2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Performance Analysis for a Class of Decentralized Control Systems

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Abstract— **This paper is concerned with decentralized controller design for large-scale interconnected systems of pseudo-hierarchical structure. Given such a system, one can use the existing techniques to design a decentralized controller for the reference hierarchical model, which is obtained by eliminating certain weak interconnections of the original system. Although this indirect controller design is often fascinating as far as the computational complexity is concerned, it may not provide a satisfactory performance for the original pseudo-hierarchical system. A LQ cost function is defined in order to evaluate the discrepancy between the pseudo-hierarchical system and its reference hierarchical model under the designed decentralized controller. A discrete Lyapunov equation should then be solved to compute this performance index. However, due to the large-scale nature of the system, this equation can by no means be handled for many real-world systems. Thus, attaining an upper bound on this cost function can be much more desirable than finding its exact value. For this purpose, a novel technique is proposed, which only requires solving a simple LMI optimization problem with three variables. The problem is then reduced to a scalar optimization problem, for which an explicit solution is provided. It is also proved that as the pseudo-hierarchical system approaches its reference hierarchical model, the bounds obtained from the LMI and scalar optimization problems will both go to zero. In the particular case, when the two models are identical (i.e., the original system is exactly hierarchical), both upper bounds will be zero.**

I. Introduction

Many real-world plants can be modeled as large-scale interconnected systems [1]. Since such systems normally comprise several subsystems, their control is intricate. Decentralized control theory was developed to alleviate the control design and implementation problems for this type of systems. Distinctive aspects of decentralized control systems have been well-documented in the last three decades [2], [3]. A decentralized controller consists of a number of isolated local controllers corresponding to the subsystems (or control channels) of the interconnected system. For the sake of simplicity of the control design problem, it is often desirable that the large-scale system possesses a hierarchical structure [4], [5]. The control design problem for a hierarchical system can be broken down into a number of parallel design subproblems corresponding to diverse subsystems. The advantage of such design techniques is twofold; not only is the control design procedure far simpler for a number of low-order subsystems rather than a high-order system, but the parallel computation is also intriguing.

Many important physical cooperative control applications such as formation flight, underwater vehicles, automated highway systems, robotics, satellite constellation, etc. with a leader-follower configuration have a hierarchical structure [6], [7], [8]. Furthermore, it is shown in [9] that under certain conditions, a continuous-time non-hierarchical system can have a hierarchical discrete-time equivalent model. It is straightforward to show that a set of stabilizing local controllers obtained by neglecting all the interconnections between the subsystems constitute a stabilizing decentralized controller for the original hierarchical system [5]. This significant result is quite beneficial in the sense that it provides a simple technique for decentralized control design. In addition, a technique is given in [10] to design a near-optimal decentralized controller for hierarchical systems. This idea is further developed in [11] to decentralize any given centralized controller without losing its fundamental properties. While decentralized control design for hierarchical systems to achieve various design specifications has been widely investigated in the past several years, there are only a few fledgling control design techniques for general large-scale systems, due to the complexity of the problem. Furthermore, there is no efficient performance evaluation method for a closed-loop decentralized system when the controller is designed for the system after some structural modifications.

On the other hand, there exist numerous non-hierarchical systems which are "close" to being hierarchical. Such systems have a few weak interconnections between their subsystems whose removal will change the structure of the system to an exact hierarchical one. This type of systems will be referred to as *pseudo-hierarchical* systems, and the hierarchical model obtained by eliminating a minimum number of "weak" interconnections will be called *reference hierarchical* model throughout this paper. Given a pseudo-hierarchical large-scale system, a decentralized controller can be designed for the corresponding reference hierarchical model, by exploiting the available techniques. Even though this straightforward approach is appealing as far as the computational complexity is concerned, the decentralized controller obtained will not

This work has been supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant RGPIN-262127-07.

necessarily meet the design specifications for the original pseudo-hierarchical system. In fact, the controller obtained may not even stabilize the original system, whereas it definitely stabilizes the corresponding reference hierarchical model. Apart from the stabilizability issue (which is not a concern if the removed interconnections in the reference hierarchical model are sufficiently weak), the performance of the pseudo-hierarchical system under this controller can be quite poor. Thus, it is important to carry out a performance analysis for the corresponding closed-loop system in order to make certain that this indirect design technique is suitable for the given pseudo-hierarchical large-scale system.

