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Discrete-Time Nonlinear Output Feedback Control for a Class of Nonlinear Systems Using Finite Difference Approaches

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***Abstract*—In this article, the observation and control of a general class of nonlinear systems within the full linearization framework is constructed. Under step-by-step linearization procedures, the nonlinear control is determined by solving the implicit, nonlinear ordinary-differential-equation (ODE) while the observability matrix has full rank. Using the finite difference approach, the discrete-time output feedback control architecture is developed. Closed-loop simulations show that an unstable chemical reactor in the presence of input delay and unknown disturbances is successfully demonstrated.**

I. Introduction

SINCE many nonlinear model-based control frameworks required full state information, in the past decades the observer-based controller designs have been addressed in continuous-time setting [1,2], and in discrete-time setting [3, 4]. However, these nonlinear observers were open-loop state estimator in regard to consistent initialization. Inspired by Luenberger-type observer design for nonlinear systems [5], Valluri and Soroush [6] and Kazantzis et al. [7] proposed the extension of nonlinear observers to precisely estimate the states of unstable nonlinear systems. Wu et al. [8] and Jana et al. [9] developed the extended observer-based control to ensure the performance in terms of set point tracking and disturbance rejection. Regarding the nonlinear control synthesis in discrete-time, Henson and Seborg [3] provided a theoretical analysis for the discrete-time model-based design, and Sistu and Bequette [10] showed that the forward difference discretization could affect the closed-loop stability of a nonlinear process connected to a discrete controller. Recently, Soroush et al. [11] developed a discrete-time modified internal model controller for a discrete-time mathematical model with implicit inputs.

II. Observation and Control of Nonlinear Systems

Consider a general class of SISO nonlinear processes with implicit manipulated input:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in X \subset \Re^n$ is the process state, $y \in \Re$ is the measured output and $u \in M \subset \Re$ is the manipulated variable. The maps F and h are smooth (infinitely differentiable) in the set $X \times M$. Inspired by the result in [8], if the full rank condition is satisfied, Eq. (1) can be transformed into the fully linearization form with new coordinates ξ:

$$\begin{aligned} \dot{\xi}_1 &= \frac{\partial h}{\partial x} F(x,u) = \xi_2(x) \\ &\vdots \\ \dot{\xi}_r &= \frac{\partial \xi_r}{\partial x} F(x,u) = \xi_{r+1}(x,u) \\ \dot{\xi}_{r+1} &= \frac{\partial \xi_{r+1}}{\partial x} F(x,u) + \frac{\partial \xi_{r+1}}{\partial u}\dot{u} = \xi_{r+2}(x,u,\dot{u}) \\ &\vdots \\ \dot{\xi}_n &= \frac{\partial \xi_n}{\partial x} F(x,u) + \sum_{i=1}^{\alpha}\left[\frac{\partial \xi_n}{\partial u^{(i-1)}}\right] u^{(i)} = \chi(x,u,\dot{u},\ldots,u^{(\alpha)}) \end{aligned} \tag{2}$$

and the Luenberger-type transformed estimator is written by

$$\dot{\hat{\xi}} = A_n\hat{\xi} + B_n\left[\chi(\hat{x},u,\dot{u},\ldots,u^{(\alpha)})\right] + K\left(y - C_n\hat{\xi}\right) \tag{3}$$

where $\hat{\xi}$ is the estimated value of the transformed variable ξ ; $\hat{x} \in \overline{X}$ is the estimated value of the state x; $K = [\iota_1, \iota_2, \ldots, \iota_n]^T \in \Re^n$ is the observer gain,

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \Re^{n\times n},\ B_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re^{n\times 1}, \text{ and}$$

$$C_n = [1 0 \ldots 0] \in \Re^{1\times n} \tag{4}$$

Moreover, the transformed error dynamic by Eqs (4) and (5) is governed by

$$\begin{aligned} \dot{\hat{e}} = A_n\hat{e} + B_n\left[\chi(\hat{x},u,\dot{u},\ldots,u^{(\alpha)}) - \chi(x,u,\dot{u},\ldots,u^{(\alpha)})\right] \\ + K\left(y - C_n\hat{\xi}\right) \end{aligned} \tag{5}$$

where $\hat{e} \triangleq \hat{\xi} - \xi$ is the estimation error.

