can be used to obtain state feedback control laws for solving problems of tracking and disturbance attenuation. We also give a numerical example of set-point control for a controlled Chafee-Infante diffusion reaction equation which involves the consideration of a bounded input operator and an unbounded (point evaluation) output operator.**

I. Introduction

An important goal in the development of a theory of nonlinear output regulation is to establish a theory as parallel as possible that which has been established for finite dimensional linear [15]–[17] and nonlinear systems ([24], [23], [1], [20], [25]). In this direction, we note that for a large class of linear DPS problems those state feedback control laws which solve the problem of output regulation for a stable linear system with bounded inputs and outputs can also be characterized in an appealing systems theoretic fashion [27], [28], [29], [30] [6].

Output regulation is an asymptotic theory and the long time existence of solutions to open-loop nonlinear distributed parameter systems remains extremely challenging Nonetheless, we have been successful in establishing long time existence and asymptotic behavior for certain examples or system classes using particular feedback design methods (see, e.g., [4]–[9]). For example, our current efforts are primarily focused on local results for output regulation with respect to signals and disturbances generated by finite-dimensional exogeneous systems (see, however, [21] for a discussion of infinite-dimensional exosystems). In our setting, the exosystem is both finite dimensional and *neutrally stable* [24] and we can appeal to powerful center manifold methods to obtain some nontrivial insights and results. We emphasize the fact that these local techniques are not simply an appeal to linearization. Even in the lumped nonlinear case, elementary examples [11] show that a solution to the problem of output regulation for the linearization does not solve the output regulation problem for the nonlinear problem.

In general, we consider a system in the form

$$\frac{dz}{dt}(t) = F(z,d,u) \tag{I.1}$$

$$y(t) = c(z(t)), \text{ (measured output)} \tag{I.2}$$

$$z(0) = z_0, \tag{I.3}$$

where z is the state of the system in the infinite dimensional Hilbert space $\mathcal{Z}$; u is a control; y is the output and $z_0 \in \mathcal{Z}$ is the initial state of the system and d is a disturbance. In addition we assume there exists a *neutrally stable* [24] finite dimensional exogenous system

$$\frac{dw}{dt} = s(w) \tag{I.4}$$

$$w(0) = w_0 \in \mathcal{W}, \tag{I.5}$$

(here we assume that $\mathcal{W}$ is a finite dimensional Hilbert vector space) that generates both a reference signal y_r and the disturbance d. Namely, we assume

$$y_r(t) = q(w(t)) \qquad q : \mathcal{W} \mapsto Y. \tag{I.6}$$

$$d(t) = p(w(t)) \qquad p : W \mapsto Z. \tag{I.7}$$

We assume that $(0,0,0)$ is an equilibrium point, i.e., $F(0,0,0) = 0$ and that the linearization of F about this

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equilibrium is

$$F(z,w,u) = \left.\frac{\partial F}{\partial z}\right|_0 z + \left.\frac{\partial F}{\partial w}\right|_0 w + \left.\frac{\partial F}{\partial u}\right|_0 u + f(z,w,u),$$

where the nonlinear term $f(z,w,u)$ satisfies $f(0,0,0)=0$, $f_z(0,0,0)=0$, $f_u(0,0,0)=0$ and $f_w(0,0,0)=0$. Next we define the linear operators

$$A = \left.\frac{\partial F}{\partial z}\right|_0 \in \mathcal{L}(\mathcal{Z}), \quad B = \left.\frac{\partial F}{\partial u}\right|_0 \in \mathcal{L}(\mathcal{U},\mathcal{Z}).$$

We also have

$$p(w) = \left.\frac{\partial p}{\partial w}\right|_0 + \widehat{p}(w), \quad \widehat{p}(0)=0, \quad \frac{\partial \widehat{p}}{\partial w}(0)=0.$$

$$q(w) = \left.\frac{\partial q}{\partial w}\right|_0 + \widehat{q}(w), \quad \widehat{q}(0)=0, \quad \frac{\partial \widehat{q}}{\partial w}(0)=0.$$