This paper deals with the performance analysis for pseudo-hierarchical decentralized large-scale systems. It is assumed that a decentralized controller has been designed for the reference hierarchical model of a pseudo-hierarchical system to meet certain control objectives. Moreover, it is supposed that the above-mentioned controller stabilizes the pseudo-hierarchical system, while it may deteriorate the overall performance. A LQ cost function is appropriately defined to assess the discrepancy between the pseudo-hierarchical system and the corresponding reference hierarchical model under this decentralized controller. The smaller this performance index is, the closer the two closed-loop systems are to each other. Obtaining this cost function involves solving a discrete Lyapunov equation. As a result of the large-scale nature of the system, this equation by no means can be handled efficiently. Alternatively, it would be very useful to attain an upper bound on this cost function. A novel technique is proposed to address this objective, and it is subsequently shown that a LMI optimization problem with only three variables needs to be solved in order to compute this bound. This problem is also simplified and an explicit bound is proposed, without having to solve any optimization problem. The main distinguishing feature of this work is that it presents a simple technique for performance evaluation of pseudo-hierarchical decentralized systems. To elucidate that the obtained bounds are not too conservative in general, it is proved that as the pseudo-hierarchical system approaches the corresponding reference hierarchical model, these bounds go to zero, and if the original model is exactly hierarchical, then the bounds are both equal to zero.

The organization of this paper is as follows. In Section II, some preliminary results are provided and the problem is also formulated. The main results and developments are derived in Section III, which are illustrated in two numerical examples in Section IV. Finally, some concluding remarks are drawn in Section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a large-scale interconnected system $\mathcal{S}$ consisting of ν subsystems, where its i^{th} subsystem S_i is represented by:

$$\begin{aligned} x_i[k+1] &= \sum_{j=1}^{\nu} A_{ij}x_j[k] + B_iu_i[k] \\ y_i[k] &= C_ix_i[k], \quad i \in \bar{\nu} := \{1, 2, \cdots, \nu\} \end{aligned} \tag{1}$$

In the above equation, $x_i[k] \in \mathbb{R}^{n_j}$, $u_i[k] \in \mathbb{R}^{m_i}$ and $y_i[k] \in \mathbb{R}^{r_i}$ are the state, the input and the output of S_i, respectively. Define n to be $n_1 + n_2 + \cdots + n_\nu$. Sketch now a digraph $\mathcal{G}$ associated with the system $\mathcal{S}$ as follows:

- Assign ν vertices, one for each subsystem of $\mathcal{S}$.
- For any $i, j \in \bar{\nu},\ i \neq j$, connect vertex i to vertex j with a directed edge if $A_{ij} \neq 0$.
- For any $i, j \in \bar{\nu}$, if there is an edge between vertex i and vertex j, attribute the weight $\|A_{ij}\|_F$ to that edge, where $\|\cdot\|_F$ represents the Frobenius norm operator.