Corollary 1: Suppose that (i) the observer gain K is chosen such that all the eigenvalues of the matrix $(A_n - KC_n)$ lie strictly in the left half of the complex plane; (ii) $\chi(\bullet)$ satisfies the local Lipschitz condition, i.e. there is a $\gamma_0 > 0$ such that

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$$\left\| \begin{bmatrix} \chi(x,u,\dot{u},\ldots,u^{(\alpha)})\big|_{x=T^{-1}} \\ -\chi(\hat{x},u,\dot{u},\ldots,u^{(\alpha)})\big|_{\hat{x}=T^{-1}} \end{bmatrix} \right\| \le \gamma_0 \|\hat{e}\| \quad (6)$$

Then, the local convergence decay of the error dynamic in Eq. (7) can be asymptotically achieved.

Remark 1: The similar proof of Corollary 1 has been shown in [8]. By virtue of the nonlinear inversion of transformation in Eq. (2), the original nonlinear observer as the state estimation is shown by

$$\dot{\hat{x}} = F(\hat{x},u) + Q^{-1}(\hat{x},u,\dot{u},\ldots,u^{(\alpha-1)})K\left(y-h(\hat{x})\right) \quad (7)$$

Since $Q^{-1}K\left(y-\hat{\xi}_1\right)$ can be denoted as the compensation design, the closed-loop observer design can admit some initial errors, i.e. the non-consistent initialization, $x(0) \neq \hat{x}(0)$.

Assume that the implicit, nonlinear ODE can be modified to

$$\chi(x,u,\dot{u},\ldots,u^{(\alpha)}) + Ge = 0 \quad (8)$$

where $e \equiv [(\xi_1 - y_R), \xi_2, \ldots, \xi_n]^T$, $y_R \in \Re$ is the constant, and $G = [g_1, g_2, \ldots, g_n] \in \Re^{1\times n}$ should satisfy the Hurwitz condition. Moreover, the continuous-time observer-based controller is directly obtained:

$$\dot{\hat{x}} = F[\hat{x},u(t)] + \tilde{Q}(\hat{x},t)K[y-h(\hat{x})] \quad (9a)$$

$$\chi(\hat{x},u,\dot{u},\ldots,u^{(\alpha)}) + G\tilde{e} = 0 \quad (9b)$$

where the control solution by solving the nonlinear ODE in Eq. (9b).

Remark 2: The exact control law depends on the solvability problem of the implicit, nonlinear ODE. Although the numerical solution may evaluate the accurate control action, the continuous-time nonlinear control law is still vague.

Furthermore, the auxiliary closed-loop systems with respect to prescribed coordinates are of the form:

$$\dot{e} = \overline{A}_n e + B_n\left[\tilde{\chi}_1(x,\hat{x},t) - \tilde{\chi}_2(\hat{x},t) + G\hat{e}\right]$$

$$\dot{\hat{e}} = \hat{A}_n \hat{e} + B_n\left[\tilde{\chi}_1(x,\hat{x},t) - \tilde{\chi}_2(\hat{x},t)\right] \quad (10)$$

where $\tilde{\chi}_1(x,\hat{x},t) = \chi(x,u,\dot{u},\ldots,u^{(\alpha-1)})\big|_{u=\Phi(\hat{x},t)}$, $\tilde{\chi}_2(\hat{x},t) = \chi(\hat{x},u,\dot{u},\ldots,u^{(\alpha-1)})\big|_{u=\Phi(\hat{x},t)}$,

$$\overline{A}_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -g_1 & -g_2 & & \ldots & -g_n \end{bmatrix},$$

and

$$\hat{A}_n = \begin{bmatrix} -\iota_1 & 1 & 0 & \ldots & 0 \\ -\iota_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -\iota_n & 0 & & \ldots & 0 \end{bmatrix}.$$