We set

$$P = \left.\frac{\partial p}{\partial w}\right|_0 \in \mathcal{L}(\mathcal{W},\mathcal{Z}), \quad Q = \left.\frac{\partial q}{\partial w}\right|_0 \in \mathcal{L}(\mathcal{W},\mathcal{Y}).$$

For the measured output we have

$$c(w) = \left.\frac{\partial c}{\partial w}\right|_0 + \widehat{c}(w), \quad \widehat{c}(0)=0, \quad \frac{\partial \widehat{c}}{\partial w}(0)=0,$$

and we set $C = \partial c/\partial w\big|_0$ so that $C \in \mathcal{L}(\mathcal{W},\mathcal{Y})$ and

$$y(t) = Cw(t) + \widehat{c}(w(t)).$$

Finally we also have

$$s(w) = Sw + \widehat{s}(w) \quad \widehat{s}(0)=0, \quad \frac{\partial \widehat{s}}{\partial w}(0)=0.$$

Here $S \in \mathcal{L}(\mathcal{W})$ (so in a fixed basis S is given as an $N_W \times N_W$ matrix).

The objective of output regulation is to find a control law

$$u = \gamma(w) = \Gamma w + \widetilde{\gamma}(w),$$

$$\Gamma \in \mathcal{L}(\mathcal{W},\mathcal{U}), \quad \widetilde{\gamma}(0)=0, \quad \frac{\partial \widehat{\gamma}}{\partial w}(0)=0.$$

so that closed-loop trajectories exist and so that the error

$$e(t) = y(t) - y_r(t) = c(z(t)) - q(z(t)).$$

exists as $t \to +\infty$ and tends to 0.

II. The Basic Hypotheses

We consider a special class of problems in which we impose the following assumptions. First, recall that An operator is *accretive* if the numerical range lies in the half plane, i.e., Re $(\Theta(A))$ = Re $\langle A\varphi,\varphi\rangle \geq 0$ for $\varphi \in D(A) \subset Z$. An accretive operator A is *m-accretive* if for all Re $(\lambda) < 0$ (in addition to accretive) we have

$$(\lambda I - A)^{-1} \in \mathcal{B}(Z), \quad \|(\lambda I - A)^{-1}\| \leq \frac{1}{|\text{Re } (\lambda)|}.$$

An operator is called quasi-accretive (or quasi m-accretive) if $(\alpha I - A)$ is accretive (or quiasi-accretive) for some scalar α. This is equivalent to the condition $\Theta(A)$ is contained in a half space Re $(\lambda) \geq$ const.

A quasi-accretive operator A is called *sectorial* if the numerical range is not only contained in a half space Re $(\lambda) \geq$ const but also is contained in a sector

$$|\arg(\lambda - \gamma)| \leq \theta < \pi/2.$$

Here γ is called the vertex and θ is the semi-angle. A is called *m-sectorial* if it is sectorial and A is quasi-m-accretive.

Finally, following Pazy [26], we say that A is dissipative provided $(-A)$ is (maximal) accretive.

Assumption 1: 1) $(-A)$ is a sectorial operator with compact resolvent. Therefore $(-A)$ generate a Hilbert scale Z_α.

2) The analytic semigroup $T(t) = e^{At}$ is exponentially stable (Notice that it is also a contraction semigroup).
3) $B \in \mathcal{B}(\mathcal{U},\mathcal{Z})$ and $P \in \mathcal{L}(\mathcal{W},\mathcal{Z})$ are bounded.
4) $C \in \mathcal{B}(Z_\alpha, Y)$ for some $\alpha > 0$, i.e., there is a constant c_α so that

$$\|C\varphi\|_{\mathcal{Y}} \leq c_\alpha \|\varphi\|_\alpha.$$

5) We assume that the exosystem has the origin as a neutrally stable equilibrium, i.e., $w=0$ is a fixed point which is Lyapunov stable but not attracting. A center is an example of such a fixed point. This, in particular, implies $\sigma(S) \subset i\mathbb{R}$ (i.e., the spectrum of S is on the imaginary axis) and has no non-trivial Jordan blocks.