The graph $\mathcal{G}$ specifies the topology of information transmission between the subsystems. From this perspective, it plays an important role in the stability and stabilizability analysis of the system. If the graph $\mathcal{G}$ has no directed cycles, then the system $\mathcal{S}$ is said to be hierarchical. For any $i \in \bar{\nu}$, define the isolated subsystem $\bar{S}_i$ as:

$$\begin{aligned} \bar{x}_i[k+1] &= A_{ii}\bar{x}_i[k] + B_i\bar{u}_i[k] \\ \bar{y}_i[k] &= C_i\bar{x}_i[k] \end{aligned} \tag{2}$$

For the case when the graph $\mathcal{G}$ is acyclic, a stabilizing decentralized controller can be obtained by designing ν local controllers separately such that the i^{th} local controller stabilizes the isolated subsystem $\bar{S}_i$, for all $i \in \bar{\nu}$. This simple, but important fact implies that when the graph $\mathcal{G}$ is acyclic, the decentralized controller design procedure can be quite straightforward (as far as the stability is concerned). It is to be noted that as discussed earlier, several methods are proposed in the literature to design a LTI decentralized controller for a hierarchical system in order to achieve predetermined objectives.

In the general case, when the graph $\mathcal{G}$ is not acyclic, one can remove certain edges of $\mathcal{G}$ to obtain an acyclic graph, and design the local controllers for the resultant system as described before. However, the controller obtained may not perform satisfactorily when applied to the original system, if the interconnections neglected in the control design are not sufficiently weak.

Assume now that some of the edges are removed to obtain a hierarchical model, and a LTI decentralized controller K is designed for the resultant model using the available approaches. Once this controller is applied to the original system $\mathcal{S}$, the closed-loop system may perform poorly, and may even be unstable. Therefore, it is desired in this paper to evaluate the performance of the system $\mathcal{S}$ under the controller K, with respect to its hierarchical counterpart (i.e. the hierarchical model under the same controller K). To this end, it is assumed that the closed-loop system is stable, which is a requirement for performance degradation analysis in this work. It is worth mentioning that the closed-loop stability required here is guaranteed if the interconnections neglected in control design procedure are sufficiently weak.

In the sequel, the hierarchical system $\mathcal{S}_h$ under the LTI decentralized controller K is represented as:

$$x_h[k+1] = A_hx_h[k] \tag{3}$$

and the original system $\mathcal{S}$ under the same controller as:

$$x_c[k+1] = A_cx_c[k] \tag{4}$$

Furthermore, set $x_c[0] = x_h[0]$. With no loss of generality, the matrix A_h can be assumed to be lower-block triangular.

Remark 1: In the case when a decentralized overlapping controller is to be designed for an overlapping system by mean of the inclusion principle, the expanded system (obtained from the original overlapping system) under the decentralized controller designed after neglecting all the interconnections is expressed by (4). The expanded system with nullified interconnections under the above-mentioned decentralized controller can then be described by (3), as a special case of a hierarchical model.

In order to assess the closeness of the systems given in (3) and (4), one can measure the discrepancy between the states $x_h[k]$ and $x_c[k]$. This can be evaluated through the following performance index:

$$J_d = \sum_{k=0}^{\infty} (x_c[k] - x_h[k])^T (x_c[k] - x_h[k]) \tag{5}$$

Definition 1: Define the performance indices J_c and J_h as below:

$$J_c = \sum_{k=0}^{\infty} x_c[k]^T Q x_c[k], \quad J_h = \sum_{k=0}^{\infty} x_h[k]^T Q x_h[k] \tag{6}$$

With no loss of generality, it will be assumed hereafter that $Q = I$.

Definition 2: The controller K is said to be μ-suboptimal, if the inequality $\frac{J_d}{J_h} < \mu$ holds.

Some works such as [12], define the suboptimality based on the ratio $\frac{J_c}{J_h}$, as opposed to $\frac{J_d}{J_h}$. However, it is manifest that the smallness of $\frac{J_c}{J_h}$ does not necessarily imply the closeness of $x_c[k]$ and $x_h[k]$. The objective here is to obtain a proper and easy-to-compute μ by which the controller K is suboptimal. Nevertheless, the following practical restrictions are also made.