Corollary 2: Suppose that (i) the observer design by Corollary 1 can hold; (ii) the control solution exists; (iii) inequalities for bounds of nonlinearities are satisfied by

$$\left\|2P_cB_n\left[\tilde{\chi}_1(x,\hat{x},t) - \tilde{\chi}_2(\hat{x},t) - G\hat{e}\right]\right\| \le \ell_1\|e\| + \ell_2\|\hat{e}\| \quad (11a)$$

$$\left\|2P_kB_n\left[\tilde{\chi}_1(x,\hat{x},t) - \tilde{\chi}_2(\hat{x},t)\right]\right\| \le \ell_3\|\hat{e}\| \quad (11b)$$

where P_c and P_k are the solution of the following Lyapunov equations

$$P_cA_n + A_n^TP_c = -I$$
$$P_k\overline{A}_n + \overline{A}_n^TP_c = -I \quad (12)$$

Then the asymptotic output regulation can be achieved.

$$\lim_{t\to\infty}\|e(t)\| = 0 \quad (13)$$

III. Discrete-Time Nonlinear Output Feedback Controllers

Referring to the finite difference approach [12], the relationship between forward finite differences and differential operators are introduced,

$$h^mD^m = \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right)^m, \ m=1,\ 2,\ \ldots,\ \alpha \quad (14)$$

where the differential operator $D \triangleq d/dt$; Δ represents the forward difference operator, e.g. $\Delta u(t) = u(t+h) - u(t)$; $0 \le h < 1$ represents the small time interval. Using the differential operator, assuming Eq. (10) is reduced into

$$D^\alpha u = \Im(x,u,Du,\ldots,D^{\alpha-1}u) \quad (15)$$

It is rearranged by

$$h^mD^m = \Delta^m + O_m(\Delta^{m+1}), \ m=1,\ 2,\ \ldots,\ \alpha \quad (16)$$

In terms of the forward finite differences the nonlinear difference equation is described by

$$\Delta^\alpha u = \Im_\alpha(x,u,\Delta u,\ldots,\Delta^{\alpha-1}u) + O_\alpha(x,\Delta^2u,\Delta^3u,\ldots) \quad (17)$$

where $O_m(\bullet)$ and $O_\alpha(\bullet)$ represent the remainders of the expansion of Eqs (16) and (17), respectively. Moreover, Eq. (17) is reduced as the discrete-time formulation

$$u(k+\alpha) = \Psi(x(k),u(k),...,u(k+\alpha-1)) \quad (18)$$

Remark 3: The forward finite difference combination is recommended because of the time discretization for $t \ge 0$. When h<1, the truncated portion with higher-order difference term in Eq. (18) is carefully eliminated such that the nonlinear ODE is reduced into the difference equation. Similarly, the closed-loop observer is approximated as the discrete-time formulation

$$\hat{x}(k+1) = F_d(\hat{x}(k), u(k), y(k)) \quad (19)$$

and the discrete-time observer-based feedback control is also solved by

$$u(k+\alpha) = \Psi(\hat{x}(k), u(k), ..., u(k+\alpha-1)) \quad (20)$$

TABLE 1
NOMINAL PARAMETER VALUES FOR CSTR MODEL

$C_{Ai} = 10$	kmol/m^3
$T_i = 295.2$	K
$z_1 = 2000$	$\text{m}^6/\text{kmol}^2\text{s}$
$z_2 = 3.4\times10^6$	$\text{kmol}^{0.5}/\text{m}^{1.5}\text{s}$
$z_d = 2.63\times10^5$	s^{-1}
$E_1 = 4.9\times10^4$	kcal/kmol
$E_2 = 6.5\times10^4$	kcal/kmol
$E_d = 5.7\times10^4$	kcal/kmol
R=8.354	$\text{kJ/kmol}\cdot\text{K}$
$-\Delta H_1 = 4.5\times10^4$	kcal/kmol
$-\Delta H_2 = 5\times10^4$	kcal/kmol
$-\Delta H_d = 6\times10^4$	kcal/kmol
$\rho = 1000$	kg/m^3
$c_p = 4.2$	$\text{kJ/kg}\cdot\text{K}$
$V = 0.01$	m^3

where $\hat{x}(k)$ represents the current estimated state.

Remark 4: Under the finite difference method, the accuracy of discrete-time observer plus control design is affected by the approximation errors including the finite sampling period (T_s) as well as the higher-order difference terms (truncation errors). Moreover, the convergence property of the discrete-time closed-loop system will be addressed as follows.