Remark 1: In our examples, and quite often in practice, A is self-adjoint. We also note that, since we assume $(-A)$ is sectorial, we work with the semigroup $\exp(At)$ rather than $\exp(-At)$ as is done in Henry [19].

In order to simplify the exposition in this paper we will impose the following simplifying assumptions.

Assumption 2: We will assume that the input and measured output are linear functions of the state of the plant and reference signal and disturbance are linear functions of the state of the exosystem. We also assume that $f(z,w,u) = f(z)$ so that, in particular, the uncontrolled problem is autonomous. Thus we assume $\widehat{q} = 0$, $\widehat{p} = 0$, $\widehat{q} = 0$, $\widehat{c} = 0$ and we have

$$c(w) = Cw, \ q(w) = Qw.$$

III. A Local State Feedback Result

Under the assumptions made in the previous section, we obtain an abstract nonlinear system

$$\dot{z} = Az + f(z) + Bu + Pw \tag{III.1}$$

$$\dot{w} = s(w) \tag{III.2}$$

$$z(0) = z_0, \quad w(0) = w_0, \tag{III.3}$$

$$y = c(z), \quad y_r = q(w) \tag{III.4}$$

$$e = y - y_r.$$

Theorem 1: Under assumptions 1 and 2, the (local) state feedback regulator problem for (III.1)-(III.4) *is solvable if, and only if, there exist mappings* $\pi : \mathcal{W} \to D(A) \subset \mathcal{Z}$ and $\gamma : \mathcal{W} \to \mathcal{Y}$ *satisfying the "regulator equations,"*

$$\frac{\partial \pi}{\partial w} s(w) = A\pi(w) + f(\pi(w)) + B\gamma(w) + Pw \tag{III.5}$$

$$c(\pi(w)) = q(w). \tag{III.6}$$

In this case a feedback law solving the state feedback regulator problem is given by

$$u(t) = \gamma(w)(t). \tag{III.7}$$

Modulo the inherent technical difficulties that arise in infinite dimensions, Theorem 1 can be obtained using an argument similar to that given in [24]. Indeed, under the assumptions on A, B and C, we can appeal to a version of the Center Manifold Theorem to aid in the proof.

Proof: To be able to adapt the necessary results from center manifold theory, it is useful to formulate the problem in the state space $\mathcal{X} = \mathcal{Z} \times \mathcal{W}$. Namely, with $u = \gamma(w)$ we have

$$\dot{X} = \mathcal{A}X + \mathcal{F}(X), \ X(0) = X_0, \tag{III.8}$$

$$X = \begin{pmatrix} z \\ w \end{pmatrix}, \quad \mathcal{F}(X) = \begin{pmatrix} f(z) + \widehat{\gamma}(w) \\ \widehat{s}(w) \end{pmatrix}$$

$$\mathcal{A} = \begin{pmatrix} A & (B\Gamma + P) \\ 0 & S \end{pmatrix}$$

$$e = c(z) - q(w).$$

Under our Assumption 1 we have, for any fixed, continuously differentiable γ and that $\mathcal{F}$ is continuously differentiable in $\mathcal{X}_\alpha$ for some $\alpha > 0$,

$$\mathcal{F}(0) = 0, \quad \mathcal{F}_X(0) = 0$$

and the operator $(-\mathcal{A})$ is a sectorial operator and therefore generates an analytic semigroup $\mathcal{T}(t)$. Furthermore,

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}) \cup \sigma(S), \quad \sigma(\mathcal{A}) \cap \sigma(S) = \emptyset,$$

and for some $\beta > 0$,

$$\sigma(\mathcal{A}) \subset \mathbb{C}^-_{-\beta} = \{\zeta : \text{Re } (\zeta) \leq -\beta\}, \sigma(S) \subset \mathbb{C}_0 = i\mathbb{R}.$$

