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Recent advances on the reachability of single-input positive switched systems

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Abstract— **In this paper the reachability property for single-input continuous-time positive switched systems is investigated. By referring to an existing (and hard to check) characterization of the reachability of the class of positive switched systems which commute among n single-input n-dimensional systems [9], we develop new algebraic tools which allow us to derive sufficient reachability conditions which are easier to evaluate.**

I. Introduction

Modeling of physical phenomena typically comes as the result of a pondered balance among different, often conflicting, needs. The first natural goal one pursues, when describing a physical system, is accuracy, which is generally ensured by resorting to computationally demanding solutions. As a consequence, this requirement is often weakened in order to achieve feasible solutions, which are more suitable to real-time implementation. Under this point of view, the case often occurs that a complex nonlinear model, which provides a good description of the real system dynamics, can be efficiently replaced by a family of simpler and possibly linear models, each of them appropriate for describing the system evolution under specific working conditions.

This simple fact stimulated, in the last ten-fifteen years, a long stream of research concerned with the analysis and design of "switched linear systems", by this meaning systems whose describing equations change, according to some switching law, within a (possibly infinite) family of (linear) subsystems. Research efforts in this area were first oriented to the investigation of stability and stabilizability issues [7], and it was only a few years later that structural properties, like reachability, controllability and observability, were initially addressed [4], [13], [14].

On the other hand, the positivity requirement is often introduced in the system models whenever the physical nature of the describing variables constrains them to take only positive (or at least nonnegative) values. Positive linear systems naturally arise in various fields such as bioengineering (compartmental models), economic modelling, behavioral science, and stochastic processes (Markov chains), where the state variables represent quantities, like pressures, population levels, concentrations, etc., that have no meaning unless nonnegative [3].

In this perspective, switched positive systems are mathematical models which keep into account two different needs: the need for a system model which is obtained as a family of simple subsystems, each of them accurate enough to capture the system laws under specific operating conditions, and the need to introduce the nonnegativity constraint the physical variables are subject to. This is the case when trying to describe certain physiological and pharmacokinetic processes. For instance, the insulin-sugar metabolism is captured by two different compartmental models: one valid in steady-state and the other (more complex) describing the evolution under perturbed conditions, following an oral assumption or an intravenous injection.

Of course, the need for this class of systems in specific research contexts has stimulated an interest in theoretical issues related to them, and, in particular, structural properties of continuous-time positive switched systems have been recently investigated in [8], [9], [10]. In detail, necessary conditions for reachability have been investigated in [8] (monomial reachability) and in [9] (pattern reachability), while necessary and sufficient conditions for the reachability of continuous-time positive switched systems of dimension n, which commute among n single-input subsystems, have been investigated in [10]. These conditions, even though valuable from a theoretical point of view, appear quite difficult to check. This difficulty has stimulated research interest in the detailed analysis of the dominant modes of the exponential of a Metzler matrix [11].

By relaying on those results and on some new technical lemmas, we will be able to derive in this paper some new sufficient conditions for reachability of this special class of systems which are easy enough to check.

Before proceeding, we introduce some notation. For every $k \in \mathbb{N}$, we set $\langle k \rangle := \{1, 2, \dots, k\}$. In the sequel, the (i,j)th entry of a matrix A is denoted by $[A]_{i,j}$. If A is block partitioned, $\mathrm{block}_{(i,j)}[A]$ denotes its (i,j)th block. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix A with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A \geq 0$); if $A \geq 0$ and $A \neq 0$, A is a *positive matrix* ($A > 0$), while if all its entries are positive it is a *strictly positive matrix* ($A \gg 0$). The same notation is adopted for nonnegative, positive and strictly positive vectors. A *Metzler matrix*, on the other hand, is a real square matrix, whose off-diagonal entries are nonnegative.

Given any matrix $A \in \mathbb{R}^{q \times r}$, by the *nonzero pattern* of A we mean the set of index pairs corresponding to its nonzero entries, namely $\overline{\mathrm{ZP}}(A) := \{(i,j) : [A]_{i,j} \neq 0\}$. Conversely, the *zero pattern* $\mathrm{ZP}(A)$ is the set of indices corresponding to the zero entries of A. The adaptation of these concepts to the vector case is straightforward.

We let $\mathbf{e}_i$ denote the ith vector of the canonical basis in $\mathbb{R}^n$ (where n is always clear from the context), whose entries

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are all zero except for the ith which is unitary. We say that a vector $\mathbf{v} \in \mathbb{R}^n_+$ is an *ith monomial vector* if $\overline{\mathrm{ZP}}(\mathbf{v}) = \overline{\mathrm{ZP}}(\mathbf{e}_i) = \{i\}$. For any set $\mathcal{S} \subseteq \langle n \rangle$ and any vector $\mathbf{v} \in \mathbb{R}^n_+$, we let $\mathbf{v}_\mathcal{S} \in \mathbb{R}^{|\mathcal{S}|}_+$ be the restriction of $\mathbf{v}$ to its positive components.

To every $n \times n$ Metzler matrix A we associate [2], [12] a *directed graph* $\mathcal{G}(A)$ with vertices indexed by $1, 2, \ldots, n$. There is an arc (j, i) from j to i if and only if $[A]_{ij} \neq 0$. We say that vertex i is *accessible* from j if there exists a path (i.e., a sequence of adjacent arcs $(j, i_1), (i_1, i_2), \ldots, (i_{k-1}, i)$) in $\mathcal{G}(A)$ from j to i (equivalently, $\exists\ k \in \mathbb{N}$ such that $[A^k]_{ij} \neq 0$). Two distinct vertices are said to *communicate* if each of them is accessible from the other. By definition, each vertex communicates with itself. The concept of communicating vertices allows to partition the set of vertices $\langle n \rangle$ into *communicating classes*, say $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$. To any class $\mathcal{C}_i$ we associate two index sets:

$$\begin{aligned}\mathcal{A}(\mathcal{C}_i) &:= \{j : \text{ the class } \mathcal{C}_j \text{ has access to the class } \mathcal{C}_i\}\\ \mathcal{D}(\mathcal{C}_i) &:= \{j : \text{ the class } \mathcal{C}_j \text{ is accessible from the class } \mathcal{C}_i\}.\end{aligned}$$

Each class $\mathcal{C}_i$ is assumed to access to itself. If i is a vertex in $\mathcal{G}(A)$, we denote by $\mathcal{C}(i)$ the class i belongs to.