Assumption 1: The order of the entire interconnected system $\mathcal{S}$ is substantially higher than the order of each individual subsystem S_i; i.e., $n \gg n_i,\ i \in \nu$. As a result, a discrete Lyapunov equation corresponding to the entire system cannot be solved efficiently, whereas a discrete Lyapunov equation corresponding to any individual subsystem can be easily solved.

Assumption 2: Although a Lyapunov equation of order n cannot be computed efficiently, lower and upper bounds on the eigenvalues of any matrix of order n can be obtained.

It is quite important to note in Assumption 2 that solving a Lyapunov equation of order n is much more difficult than estimating the eigenvalues of a matrix of order n, as the former problem involves n^2 variables while the latter one only $n + 1$ (regardless of their linearity or bilinearity).

It is evident that J_h and J_c in (6) satisfy the relations:

$$J_h = x_h[0]^T P_h x_h[0], \quad J_c = x_c[0]^T P_c x_c[0] \tag{7}$$

where:

$$A_h^T P_h A_h - P_h + I = 0, \quad A_c^T P_c A_c - P_c + I = 0 \tag{8}$$

In order to develop the main results of the paper, one more assumption is required to be made.

Assumption 3: The closed-loop system given in (4) is stable with the Lyapunov matrix P_h.

It is to be noted that Assumption 3 is more restrictive than only the stability condition for the system (4), and is met when the removed edges have sufficiently small weights. Various sufficient conditions are provided in the literature, which ensure the validity of this assumption.

III. MAIN RESULTS

In what follows, the performance deviation J_d will be formulated.

Lemma 1: The performance index J_d can be written as:

$$J_d = \begin{bmatrix} x_h[0]^T & x_c[0]^T \end{bmatrix} P_d \begin{bmatrix} x_h[0] \\ x_c[0] \end{bmatrix} \tag{9}$$

where:

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} P_d \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T - P_d + \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} = 0 \tag{10}$$

Proof: Augmenting the closed-loop systems (3) and (4) results in:

$$\begin{bmatrix} x_h[k+1] \\ x_c[k+1] \end{bmatrix} = \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_c[k] \end{bmatrix} \tag{11}$$

On the other hand, the performance index J_d can be rewritten as:

$$J_d = \sum_{k=0}^{\infty} \left[x_h[k]^T \ x_c[k]^T \right] \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_c[k] \end{bmatrix} \tag{12}$$

It is well-known that the performance index J_d, can be written as (9) where the matrix P_d satisfies the equation (10). This completes the proof. ■

Due to Assumption 1, the performance deviation J_d cannot be directly computed from Lemma 1 in order to compute the ratio $\frac{J_d}{J_h}$ precisely. Hence, the notion of μ-suboptimality is helpful here in order to obtain a reasonable upper bound on this ratio, which is carried out in the sequel.

Lemma 2: Given a matrix H of proper dimension, assume the following inequality is satisfied:

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T H \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} - H + \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} < 0 \tag{13}$$

Then, the inequality given below holds:

$$J_d < \begin{bmatrix} x_h[0]^T & x_c[0]^T \end{bmatrix} H \begin{bmatrix} x_h[0] \\ x_c[0] \end{bmatrix} \tag{14}$$

Proof: It can be concluded from the relations (10) and (13) that:

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T (H - P_d) \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} - (H - P_d) < 0 \tag{15}$$

Since both of the matrices A_c and A_h are assumed to be Schur, it results from the above inequality that $P_d < H$. The proof follows immediately from this result and the equation (9). ■

Let the following optimization problem be introduced.

Problem 1: Find the infimum of the objective function $k_1+2k_2+k_3$ for the variables k_1, k_2 and k_3, subject to:

$$\begin{bmatrix} (1-k_1)I & k_2(A_h^T P_h A_c - P_h) - I \\ k_2(A_c^T P_h A_h - P_h) - I & k_3(A_c^T P_h A_c - P_h) + I \end{bmatrix} < 0 \tag{16}$$

Remark 2: Problem 1 is a LMI optimization which can be efficiently handled using proper software tools.