According to [19, Theorem 6.2.1], for every γ (as above), $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$ where $\mathcal{X}_j$ are $\mathcal{A}$ invariant subspaces. $X_1 = \mathcal{P}(\mathcal{X}) \cong \mathcal{W}$ is the eigenspace spanned by the finitely many eigenvalues of $\mathcal{A}_1 = \mathcal{A}\big|_{\mathcal{X}_1}$ (from [24] we recall that neutral stability of the exosystem implies $\sigma(\mathcal{A}) \cap \mathbb{C}_0 = \sigma(S)$ consists of finitely many eigenvalues on the imaginary axis each having geometric multiplicity one) and $\mathcal{X}_2 = \mathcal{P}_2\mathcal{X} = (I - \mathcal{P})\mathcal{X}$ corresponds to the spectrum of $\mathcal{A}_2 = \mathcal{A}\big|_{\mathcal{X}_2}$.

In particular, according to [19, Theorem 6.2.1] there is a C^1 local invariant manifold Σ defined in a neighborhood U of the origin in X^α and a mapping $\pi : \mathcal{X}_1 \to \mathcal{X}$ so that, after identifying $\mathcal{W}$ and $\mathcal{X}_1$, we have

$$\Sigma = \left\{ \begin{pmatrix} \pi(w) \\ w \end{pmatrix} : w \in \mathcal{W} \right\}. \tag{III.9}$$

Also $\mathcal{A}_2$ generates an exponentially stable analytic semigroup $\mathcal{T}_2$ in $\mathcal{X}_2$.

Invariance of Σ implies that for initial data $X(0) = (\pi(w), w)^\top \in \Sigma$ the solution to (III.8) satisfies $z = \pi(w)$.

In particular, on Σ we have

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \pi(w) \\ w \end{pmatrix}$$

and from the equations of motion we have

$$\frac{dz}{dt} = A\pi(w) + f(\pi(w)) + B\gamma(w) + Pw$$

and

$$\frac{dz}{dt} = \frac{d\pi(w)}{dt} = \frac{\partial \pi}{\partial w}\frac{dw}{dt} = \frac{\partial \pi}{\partial w}s(w).$$

Therefore, invariance of Σ is equivalent to

$$\frac{\partial \pi}{\partial w}s(w) = A\pi(w) + f(\pi(w)) + B\gamma(w) + Pw. \quad \text{(III.10)}$$

Furthermore, for initial data $(z_0, w_0)^\top \in U$, i.e., sufficiently small, the solution $(z, w)^\top$ satisfies

$$\|z(t) - \pi(w(t))\|_\alpha \le K e^{-\alpha_0 t}\|z_0 - \pi(w_0)\|_\alpha \quad \text{(III.11)}$$

where K depends on α. That is, under our assumptions, the center manifold is locally exponentially attractive.

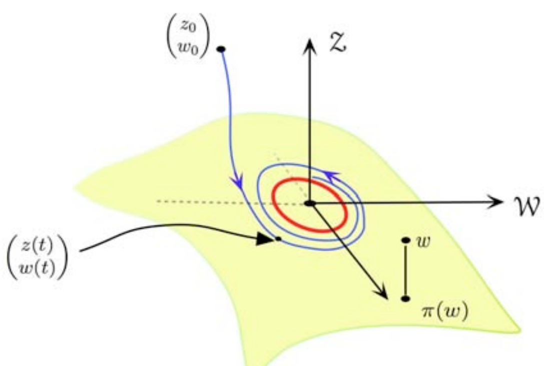

Figure 1: Center Manifold is Attractive

Remark 2: In the general theory of dynamical systems, center manifolds are not unique. However, as in [24], an ω-limit argument using neutral stability of the exosystem implies uniqueness of the center manifold Σ, given any choice of γ.