The *reduced graph* $\mathcal{R}(A)$ [12] associated with A (with $\mathcal{G}(A)$) is the (acyclic) graph having the classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ as vertices. There is an arc (j, i) in $\mathcal{R}(A)$ if and only if $i \in \mathcal{D}(\mathcal{C}_j)$. Any (acyclic) path $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ in $\mathcal{R}(A)$ identifies a *chain of classes* $(\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \ldots, \mathcal{C}_{i_k})$, having $\mathcal{C}_{i_1}$ as *initial class* and $\mathcal{C}_{i_k}$ as *final class*.

An $n \times n$ Metzler matrix A is *reducible* if there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*. It follows that 1×1 matrices are always irreducible. In general, given a square Metzler matrix A, a permutation matrix P can be found such that

$$P^T A P = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1\ell} \\ & A_{22} & \ldots & A_{2\ell} \\ & & \ddots & \vdots \\ & & & A_{\ell\ell} \end{bmatrix}, \qquad (1)$$

where each A_{ii} is irreducible. (1) is usually known as *Frobenius normal form* of A [2]. Clearly, the irreducible matrices $A_{11}, A_{22}, \ldots, A_{\ell\ell}$ correspond to the communicating classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ of $\mathcal{G}(P^T A P)$ (coinciding with those of $\mathcal{G}(A)$, after a suitable relabelling).

When dealing with the graph of a matrix in Frobenius normal form (1), for every $i \in \langle \ell \rangle$, $\mathcal{A}(\mathcal{C}_i) \subseteq \{i, i+1, \ldots, \ell\}$, while $\mathcal{D}(\mathcal{C}_i) \subseteq \{1, 2, \ldots, i\} = \langle i \rangle$, so that $\mathcal{A}(\mathcal{C}_i) \cap \mathcal{D}(\mathcal{C}_i) = \{i\}$. On the other hand, if $i > j$ then $\mathcal{A}(\mathcal{C}_i) \cap \mathcal{D}(\mathcal{C}_j) = \emptyset$, while if $i < j$ the following conditions are equivalent

$$\mathcal{A}(\mathcal{C}_i) \cap \mathcal{D}(\mathcal{C}_j) \neq \emptyset \quad \Leftrightarrow \quad i \in \mathcal{D}(\mathcal{C}_j) \quad \Leftrightarrow \quad j \in \mathcal{A}(\mathcal{C}_i).$$

A class $\mathcal{C}_i$ is *initial* if $\mathcal{A}(\mathcal{C}_i) = \{i\}$, and it is *distinguished* [12] if $\lambda_{\max}(A_{ii}) > \lambda_{\max}(A_{jj})$ for every $j \in \mathcal{D}(\mathcal{C}_i), j \neq i$.

Basic definitions and results about cones may be found, for instance, in [1]. We recall here only the few facts used within this paper. A set $\mathcal{K} \subset \mathbb{R}^n$ is said to be a *cone* if $\alpha\mathcal{K} \subset \mathcal{K}$ for all $\alpha \geq 0$. A cone $\mathcal{K}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This amounts to saying that $k \in \mathbb{N}$ and $C \in \mathbb{R}^{n \times k}$ can be found, such that $\mathcal{K}$ coincides with the set of nonnegative combinations of the columns of C. In this case, we adopt the notation $\mathcal{K} := \mathrm{Cone}(C)$.

A polyhedral cone $\mathcal{K}$ in $\mathbb{R}^n$ is *simplicial* if it admits n linearly independent generating vectors. In other words, $\mathcal{K} := \mathrm{Cone}(C)$ for some nonsingular matrix C. When so, a vector $\mathbf{v}$ belongs to the boundary of the simplicial cone $\mathcal{K}$ if and only if $\mathbf{v} = C\mathbf{u}$ for some $\mathbf{u} > 0$, with $\overline{\mathrm{ZP}}(\mathbf{u}) \neq \langle n \rangle$.

II. Reachability of single-input positive switched systems

A *single-input continuous-time positive switched system* is described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + b_{\sigma(t)}u(t), \qquad t \in \mathbb{R}_+, \qquad (2)$$

where $\mathbf{x}(t)$ and $u(t)$ denote the n-dimensional state variable and the scalar input, respectively, at the time instant t, σ is a switching sequence, taking values in a finite set $\mathcal{P} = \{1, 2, \ldots, p\}$. As a matter of fact, in this paper we will steadily address the case when $\mathcal{P} = \langle n \rangle$, namely $p = n$.

We assume that the switching sequence is piece-wise constant, and hence in every time interval $[0, t]$ there is a finite number of discontinuities, which corresponds to a finite number of switching instants $0 = t_0 < t_1 < \ldots < t_k < t$. Also, we assume that, at the switching time t_ℓ, σ is right continuous. For each $i \in \mathcal{P}$, the pair (A_i, b_i) represents a continuous-time positive system, which means that A_i is an $n \times n$ Metzler matrix and b_i is an n-dimensional nonnegative column vector.

As a first step, we recall the definition of reachability for positive switched systems.

Definition 1: [9], [10] A state $\mathbf{x}_f \in \mathbb{R}^n_+$ is said to be *reachable* if there exist some time instant $t_f > 0$, a switching sequence $\sigma : [0, t_f[\to \mathcal{P}$ and an input $u : [0, t_f[\to \mathbb{R}_+$ that lead the state trajectory from $\mathbf{x}(0) = 0$ to $\mathbf{x}(t_f) = \mathbf{x}_f$.

A positive switched system is said to be *reachable* if every state $\mathbf{x}_f \in \mathbb{R}^n_+$ is reachable.

As a starting point, we recall here the characterization of reachability for a system described as in (2) and commuting among n single-input subsystems. For the sake of compactness, we restate it by making use of some new notation. Given any set $\mathcal{S} \subseteq \langle n \rangle$, we introduce the family of indices

$$\mathcal{I}_\mathcal{S} := \{i \in \mathcal{P} : \overline{\mathrm{ZP}}(e^{A_i}\mathbf{e}_\mathcal{S}) = \mathcal{S}\}. \qquad (3)$$

Proposition 1: [9] Given an n-dimensional continuous-time positive switched system (2), commuting among n single-input subsystems $(A_i, b_i), i = 1, 2, \ldots, n$, the following facts are equivalent:

i) the switched system (2) is reachable;
ii) for every proper subset[1] $\mathcal{S} \subset \langle n \rangle$ we have:

iia) [2] if $|\mathcal{S}| = 1$, then $\exists\, j(\mathcal{S}) \in \mathcal{I}_{\mathcal{S}}$ such that $\overline{\mathrm{ZP}}(b_{j(\mathcal{S})}) = \mathcal{S}$;
iib) if $|\mathcal{S}| > 1$, then $\mathcal{I}_{\mathcal{S}} \neq \emptyset$ and either

1. $\exists\, j(\mathcal{S}) \in \mathcal{I}_{\mathcal{S}}$ such that $\overline{\mathrm{ZP}}(b_{j(\mathcal{S})}) \subset \mathcal{S}$,

or

2. for every $\mathbf{v} \in \mathbb{R}^n_+$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, there exist $m \in \mathbb{N}$, $(\tau_1, \dots, \tau_m) \in \mathbb{R}^n_+$ and $i_1, \dots, i_m \in \mathcal{I}_{\mathcal{S}}$, such that $\mathbf{v}$ can be obtained as the nonnegative combination of no more than $|\mathcal{S}| - 1$ columns of $e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m} P_{\mathcal{S}}$, where $P_{\mathcal{S}}$ is the selection matrix which selects all the columns corresponding to the indices appearing in $\mathcal{S}$.