Theorem 1: The controller K is μ-suboptimal, where μ denotes the infimum obtained by solving Problem 1.

Proof: Consider any real scalars k_1, k_2 and k_3 satisfying the inequality (16) given in Problem 1. Using the equations given in (8), this inequality can be rewritten as:

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T H \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} - H + \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} < 0 \tag{17}$$

if the matrix H is chosen as:

$$H = \begin{bmatrix} k_1 P_h & k_2 P_h \\ k_2 P_h & k_3 P_h \end{bmatrix} \tag{18}$$

Therefore, it can be inferred from Lemma 2 that:

$$\begin{aligned} J_d &< \begin{bmatrix} x_h[0]^T & x_c[0]^T \end{bmatrix} H \begin{bmatrix} x_h[0] \\ x_c[0] \end{bmatrix} = \\ &(k_1+2k_2+k_3)x_h[0]^T P_h x_h[0] = \\ &(k_1+2k_2+k_3)J_h \end{aligned} \tag{19}$$

(note that $x_h[0] = x_c[0]$). Thus:

$$\frac{J_d}{J_h} < k_1 + 2k_2 + k_3 \tag{20}$$

The above inequality explains why the objective function $k_1+2k_2+k_3$ is to be minimized, and hence it completes the proof. ■

Theorem 1 states that the solution of Problem 1 provides an upper bound on the ratio $\frac{J_d}{J_h}$. It is interesting to note that the inequality constraint of this optimization problem is always feasible. To prove this, it suffices to choose $k_1 = 2$, $k_2 = 0$ and to let k_3 be a very large number. Since it is assumed in Assumption 3 that the Lyapunov matrix P_h determines the stability or instability of the system (3), the matrix $A_c^T P_h A_c - P_h$ is negative definite, and thus the inequality (16) holds.

Due to Assumption 1 and the large-scale nature of the system $\mathcal{S}$, Problem 1 may not be handled efficiently by existing techniques. This is mainly because of the matrix constraint (16) which becomes sophisticated for large-scale systems. Thus, it is desirable to convert the matrix inequality (16) into a scalar one. This objective will be addressed in the sequel.

Problem 2: Find the infimum of the objective function $k_1+2k_2+k_3$ for the variables k_1, k_2 and k_3 subject to the scalar inequalities $k_1 > 1$ and:

$$(k_1-1)(-1+k_3 m_1) - 1 - k_2^2 m_2 - k_2 m_3 > 0 \tag{21}$$

where:

$$\begin{aligned} &m_1 = \underline{\lambda}(-R_2), \quad m_2 = \bar{\lambda}\left(R_1 R_1^T\right), \\ &m_3 = \bar{\lambda}\left(-R_1 - R_1^T\right) \end{aligned} \tag{22}$$

(the symbols $\bar{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ represent the maximum and minimum magnitudes of the eigenvalues of a matrix, respectively).

Theorem 2: Denote with μ the infimum obtained by solving Problem 2. Then, the controller K is μ-suboptimal.

Proof: The inequality constraint of Problem 1 can be rearranged as:

$$\begin{bmatrix} (k_3 R_2 + I) & (k_2 R_1 - I) \\ \left(k_2 R_1^T - I\right) & (1-k_1)\,I \end{bmatrix} < 0 \tag{23}$$

where:

$$R_1 = A_c^T P_h A_h - P_h, \quad R_2 = A_c^T P_h A_c - P_h \tag{24}$$

Applying the Schur complement formula to the inequality (23) results in the following inequality:

$$(k_1-1)(k_3 R_2 + I) + (k_2 R_1 - I)\left(k_2 R_1^T - I\right) < 0 \tag{25}$$

It is easy to verify that the matrix inequality (25) is guaranteed to hold, provided the scalar inequality given below is satisfied:

$$\underline{\lambda}\Big((k_1-1)(-k_3 R_2 - I)\Big) > \bar{\lambda}\Big((I - k_2 R_1)\left(I - k_2 R_1^T\right)\Big) \tag{26}$$