IV. DEMONSTRATION

According to the form of Eq. (1), an unstable CSTR example is shown by

$$\dot{x} = F(x,u) = \begin{pmatrix} F_1(x) \\ F_2(x,u) \end{pmatrix} = \begin{pmatrix} R_A(x_1,x_2) + (C_{Ai} - x_1)/\tau \\ R_H(x_1,x_2)/\rho c_p + (T_i - x_2)/\tau + \dfrac{u}{\rho c_p V} \end{pmatrix} \quad (21)$$

where $(x_1,x_2)=(C_A,T)$, and

$$R_A(C_A,T) = -\kappa_1(T)C_A^3 - \kappa_2(T)C_A^{1/2} - \kappa_d(T)C_A$$
$$R_H(C_A,T) = (-\Delta H_1)\kappa_1(T)C_A^3 + (-\Delta H_2)\kappa_2(T)C_A^{1/2} + (-\Delta H_d)\kappa_d(T)C_A \quad (22)$$

Note that the reactor temperature T is the measurable output, i.e., $h(x) = T$, and u is the manipulated input as the rate of heat input to the reactor. $\kappa_i(T) = z_i \exp(-E_i/RT)$, i=1, 2, and $\kappa_d(T) = z_d \exp(-E_d/RT)$ are the reaction rate constants; C_{A_i} and T_i are the inlet concentration and temperature of stream, respectively. Under the system parameters in Table 1, the process operation is assumed nearby the unstable region. First, the observer-based control in continuous-time setting is synthesized by

$$\dot{\hat{x}} = F(\hat{x},u) + Q^{-1}K\left(y - h(\hat{x})\right) \quad (23a)$$

$$\frac{du}{dt} = \left(\frac{\partial F_2(\hat{x},u)}{\partial u}\right)^{-1} \times \begin{bmatrix} -g_1(y-y_R) - g_2F_2(\hat{x},u) \\ -\dfrac{\partial F_2(\hat{x},u)}{\partial \hat{x}_2}F_2(\hat{x},u) - \dfrac{\partial F_2(\hat{x},u)}{\partial \hat{x}_1}F_1(\hat{x}) \end{bmatrix} \quad (23b)$$

where $K = [\iota_1\,\iota_2]^T$ and

$$Q^{-1}(\hat{x}) = \left(\frac{\partial}{\partial x}\begin{bmatrix} y \\ \dot{y} \end{bmatrix}\right)^{-1} = \left(-\frac{1}{\rho c_p}\frac{\partial R_H}{\partial x_1}\right)^{-1} \begin{bmatrix} \dfrac{1}{\rho c_p}\dfrac{\partial R_H}{\partial x_2} - \dfrac{1}{\tau} & -1 \\ -\dfrac{1}{\rho c_p}\dfrac{\partial R_H}{\partial x_1} & 0 \end{bmatrix} \quad (24)$$

Second, the synthesized discrete-time observer-based feedback controller is evaluated by the computer-assisted computation. Let all controller parameters, $(g_1,g_2) = (0.01, 0.2)$ and $(\iota_1,\iota_2) = (0.04, 0.0004)$ be fixed,

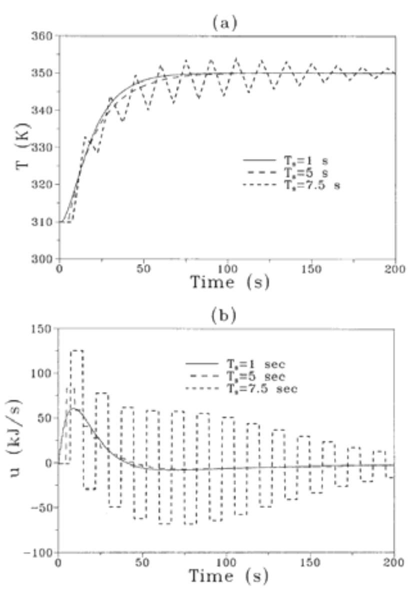

Fig. 1. Under consistent initialization, different sampling period, constant setpoint, and using the observer-based controller: (a) Setpoint ($y_R = 350$ K) tracking response (b) Corresponding control input

Fig. 1(a) shows the output tracking performance while consistent initialization ($x(0) = \hat{x}(0)$) is considered. Although the sampling rate of the real measurement devices is fast, the simulation with long sample period can claim the robustness of discrete-time nonlinear control.