Remark 1: It is worthwhile noticing that the two subcases of point iib) correspond to two different ways of reaching the final vector $\mathbf{v}$: indeed, in the former case a nonnegative forcing input is applied during the last switching interval, while in the latter the system is left to freely evolve.

Remark 2: The algebraic conditions provided in Proposition 1 cannot be easily verified. Specifically, there is no obvious way of testing whether indices $i_1, \dots, i_m$ and positive time intervals $\tau_1, \dots, \tau_m$ can be found, such that a given positive vector $\mathbf{v}$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, belongs to $\mathrm{Cone}(e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m} P_{\mathcal{S}})$ and it can be obtained by combining less than $|\mathcal{S}|$ columns of $e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m} P_{\mathcal{S}}$. This would require trying all index sequences, of increasing length, meanwhile varying the lengths of the switching intervals τ_i. Of course, as there is no obvious result about what it may be convenient to do (increasing or decreasing the τ_i's) and when one may give up (is there a maximum numbers of indices m after which no successful result can be obtained, unless it has been obtained earlier?), we need to explore alternative means for solving this problem. This will require us to find sufficient conditions for the problem solvability.

If we assume that the nonempty set $\mathcal{S} \subset \langle n \rangle$ and the indices $i_1, i_2, \dots, i_m \in \mathcal{I}_{\mathcal{S}}$ are given, the problem one has to address is the following one.

PROBLEM STATEMENT: Given any positive vector $\mathbf{v} \in \mathbb{R}^n_+$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, find conditions ensuring that $\mathbf{v}_{\mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}_+$ belongs to the boundary of the simplicial cone, $\mathrm{Cone}[P_{\mathcal{S}}^T e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m} P_{\mathcal{S}}]$, where $\mathbf{v}_{\mathcal{S}}$ is the $|\mathcal{S}|$-dimensional strictly positive vector obtained by restricting $\mathbf{v}$ to its positive entries, and $P_{\mathcal{S}}$ is the selection matrix that singles out the columns indexed on $\mathcal{S}$.

This amounts to searching for conditions that allow to obtain $\mathbf{v}$ as $\mathbf{v} = e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m}\mathbf{u}$, for some $\mathbf{u} \in \mathbb{R}^n_+$ with $\overline{\mathrm{ZP}}(\mathbf{u}) \subsetneq \mathcal{S}$, or, again, that we can obtain $\mathbf{v}_{\mathcal{S}}$ in the form $\mathbf{v}_{\mathcal{S}} = P_{\mathcal{S}}^T e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m} P_{\mathcal{S}} \mathbf{u}_{\mathcal{S}}$, for some $\mathbf{u}_{\mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}_+$ with $\mathrm{ZP}(\mathbf{u}_{\mathcal{S}}) \neq \emptyset$.

[1]The case $\mathcal{S} = \langle n \rangle$ is trivial, as either one the following two conditions in iib) is necessarily satisfied.

[2]This condition is equivalent [8] to the so-called *monomial reachability*, namely the possibility of reaching, starting from the zero initial condition and by resorting to nonnegative inputs, any monomial vector.

In the following we will address our problem in an apparently restrictive, but in fact equivalent, formulation, by assuming $\mathcal{S} = \langle n \rangle$ and $I_{\mathcal{S}} = \mathcal{P}$. As a result, $\mathbf{v}_{\mathcal{S}} = \mathbf{v}$ and

NEW PROBLEM STATEMENT: we search for conditions ensuring that $\mathbf{v} \in \mathbb{R}^n_+, \mathbf{v} \gg 0$, can be obtained as

$$\mathbf{v} = e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m}\mathbf{u},\ \exists\ \mathbf{u} \in \mathbb{R}^n_+ \text{ with } \mathrm{ZP}(\mathbf{u}) \neq \emptyset. \quad (4)$$

III. EXPONENTIAL ASYMPTOTIC CONES

To solve this new problem, we recall the concept of asymptotic exponential cone of an exponential matrix and introduce, more generally, the asymptotic cone of a product of exponential matrices along some direction in $\mathbb{R}^n_+$.

Definition 2: [10], [11] Given an $n \times n$ Metzler matrix A, we define its *asymptotic exponential cone*, $\mathrm{Cone}_\infty(e^{At})$, as the polyhedral cone generated by the vectors $\mathbf{v}_i^\infty$, which represent the asymptotic directions of the columns of e^{At}:

$$\mathbf{v}_i^\infty := \lim_{t\to\infty} \frac{e^{At}\mathbf{e_i}}{\|e^{At}\mathbf{e_i}\|}, \qquad i = 1, 2, \dots, n.$$

Analogously, given an ordered set of $n \times n$ Metzler matrices $A_{i_1}, \dots, A_{i_m}$ and a positive vector $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$, we define their *asymptotic exponential cone* along $\bar{\alpha}$

$$\mathrm{Cone}_\infty^{\bar{\alpha}}\left(e^{A_{i_1}t} \dots e^{A_{i_m}t}\right)$$

as the polyhedral cone generated by the (normalized) vectors $\mathbf{v}_i^\infty$ which represent the asymptotic directions of the columns of $e^{A_{i_1}\alpha_1 t} \dots e^{A_{i_m}\alpha_m t}$, i.e.

$$\mathbf{v}_i^\infty := \lim_{t\to\infty} \frac{e^{A_{i_1}\alpha_1 t} \dots e^{A_{i_m}\alpha_m t}\mathbf{e_i}}{\|e^{A_{i_1}\alpha_1 t} \dots e^{A_{i_m}\alpha_m t}\mathbf{e_i}\|}, \qquad i = 1, 2, \dots, n.$$