Choose some scalars k_1, k_2, k_3 satisfying the inequality (21). It can be deduced from the above discussion and the result of Theorem 1 that in order to prove Theorem 2 it suffices to substantiate the validity of the inequality (26). To show this, one can use the following equation:

$$\underline{\lambda}\Big((k_1-1)(-k_3 R_2 - I)\Big) = (k_1-1)(-1+k_3 m_1) \tag{27}$$

Moreover, it results from Lemma 2.1 in [13] that:

$$\begin{aligned} &\bar{\lambda}\Big((I - k_2 R_1)\left(I - k_2 R_1^T\right)\Big) = \\ &1 + \bar{\lambda}\big(k_2\left(-R_1 - R_1^T\right) + k_2^2 R_1 R_1^T\big) \\ &\le 1 + k_2 \bar{\lambda}\big(-R_1 - R_1^T\big) + k_2^2 \bar{\lambda}\big(R_1 R_1^T\big) \\ &\le 1 + k_2 m_3 + k_2^2 m_2 \end{aligned} \tag{28}$$

The relations (21), (27) and (28) altogether lead to the inequality (26). ■

Remark 3: Similar to the previous case, it can be shown that the constraints of Problem 2 are always feasible (by considering $k_1 = 2$, $k_2 = 0$ and choosing a sufficiently large value for k_3). It is to be noted that m_1 is positive in light of Assumption 3.

Remark 4: Since the statement of Problem 2 is attained by reducing the matrix constraint in Problem 1 to some scalar constraints, the upper bound proposed for μ in Theorem 2 is more conservative than the one given in Theorem 1.

To solve any of the two problems introduced in this paper, the Lyapunov matrix P_h needs to be obtained first. As a consequence of Assumption 1, this matrix cannot be computed using the conventional methods. However, since the matrix A_h which is required for obtaining P_h is assumed to be lower-block triangular, it can be found by solving a number of Lyapunov and Sylvester equations of subsystems' orders (as opposed to the system's order), successively. To clarify this issue, assume that there are only two subsystems (i.e. $\nu = 2$). In this case, the equation $A_h^T P_h A_h - P_h + I = 0$ can be equivalently decomposed as:

$$A_{22}^T P_3 A_{22} - P_3 + I = 0 \quad (29a)$$
$$A_{11}^T P_2 A_{22} + A_{21}^T P_3 A_{22} - P_2 = 0 \quad (29b)$$
$$A_{11}^T P_1 A_{11} + A_{11}^T P_2 A_{21} + A_{21}^T P_2 A_{11} + A_{21}^T P_3 A_{21} - P_1 + I = 0 \quad (29c)$$

where:

$$P_h = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \quad (30)$$

Since the underlying hierarchical closed-loop system is stable, the matrices A_{11} and A_{22} are both Schur. At this point, the Lyapunov equation (29a), which is of subsystem's order, can be solved to find the matrix P_3. Substitution of P_3 in the equation (29b) will yield a Sylvester equation, which has a unique solution P_2 (because the eigenvalues of A_{11} and A_{22} are all inside the unit circle). Finally, the Lyapunov equation (29c) can be solved for the matrix variable P_1 after substituting P_2 and P_3 obtained above into this equation. This illustrates that due to the special structure of A_h, the condition of Assumption 1 is not essential. It is worth mentioning that in the general case (a system with any arbitrary number of subsystems), the corresponding Lyapunov and Sylvester equations can be systematically obtained, similarly to the equations given in (29).