By the same ω-limit argument and the attractivity of the center manifold, it follows that *if* $u = \gamma(w)$ solves the output regulation problem, then the center manifold be error zeroing,

$$e = c(\pi(w)) - q(w) = 0, \quad \text{(III.12)}$$

must hold. In particular, the solvability of the output regulation problem implies the solvability of the regulator equations (IV.7), (III.6).

Conversely, if the regulator equations hold then $z = \pi(w)$ is an invariant manifold which, by reversing the arguments above, must be *the* center manifold Σ for the closed-loop system. Since Σ is error zeroing and exponentially attractive, we have

$$\begin{aligned}
\|e(t)\|_{\mathcal{Y}} &= \|c(z(t)) - q(w(t))\|_{\mathcal{Y}} \\
&= \|[c(z(t)) - q(w(t))] \\
&\quad - [c(\pi(w(t))) - q(w(t))]\|_{\mathcal{Y}} \\
&= \|c(z(t)) - c(\pi(w(t)))\|_{\mathcal{Y}} \\
&= \|C(z(t) - \pi(w(t))\|_{\mathcal{Y}} \\
&\le c_\alpha \|z(t) - \pi(w(t))\|_\alpha \\
&\le c_\alpha K e^{-\alpha_0 t}\|z_0 - \pi(w_0)\|_\alpha \\
&\to 0 \quad \text{as} \quad t \to \infty.
\end{aligned}$$

This concludes the proof of Theorem 1. ■

Remark 3: As first observed for the lumped case in [24] , the dynamics on the invariant manifold $\mathcal{M}$ is a copy, via π, of the dynamics of the exosystem and is therefore a model of the exosystem itself. This observation is the basis for the *internal model principle* for output regulation near an equilibrium, as developed in detail in [23], [1] and more recently in [20]. For lumped nonlinear systems, this has been extended to the nonequilibrium case in [12], [13], [25].

IV. A Numerical Example – Set-point Control

We consider as a numerical example a problem of set-point control for a controlled Chafee-Infante reaction diffusion equation. In particular we consider the system defined on $0 \le x \le 1$ for $t \ge 0$ given by

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + (z - z^3) + Bu \tag{IV.1}$$

$$z(0,t) = 0, \quad \frac{\partial z}{\partial x}(1,t) = 0 \tag{IV.2}$$

$$z(x,0) = \varphi(x) \tag{IV.3}$$

$$y(t) = (Cz)(t) = z(x_1, t), \; 0 < x_1 < 1, \tag{IV.4}$$

$$y_r(t) = M, \tag{IV.5}$$

The operator A in this case is

$$A = \frac{d^2}{dx^2}, \;\; D(A) = \{\varphi \in H^2(0,1) : \varphi(0) = 0, \; \varphi'(1) = 0\}.$$

The operator A is self-adjoint in $\mathcal{Z} = L^2(0,1)$ and generates an exponentially stable semigroup in $\mathcal{Z}$. Even more is true, $(-A)$ also satisfies all the conditions in Assumption 1. Namely, $(-A)$ is a sectorial operator and it generates an exponentially stable semigroup in the infinite scale of Hilbert spaces $Z_\alpha = D((-A)^\alpha)$.

In this example we consider a bounded input operator. Let us denote by I_0 the interval $[(x_0 - \nu_0), (x_0 + \nu_0)]$ where $0 < x_0 < 1$ and ν_0 is small enough so that $I_0 \subset (0,1)$. Then we define

$$(Bu)(x,t) = \frac{1}{2\nu_0} \chi_{I_0}(x) u(t). \tag{IV.6}$$

Here $\chi_{I_0}(x)$ denotes the indicator function of the set I_0, i.e.,

$$\chi_{I_0}(x) = \begin{cases} 1, & x \in I_0 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, the input operator is a bounded operator which for small ν_0 provides a bounded approximation to the dirac delta function supported at x_0. Notice that the output operator, point evaluation at x_1, is unbounded but it is bounded on $\mathcal{Z}_{1/2}$. So that all the required conditions are met.