It is not hard to prove that $\mathrm{Cone}_\infty(e^{At})$ always exists, it is a polyhedral convex cone in $\mathbb{R}^n_+$, and it is never the empty set. Moreover, except for the case of a diagonal matrix A (in which case $\mathrm{Cone}(e^{At}) = \mathrm{Cone}_\infty(e^{At}) = \mathbb{R}^n_+$ for every $t \geq 0$), we have for every $0 < t_1 < t_2 < +\infty$:

$$\mathbb{R}^n_+ \supsetneq \mathrm{Cone}(e^{At_1}) \supsetneq \mathrm{Cone}(e^{At_2}) \supsetneq \mathrm{Cone}_\infty(e^{At}).$$

Notice, also, that while $\mathrm{Cone}(e^{At})$ is a simplicial cone for every $t \geq 0$, $\mathrm{Cone}_\infty(e^{At})$ is typically not. Indeed, it may have no internal points. Similarly, $\mathrm{Cone}_\infty^{\bar{\alpha}}\left(e^{A_{i_1}t} \dots e^{A_{i_m}t}\right)$ is a polyhedral cone in $\mathbb{R}^n_+$, and it is never the empty set. However, no monotonicity property can be generally guaranteed, as it happens for a single matrix exponential.

One may wonder why there is the need for introducing a whole family of asymptotic cones corresponding to a certain index family $\{i_1, i_2, \dots, i_m\}$. The reason is that, for $m > 1$, different directions $\bar{\alpha}$ lead to different asymptotic cones. This simple example clarifies this point.

Example 1: Consider the two Metzler matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 6 & 1 \\ 0 & 4 \end{bmatrix}.$$

It is a matter of simple computation to show that, for any $\bar{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$, we get

$$e^{A_1\alpha_1 t}e^{A_2\alpha_2 t} = \begin{bmatrix} e^{(\alpha_1+6\alpha_2)t} & e^{(\alpha_1+6\alpha_2)t} + e^{(2\alpha_1+4\alpha_2)t} + \text{l.t.} \\ 0 & e^{(2\alpha_1+4\alpha_2)t} \end{bmatrix}$$

where "l.t." ("lower terms") denotes terms which are surely dominated by the two terms appearing in the $(1,2)$-entry. Consequently, we distinguish the following three cases:

1) $\alpha_1 + 6\alpha_2 > 2\alpha_1 + 4\alpha_2$, namely $\alpha_1 < 2\alpha_2$: if so,

$$\text{Cone}^{\bar{\alpha}}_\infty(e^{A_1 t}e^{A_2 t}) = \text{Cone}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right);$$

2) $\alpha_1 + 6\alpha_2 = 2\alpha_1 + 4\alpha_2$, namely $\alpha_1 = 2\alpha_2$: in this case

$$\text{Cone}^{\bar{\alpha}}_\infty(e^{A_1 t}e^{A_2 t}) = \text{Cone}\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right);$$

3) $\alpha_1 + 6\alpha_2 < 2\alpha_1 + 4\alpha_2$, namely $\alpha_1 > 2\alpha_2$, for which

$$\text{Cone}^{\bar{\alpha}}_\infty(e^{A_1 t}e^{A_2 t}) = \text{Cone}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right).$$

Notice that even if $\bar{\alpha}$ varies in $\mathbb{R}^m_+$, the number of asymptotic cones is necessarily finite, as it depends on which mode dominates each column in the matrix product $e^{A_{i_1}\alpha_1 t}e^{A_{i_2}\alpha_2 t}\dots e^{A_{i_m}\alpha_m t}$. Since the dominant modes are obtained by multiplying the dominant modes of each single entry of the various factors $e^{A_{i_h}\alpha_h t}$, the number of possible combinations as $\bar{\alpha}$ varies in $\mathbb{R}^m_+$ is necessarily finite.

In the following sections we investigate the relationship between the asymptotic exponential cones and the boundaries of the cones generated by either single exponential matrices or products of exponential matrices. By making use of these characterizations and of the fundamental result of Proposition 1, we will be able to provide a family of sufficient conditions for reachability.

IV. The single exponential case

Lemma 1: Given an $n \times n$ Metzler matrix A and a strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$, the following facts are equivalent:

i) there exists $\tau > 0$ such that $\mathbf{v}$ belongs to $\partial\text{Cone}(e^{A\tau})$;
ii) $\mathbf{v} \notin \text{Cone}_\infty(e^{At})$.

Even more, if any of the above equivalent conditions holds, there exists a unique $\tau > 0$ such that $\mathbf{v}$ belongs to $\partial\text{Cone}(e^{A\tau})$.

Proof: The equivalence i)-ii) has been proved in [11]. Suppose, by contradiction, that there exist $\tau_1, \tau_2 > 0$, with $\tau_1 \neq \tau_2$, such that $\mathbf{v} = e^{A\tau_1}\mathbf{u}_1 = e^{A\tau_2}\mathbf{u}_2$, for some positive vectors $\mathbf{u}_1, \mathbf{u}_2$ with nontrivial zero patterns. If we assume, w.l.o.g., $\tau_2 > \tau_1$, then from the previous identity one gets $\mathbf{u}_1 = e^{A(\tau_2-\tau_1)}\mathbf{u}_2$, which ensures [9] that $\overline{\text{ZP}}(\mathbf{u}_1) = \overline{\text{ZP}}(e^{A(\tau_2-\tau_1)}\mathbf{u}_2) = \overline{\text{ZP}}(e^{A\tau_2}\mathbf{u}_2) = \overline{\text{ZP}}(\mathbf{v})$, a contradiction. ■

As an immediate corollary of Lemma 1, we get

Corollary 1: Given an $n \times n$ Metzler matrix A, the following facts are equivalent:

i) every $\mathbf{v} \gg 0$ belongs to $\partial\text{Cone}(e^{A\tau})$ for some $\tau > 0$;
ii) $\text{Cone}_\infty(e^{At}) \subseteq \partial\mathbb{R}^n_+$;
iii) there exists some index $k \in \langle n \rangle$ such that $k \in \text{ZP}(\mathbf{v}_i^\infty)$ for every $i \in \langle n \rangle$.

$\text{Cone}_\infty(e^{At})$ and its generating vectors, $\mathbf{v}_i^\infty$ play a major role in our analysis, and hence it is fundamental to fully identify them. By making use of Theorem 5.4 and Proposition 6.1, in [11], we may derive this complete characterization.