Fig. 2(a) shows the asymptotic output tracking while the initial perturbation, $\Delta T(0) = x_2(0) - \hat{x}_2(0) = \pm 10$ K, and large sampling period ($T_s = 5s$) are considered. Fig. 2(c) shows that the convergence of estimation error is achieved. Moreover, the step disturbance of inlet temperature, e.g. $\Delta T_i = \pm 20$ K, is added, Fig. 3(a) shows that the output tracking exhibits the bounded offset.

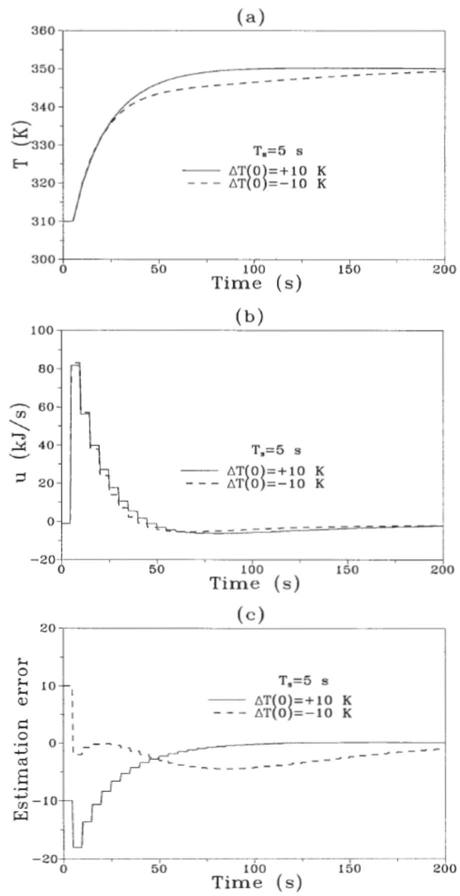

Fig. 2. Under non-consistent initialization, $\Delta T(0) = \pm 10$ K, and constant sampling period ($T_s = 5$), and using the observer-based controller: (a) Setpoint ($y_R = 350$ K) tracking response (b) Corresponding control input (c) State (x_2) estimation profile

V. Conclusions

Using the Luenberger-like nonlinear observer plus linearizing controller, the complicated nonlinear control scheme is reduced via the higher-order reduction and the finite difference approach. Two discrete-time nonlinear output feedback controllers are obtained by solving a set of difference equations. In our study, the sampling period is treated as the input delay, and the discrete-time nonlinear output feedback implementation is validated to be robust against initial perturbation and unknown disturbances.

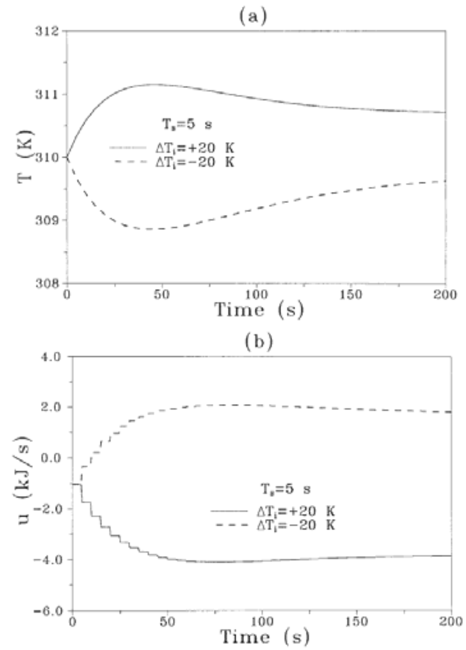

Fig. 3. Under consistent initialization, constant sampling period ($T_s = 5$), inlet temperature perturbation ($\Delta T_i = \pm 20$ K), and using the observer-based controller: (a) Disturbance rejection response (b) Corresponding control input

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