In the case of set-point control we can take $\mathcal{W} = \mathbb{R}$ and $s(w) = 0$ so that the exosystem becomes

$$\frac{dw}{dt} = 0, \quad w(0) = M$$

which implies that

$$w(t) = M \quad \text{for all } t \geq 0.$$

Thus we have

$$y_r(t) = q(w(t)) = M \quad \text{or} \quad q(w) = w.$$

For this problem we seek mappings

$$\pi : \mathcal{W} \to D(A) \subset \mathcal{Z}$$

and

$$\gamma : \mathcal{W} \to \mathcal{Y} = \mathbb{R}$$

satisfying the regulator equation (IV.7) and (III.6), which in this case become

$$0 = \frac{d^2 \pi(w)}{dx^2} + \pi(w) - \pi(w)^3 + B\gamma(w) \tag{IV.7}$$

$$\pi(w)(x_1) = w. \tag{IV.8}$$

In [3], an efficient numerical algorithm for solving these equations has been developed based on an interpretation of the regulator equations as a fixed point problem which is then solved using Newton iteration.

As a specific numerical example we take $M = .5$, $x_0 = .75$, $\nu_0 = .01$, $x_1 = .5$ and $\varphi(x) = .5\cos(\pi x)$. In this case the desired control is a nonlinear function of M but is independent of x. For this example we find

$$u = \gamma(M) = 0.7488.$$

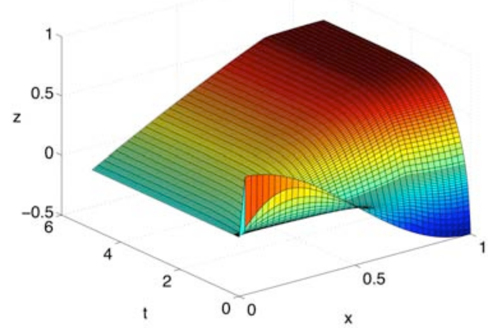

Figure 2: Surface

In Figure 2 (above) we have plotted the solution surface for the closed loop system. In Figure 3 (below) we have plotted both the measured output y for the closed loop system and the desired reference trajectory $y_r = M$. Finally in Figure 4 we have plotted the error $e(t) = y(t) - y_r(t)$. The figures clearly suggest the desired result that the error approaches (quite rapidly) zero as t tends to infinity.

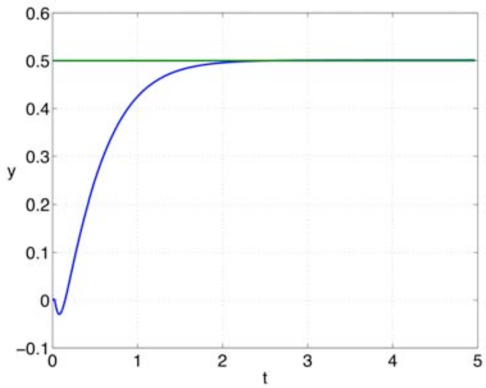

Figure 3: y and y_r

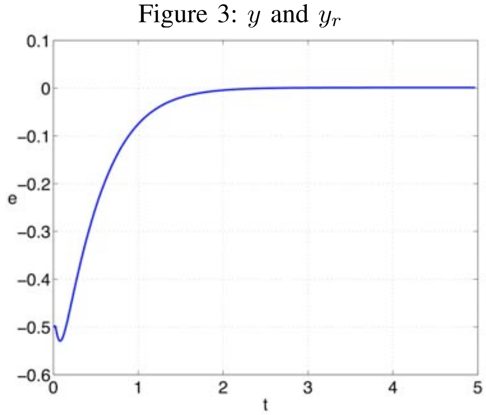

Figure 3: $e = y - y_r$

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