Lemma 2: Given an $n \times n$ Metzler matrix A,
i) for every $i \in \langle n \rangle$ $\mathbf{v}_i^\infty$ is a positive eigenvector (of unitary norm) of A, corresponding to the dominant eigenvalue of some distinguished class;
ii) a positive eigenvector $\mathbf{v}$ of A, corresponding to some eigenvalue $\lambda \in \sigma(A)$, can be expressed as the nonnegative combination of all those eigenvectors $\mathbf{v}_i^\infty$ which correspond to the eigenvalue λ, and hence $\mathbf{v}$ belongs to $\text{Cone}_\infty(e^{At})$;
iii) $\text{Cone}_\infty(e^{At})$ coincides with the (polyhedral) cone in $\mathbb{R}^n_+$ generated by the set of positive eigenvectors of A. Even more, $\text{Cone}_\infty(e^{At})$ is the polyhedral cone generated by a full column rank positive matrix.

Proof: i) has been proved in [11], and, indeed, it immediately follows from Proposition 6.1 in [11].
ii) Suppose w.l.o.g. that $\|\mathbf{v}\| = 1$ and that A is in Frobenius normal form (1). Since $e^{At}\mathbf{v} = e^{\lambda t}\mathbf{v}$, it is easily seen that

$$\lim_{t\to+\infty} \frac{e^{At}\mathbf{v}}{\|e^{At}\mathbf{v}\|} = \lim_{t\to+\infty} \frac{e^{\lambda t}\mathbf{v}}{\|e^{\lambda t}\mathbf{v}\|} = \lim_{t\to+\infty} \mathbf{v} = \mathbf{v}.$$

On the other hand, by resorting to Proposition 6.1 in [11], we may say that, when t tends to $+\infty$, then

$$e^{At}\mathbf{v} \approx \sum_{i\in I} \mathbf{v}_i^\infty[\mathbf{v}]_i \; m(t),$$

where

- $m(t)$ is the dominant mode within the set $\left\{e^{\lambda_j t}\frac{t^{m_j}}{m_j!} : j \in \overline{\text{ZP}}(\mathbf{v})\right\}$, with $\lambda_j = \max\{\lambda_{\max}(A_{kk}) : k \in \mathcal{D}(\mathcal{C}(j))\}$ and $m_j + 1$ the maximum number of classes $\mathcal{C}_k$ with $\lambda_{\max}(A_{kk}) = \lambda_j$ that lie in a single chain starting from $\mathcal{C}(j)$ in $\mathcal{R}(A)$;
- $I := \{i \in \overline{\text{ZP}}(\mathbf{v}) : m_i(t) = m(t)\}$.

Consequently, $\lim_{t\to+\infty} \frac{e^{At}\mathbf{v}}{\|e^{At}\mathbf{v}\|} = \frac{\sum_{i\in I}\mathbf{v}_i^\infty[\mathbf{v}]_i}{\|\sum_{i\in I}\mathbf{v}_i^\infty[\mathbf{v}]_i\|}$. So, it must be $\mathbf{v} = \frac{\sum_{i\in I}\mathbf{v}_i^\infty[\mathbf{v}]_i}{\|\sum_{i\in I}\mathbf{v}_i^\infty[\mathbf{v}]_i\|}$.
iii) Let V be the set of all positive eigenvectors of A. By the previous point ii), $\text{Cone}(V) \subseteq \text{Cone}_\infty(e^{At})$. On the other hand, $\text{Cone}_\infty(e^{At}) = \text{Cone}(\mathbf{v}_1^\infty, \dots, \mathbf{v}_n^\infty) \subseteq \text{Cone}(V)$, and hence $\text{Cone}(V) = \text{Cone}_\infty(e^{At})$. Since, given a subset of linearly dependent positive eigenvectors in $\{\mathbf{v}_1^\infty, \mathbf{v}_2^\infty, \dots, \mathbf{v}_n^\infty\}$, there is at least one vector which can be expressed as the nonnegative combination of the remaining eigenvectors, by getting rid of one eigenvector at a time, we can obtain a family of linearly independent generators for $\text{Cone}_\infty(e^{At})$ whose coordinates make up a full column rank matrix. ■

Remark 3: Notice that when A is an irreducible matrix, it admits only one positive eigenvector of unitary norm, which is strictly positive and corresponds to the dominant eigenvalue [1]. Therefore $\text{Cone}_\infty(e^{At})$ collapses into a one dimensional cone (a ray) which lies in the interior of the positive orthant.

We finally get a characterization of the condition $\mathrm{Cone}_\infty(e^{At}) \not\subseteq \partial\mathbb{R}^n_+$.

Proposition 2: Let A be an $n \times n$ Metzler matrix in Frobenius normal form (1). $\mathrm{Cone}_\infty(e^{At}) \not\subseteq \partial\mathbb{R}^n_+$ if and only if every initial class is distinguished, namely if $\mathcal{A}(\mathcal{C}_j) = \{j\}$ then $\lambda_{\max}(A_{jj}) > \lambda_{\max}(A_{kk})$ for every $k \in \mathcal{D}(\mathcal{C}_j)$.

Proof: [Sufficiency] Suppose that for every class $\mathcal{C}_j, j \in \langle \ell \rangle$, which is initial $\lambda_{\max}(A_{jj}) > \lambda_{\max}(A_{kk})$ for every $k \in \mathcal{D}(\mathcal{C}_j)$. Let j be an arbitrary index in $\langle \ell \rangle$. If $\mathcal{C}_j$ is an initial class, then for every index i such that $\mathcal{C}(i) = \mathcal{C}_j$, $\mathrm{block}_j[\mathbf{v}_i^\infty] \gg 0$. On the other hand, when $\mathcal{C}_j$ is not an initial class, and we let $\mathcal{C}_h$ be an initial class accessing $\mathcal{C}_j$, then for every index i such that $\mathcal{C}(i) = \mathcal{C}_h$, $\mathrm{block}_j[\mathbf{v}_i^\infty] \gg 0$. This proves that for every $j \in \langle \ell \rangle$ there is at least one vector $\mathbf{v}_i^\infty$ with $\mathrm{block}_j[\mathbf{v}_i^\infty] \gg 0$, and this ensures that $\mathrm{Cone}_\infty(e^{At}) \not\subseteq \partial\mathbb{R}^n_+$.

[Necessity] Assume, by contradiction, that there is one initial class $\mathcal{C}_j, j \in \langle \ell \rangle$, such that $\lambda_{\max}(A_{jj}) \le \lambda_{\max}(A_{kk})$ for some $k \in \mathcal{D}(\mathcal{C}_j)$. Let i be an arbitrary index in $\langle n \rangle$. If $i \notin \mathcal{C}_j$ then $\mathrm{block}_j[e^{At}\mathbf{e}_i] = 0$ and hence $\mathrm{block}_j[\mathbf{v}_i^\infty] = 0$ (see Theorem 5.4 in [11]). On the other hand, if $i \in \mathcal{C}_j$ then there exists $h < i$ such that $\mathrm{block}_h[e^{At}\mathbf{e}_i]$ strictly dominates $\mathrm{block}_j[e^{At}\mathbf{e}_i]$. Consequently, $\mathrm{block}_j[\mathbf{v}_i^\infty] = 0$. This ensures that all vectors $\mathbf{v}_i^\infty$ have the jth block identically zero, and this implies that $\mathrm{Cone}_\infty(e^{At}) \subseteq \partial\mathbb{R}^n_+$. ∎

At this point, by putting together Proposition 1 with Corollary 1 and Proposition 2, we get the following sufficient condition for reachability.

Proposition 3: Consider an n-dimensional continuous-time positive switched system (2), commuting among n single-input subsystems $(A_i, b_i), i = 1, 2, \ldots, n$, and suppose that $|\mathcal{I}_\mathcal{S}| = 1$ for every proper subset $\mathcal{S} \subset \langle n \rangle$, namely that there exists a unique index $j(\mathcal{S}) \in \langle n \rangle$ such that $\overline{\mathrm{ZP}}(e^{A_{j(\mathcal{S})}}\mathbf{e}_\mathcal{S}) = \mathcal{S}$. Then the system is reachable if and only if the following two conditions hold:

a) the system is monomially reachable;
b) for every $\mathcal{S}$, with $|\mathcal{S}| > 1$, either $\overline{\mathrm{ZP}}(b_{j(\mathcal{S})}) \subseteq \mathcal{S}$ or $\mathrm{Cone}_\infty\left(P_\mathcal{S}^T\, e^{A_{j(\mathcal{S})}t} P_\mathcal{S}\right) \subseteq \partial\mathbb{R}^n_+$, where $P_\mathcal{S}$ is the selection matrix which selects all the columns corresponding to the indices belonging to $\mathcal{S}$.

Proof: [Sufficiency] Notice, first, that if $\mathrm{Cone}_\infty\left(P_\mathcal{S}^T\, e^{A_{j(\mathcal{S})}t} P_\mathcal{S}\right) \subseteq \partial\mathbb{R}^n_+$, then, by Corollary 1, for every strictly positive vector $\mathbf{v}_\mathcal{S} \in \mathbb{R}^{|\mathcal{S}|}_+$ there exists $\tau > 0$ such that $\mathbf{v}_\mathcal{S} \in \partial\mathrm{Cone}\left(P_\mathcal{S}^T\, e^{A_{j(\mathcal{S})}\tau} P_\mathcal{S}\right)$. So, as a consequence of condition $\overline{\mathrm{ZP}}(e^{A_{j(\mathcal{S})}}\mathbf{e}_\mathcal{S}) = \mathcal{S}$, for every positive vector $\mathbf{v} \in \mathbb{R}^n_+$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, there exists $\tau > 0$ such that $\mathbf{v} = e^{A_{j(\mathcal{S})}\tau} P_\mathcal{S} \mathbf{u}_\mathcal{S}$, with $\mathrm{ZP}(\mathbf{u}_\mathcal{S}) \neq \emptyset$. Consequently, assumptions a) and b) imply conditions iia) and iib) of Proposition 1, and reachability follows.

[Necessity] By comparing the proposition's statement with the one of Proposition 1, it remains to prove that if the system is reachable and for every $\mathcal{S} \subset \langle n \rangle$ there is a single index $j(\mathcal{S})$ such that $\overline{\mathrm{ZP}}(e^{A_{j(\mathcal{S})}}\mathbf{e}_\mathcal{S}) = \mathcal{S}$, then condition[3] $\emptyset \neq \overline{\mathrm{ZP}}(b_{j(\mathcal{S})}) \not\subseteq \mathcal{S}$ implies $\mathrm{Cone}_\infty\left(P_\mathcal{S}^T\, e^{A_{j(\mathcal{S})}} P_\mathcal{S}\right) \subseteq \partial\mathbb{R}^n_+$.

Indeed, let $\mathbf{v}$ be a positive vector with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$. If the vector $\mathbf{v}$ is reachable, then [9] it can be expressed as $\mathbf{v} = e^{A_{j(\mathcal{S})}\tau_k}\mathcal{B}_k$, with $\tau_k > 0$ and

$$\begin{aligned}\mathcal{B}_k \;&:=\; e^{A_{i_{k-1}}\tau_{k-1}} \ldots e^{A_{i_1}\tau_1} e^{A_{i_0}\tau_0} b_{i_0} c_{i_0} + \ldots \\ &+\; e^{A_{i_{k-1}}\tau_{k-1}} b_{i_{k-1}} c_{i_{k-1}},\end{aligned} \tag{5}$$

for suitable indices i_ℓ (with $i_\ell \neq i_{\ell+1}$), nonnegative time intervals τ_ℓ and nonnegative coefficients c_ℓ. But then [9] $\mathcal{S}_k := \overline{\mathrm{ZP}}(\mathcal{B}_k) \subseteq \mathcal{S}$, and the uniqueness of $j(\mathcal{S})$ ensures that $\mathcal{S}_k \subsetneq \mathcal{S}$. So, $\mathbf{v} = e^{A_{j(\mathcal{S})}\tau_k} P_\mathcal{S}\mathbf{u}_\mathcal{S}$, $\exists\ \mathbf{u}_\mathcal{S} \ge 0$, with $\mathrm{ZP}(\mathbf{u}_\mathcal{S}) \neq \emptyset$. But since this must be true for every vector $\mathbf{v} \in V_\mathcal{S} := \{v : \overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}\}$, then every $\mathbf{v}_\mathcal{S} \in \mathbb{R}^{|\mathcal{S}|}_+$, with $\mathbf{v}_\mathcal{S} \gg 0$, must lie on the boundary of $\mathrm{Cone}(P_\mathcal{S}^T e^{A_{j(\mathcal{S})}\tau} P_\mathcal{S})$ for some $\tau = \tau(\mathbf{v}_\mathcal{S}) > 0$. By Corollary 1, then, it must be $\mathrm{Cone}_\infty\left(P_\mathcal{S}^T\, e^{A_{j(\mathcal{S})}} P_\mathcal{S}\right) \subseteq \partial\mathbb{R}^n_+$. ∎

Proposition 4: Consider an n-dimensional positive switched system (2), commuting among n single-input subsystems $(A_i, b_i), i = 1, 2, \ldots, n$, and suppose that the system is monomially reachable. If for every proper subset $\mathcal{S} \subset \langle n \rangle$, with $|\mathcal{S}| \ge 2$,

$$\cap_{i \in \mathcal{I}_\mathcal{S}} \mathrm{Cone}_\infty(e^{P_\mathcal{S}^T A_i P_\mathcal{S} t}) \subseteq \partial\mathbb{R}^{|\mathcal{S}|}_+, \tag{6}$$

then the system is reachable.

Proof: Monomial reachability ensures that all monomial vectors are reachable. On the other hand, consider the case of any vector $\mathbf{v}$ with $\mathcal{S} = \overline{\mathrm{ZP}}(\mathbf{v})$ of cardinality greater than 1 and let $\mathbf{v}_\mathcal{S}$ be the restriction of $\mathbf{v}$ to the indices corresponding to $\mathcal{S}$. If (6) holds, than, by the strict positivity of $\mathbf{v}_\mathcal{S}$, there exists at least one index $j = j(\mathbf{v}_\mathcal{S}) \in \mathcal{I}_\mathcal{S}$ such that $\mathbf{v}_\mathcal{S} \notin \mathrm{Cone}_\infty(e^{P_\mathcal{S}^T A_j P_\mathcal{S} t})$. Thus, by Lemma 1, there exists $\tau > 0$ such that $\mathbf{v}_\mathcal{S} \in \partial\mathrm{Cone}(e^{P_\mathcal{S}^T A_j P_\mathcal{S}\tau})$ and $\mathbf{v}$ is reachable. ∎

V. The multiple exponential case

In this section we explore the properties of the cones generated by an ordered family of exponential matrices, along certain directions. As illustrated in Example 1, once the indices $i_1, i_2, \ldots, i_m \in \mathcal{P}$ have been chosen, we are dealing with a family of asymptotic exponential cones, and not a single one. Nonetheless this is always a finite family.

Unfortunately, the result of Lemma 1 for the asymptotic cone of a single exponential matrix can be only partially extended, thus getting the following proposition, which represents an extended of version a similar result in [9].

Proposition 5: Given a set $\mathcal{A} = \{A_1, \ldots, A_p\}$ of Metzler matrices and a strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$, let m be in $\mathbb{N}$ and let $i_1, i_2, \ldots, i_m$ be indices in $\mathcal{P}$. If $\mathbf{v} \notin \mathrm{Cone}_\infty^{(1,1,\ldots,1)}(e^{A_{i_1}t} \ldots e^{A_{i_m}t})$, then $\exists\ \tau_1, \ldots, \tau_m > 0$ such that $\mathbf{v} \in \partial\,\mathrm{Cone}(e^{A_{i_1}\tau_1} \ldots e^{A_{i_m}\tau_m})$. Consequently, if $\mathbf{v} \notin \mathrm{Cone}_\infty^{\bar\alpha}(e^{A_{i_1}t} \ldots e^{A_{i_m}t})$, for some $\bar\alpha =$

[3] Notice that reachability ensures monomial reachability and this implies [9], for this class of systems, that all vectors $b_i, i \in \langle n \rangle$, are monomial.

$(\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m_+$, then $\exists\ \bar{\tau}_1, \dots, \bar{\tau}_m > 0$ such that $\mathbf{v} \in \partial\, \mathrm{Cone}(e^{A_{i_1}\bar{\tau}_1} \dots e^{A_{i_m}\bar{\tau}_m})$.

Proof: The first part was proved in [9]. The final part follows from the previous one, by assuming $\bar{A}_{i_h} := A_{i_h}\alpha_h$ and $\tau_h = \alpha_h \bar{\tau}_h$. ∎

This sufficient condition for the solvability of the NEW PROBLEM immediately brings, as a corollary, a sufficient condition for the original PROBLEM solution.

Corollary 2: Consider a set $\mathcal{A} = \{A_1, \dots, A_p\}$ of Metzler matrices and a nonnegative vector $\mathbf{v} \in \mathbb{R}^n_+$. Set $\mathcal{S} := \overline{\mathrm{ZP}}(\mathbf{v})$, and let $P_{\mathcal{S}}$ denote the (column) selection matrix corresponding to the indices in $\mathcal{S}$, and $\mathbf{v}_{\mathcal{S}} = P_{\mathcal{S}}^T \mathbf{v}$ the subvector obtained by restricting $\mathbf{v}$ to the entries corresponding to $\mathcal{S}$. If

$$\mathbf{v}_{\mathcal{S}} \notin \bigcap_{m\geq 1} \bigcap_{i_1,\dots,i_m \in \mathcal{I}_{\mathcal{S}}} \bigcap_{\bar{\alpha}\in\mathbb{R}^m_+} \mathrm{Cone}^{\bar{\alpha}}_{\infty}(P_{\mathcal{S}}^T e^{A_{i_1}t} \dots e^{A_{i_m}t} P_{\mathcal{S}}),$$

then $\exists\, i_1, i_2, \dots, i_m \in \mathcal{I}_{\mathcal{S}}$ and $\tau_1, \dots, \tau_m > 0$ such that $\mathbf{v} = e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m}\mathbf{u}$, with $\overline{\mathrm{ZP}}(\mathbf{u}) \subsetneq \mathcal{S}$.

Unfortunately, up to now, we have not been able to reverse the statement of Proposition 5. However, examples have been given showing that $\mathbf{v} \in \mathrm{Cone}^{(1,\dots,1)}_{\infty}(e^{A_{i_1}t} \dots e^{A_{i_m}t})$ does not necessarily mean that $\mathbf{v}$ cannot be expressed as $\mathbf{v} = e^{A_{i_1}\tau_1} \dots e^{A_{i_m}\tau_m}\mathbf{u}$, for some positive vector $\mathbf{u}$ with $\overline{\mathrm{ZP}}(\mathbf{u}) \subsetneq \mathcal{S}$. There are some special cases, though, when we are able to forecast that each strictly positive vector lies in the boundary of some cone generated by the product of two exponential matrices.

Lemma 3: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be two Metzler matrices. Suppose that A_2 is irreducible with dominant eigenvector (of unitary norm) $\mathbf{v} \gg 0$, so that $\mathrm{Cone}(\mathbf{v}) = \mathrm{Cone}_{\infty}(e^{A_2 t})$. If $\mathbf{v} \in \mathrm{Cone}_{\infty}(e^{A_1 t})$, but it is not an eigenvector of A_1, then $\forall\ \tau_1 > 0, \exists\ \tau_2 > 0$ such that $\mathbf{v} \in \partial\mathrm{Cone}(e^{A_1\tau_1}e^{A_2\tau_2})$.

Proof: This amounts to proving that for every $\tau_1 > 0$ there exists $\tau_2 > 0$ such that

$$\mathbf{v} = e^{A_1\tau_1}e^{A_2\tau_2}\mathbf{u} \;\Leftrightarrow\; e^{-A_1\tau_1}\mathbf{v} = e^{A_2\tau_2}\mathbf{u},$$

for some $\mathbf{u} > 0$ with $\mathrm{ZP}(\mathbf{u}) \neq \emptyset$. We first observe that for every $\tau_1 > 0$, $\mathbf{w} := e^{-A_1\tau_1}\mathbf{v}$ is not a multiple of $\mathbf{v}$ and hence it does not belong to $\mathrm{Cone}_{\infty}(e^{A_2 t})$. On the other hand, since $\mathbf{v} \in \mathrm{Cone}_{\infty}(e^{A_1 t})$, then $\mathbf{v}$ is an internal point of $\mathrm{Cone}(e^{A_1 t})$, for every $t \geq 0$. So, in particular, $\mathbf{v}$ is an internal point of $\mathrm{Cone}(e^{A_1\tau_1})$, which amounts to saying that $\mathbf{v} = e^{A_1\tau_1}\mathbf{u}_1$ for some $\mathbf{u}_1 \gg 0$. Clearly, by the invertibility of the exponential matrix, $\mathbf{w} = \mathbf{u}_1 \gg 0$. So, we have shown that $\mathbf{w}$ is a strictly positive vector which does not belong to $\mathrm{Cone}_{\infty}(e^{A_2 t})$. This implies that $\mathbf{w} \in \partial\mathrm{Cone}(e^{A_2\tau_2})$ for some $\tau_2 > 0$, and hence $\mathbf{w} = e^{A_2\tau_2}\mathbf{u}$, for some $\mathbf{u} > 0$ with $\mathrm{ZP}(\mathbf{u}) \neq \emptyset$. This completes the proof. ∎

The previous technical result leads to the following sufficient condition for reachability.

Proposition 6: An n-dimensional continuous-time positive switched system (2), commuting among n single-input subsystems $(A_i, b_i), i = 1, 2, \dots, n$, is reachable if for every proper subset $\mathcal{S} \subset \langle n\rangle$ we have:

a) if $|\mathcal{S}| = 1$, then $\exists\, j(\mathcal{S}) \in \mathcal{I}_{\mathcal{S}}$ such that $\overline{\mathrm{ZP}}(b_{j(\mathcal{S})}) = \mathcal{S}$;
b) if $|\mathcal{S}| > 1$, then either

1. $\exists\, j(\mathcal{S}) \in \mathcal{I}_{\mathcal{S}}$ such that $\overline{\mathrm{ZP}}(b_{j(\mathcal{S})}) \subset \mathcal{S}$,

or

2. $\exists\, j_i(\mathcal{S}), j_k(\mathcal{S}) \in \mathcal{I}_{\mathcal{S}}$ such that $P_{\mathcal{S}}^T A_{j_i(\mathcal{S})} P_{\mathcal{S}}$ is irreducible and its strictly positive eigenvector (of unitary modulus) is not an eigenvector of $P_{\mathcal{S}}^T A_{j_k(\mathcal{S})} P_{\mathcal{S}}$.

Proof: We only need to show that condition b) - 2. implies condition iib - 2) in Proposition 1.

To this end, let $\mathbf{v}$ be a positive vector with $\mathcal{S} = \overline{\mathrm{ZP}}(\mathbf{v})$ of cardinality greater than 1, and notice that, under assumption b)-2., there exists $j_i \in \mathcal{I}_{\mathcal{S}}$ such that $\mathrm{Cone}_{\infty}(P_{\mathcal{S}}^T A_{j_i} P_{\mathcal{S}})$ coincides with the cone generated by an eigenvector $\mathbf{w} \gg 0$.

Let $\mathbf{v}_{\mathcal{S}}$ be the restriction of $\mathbf{v}$ to the indices corresponding to $\mathcal{S}$. If $\mathbf{v}_{\mathcal{S}} \neq \mathbf{w}$, then $\mathbf{v}_{\mathcal{S}} \notin \mathrm{Cone}_{\infty}(e^{P_{\mathcal{S}}^T A_{j_i} P_{\mathcal{S}} t})$, and hence there exists $\tau > 0$ such that $\mathbf{v}_{\mathcal{S}} \in \partial\mathrm{Cone}(e^{P_{\mathcal{S}}^T A_{j_i} P_{\mathcal{S}} \tau})$. This ensures that $\mathbf{v}$ is reachable. If $\mathbf{v}_{\mathcal{S}} = \mathbf{w}$, then either $\mathbf{v}_{\mathcal{S}} \notin \mathrm{Cone}_{\infty}(e^{P_{\mathcal{S}}^T A_j P_{\mathcal{S}} t})$ for some other $j \in \mathcal{I}_{\mathcal{S}}$ (and if so, by repeating the previous argument, we may say that $\mathbf{v}_{\mathcal{S}} \in \partial\mathrm{Cone}(e^{P_{\mathcal{S}}^T A_j P_{\mathcal{S}} \tau}), \exists\ \tau > 0$, and hence $\mathbf{v}$ is reachable), or for every $j \neq i, j \in \mathcal{I}_{\mathcal{S}}$, we have $\mathbf{v}_{\mathcal{S}} \in \mathrm{Cone}_{\infty}(e^{P_{\mathcal{S}}^T A_j P_{\mathcal{S}} t})$. For one such index $j_k \in \mathcal{I}_{\mathcal{S}}$, though, $\mathbf{v}_{\mathcal{S}}$ is not an eigenvector of $P_{\mathcal{S}}^T A_{j_k} P_{\mathcal{S}}$. So, by applying Lemma 3, we may say that there exist $\tau_k, \tau_i > 0$ such that $\mathbf{v}_{\mathcal{S}} = e^{P_{\mathcal{S}}^T A_{j_k} P_{\mathcal{S}} \tau_k} e^{P_{\mathcal{S}}^T A_{j_i} P_{\mathcal{S}} \tau_i} \mathbf{u}_{\mathcal{S}}$, for some positive vector $\mathbf{u}_{\mathcal{S}}$, with $\mathrm{ZP}(\mathbf{u}_{\mathcal{S}}) \neq \emptyset$. This ensures, again, that $\mathbf{v}$ is reachable. ∎

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