Theorem 3: Denote the optimal values of the variables k_1, k_2, k_3 in Problem 2 with k_1^*, k_2^*, k_3^*. The triple (k_1^*, k_2^*, k_3^*) satisfies either the set of equations:

$$k_1^* = 1 \quad (31a)$$
$$I - k_2^* R_1 = 0 \quad (31b)$$
$$k_3^* = \frac{1}{m_1} \quad (31c)$$

or the following ones:

$$(4m_2^2 - 4m_1m_2)(k_2^*)^2 + (4m_2m_3 - 4m_1m_3)k_2^* + (m_3^2 - 4m_1) = 0 \quad (32a)$$
$$k_3^* = \frac{-2m_2k_2^* + m_3}{2m_1} + \frac{1}{m_1} \quad (32b)$$
$$k_1^* = \frac{m_2(k_2^*)^2 + m_3k_2^* + 1}{k_3^*m_1 - 1} + 1 \quad (32c)$$

Proof: Since the objective function $k_1 + 2k_2 + k_3$ has no minimum point (due to the inequality constraints), the solution of Problem 2 will occur at some point on the boundary of the region defined by the corresponding constraints. Thus, there are two possibilities as follows:

- Case 1: *The equation $k_1^* - 1 = 0$ holds:* In this case, the optimization problem reduces to finding the infimum of $1 + 2k_2 + k_3$ under the constraint $-1 - k_2^2m_2 - k_2m_3 > 0$. Since the resultant objective function has no local minimum as noted above, the optimal solution occurs at some point on the remaining boundary, i.e. $-1-(k_2^*)^2m_2-k_2^*m_3 = 0$. On the other hand, it follows from (28) that:
$$0 \le \bar{\lambda}\big((I - k_2^*R_1)\,(I - k_2^*R_1^T)\big) \le 1 + k_2^*m_3 + (k_2^*)^2m_2 \quad (33)$$
The above relation along with the equation $-1 - (k_2^*)^2m_2 - k_2^*m_3 = 0$ signifies that the matrix $I - k_2^*R_1$ is equal to zero. Taking this result into account, it can be concluded from the constraint of Problem 2 and the equation $k_1^* = 1^+$ that $k_3^*m_1 - 1$ is nonnegative. This implies that in order for the objective function to be minimized, k_3^* should be chosen as $\frac{1}{m_1}$. The relations obtained above satisfy the set of equations given in (31).
- Case 2: *The equation $(k_1^* - 1)(-1 + k_3^*m_1) - 1 - (k_2^*)^2m_2 - k_2^*m_3 = 0$ holds:* It can be inferred in this case, that:
$$k_1^* = \frac{m_2(k_2^*)^2 + m_3k_2^* + 1}{k_3^*m_1 - 1} + 1 \quad (34)$$
If $k_1^* - 1$ is equal to zero, this case turns out to be the same as Case 1. Hence, with no loss of generality, assume that $k_1^* - 1$ is strictly positive. This yields that solving Problem 2 is equivalent to finding the lowest minimum point of the function:
$$\frac{m_2k_2^2 + m_3k_2 + 1}{k_3m_1 - 1} + 1 + 2k_2 + k_3 \quad (35)$$
for which k_1^* obtained in (34) is greater than or equal to 1. Taking the gradient of the function in (35) and equating it to zero will lead to the equations:
$$\frac{m_2(k_2^*)^2 + m_3k_2^* + 1}{(k_3^*m_1 - 1)^2} \times (-m_1) + 1 = 0 \quad (36a)$$
$$2 + \frac{2m_2k_2^* + m_3}{m_1k_3^* - 1} = 0 \quad (36b)$$
One can combine these two equations to arrive at the relation (32a). The proof follows from the fact that the equations (36b) and (34) are identical to (32b) and (32c), respectively. ■

Theorem 3 presents a solution to Problem 2 which, according to Theorem 2, provides a value for μ (i.e., the suboptimality degree of controller K). Regarding the set of equations (32) in this theorem, one should note that the quadratic equation (32a) needs to be solved first; the result obtained should then be substituted into the equations (32b) and (32c) to find all other parameters.

The question arises as to how conservative the values of μ obtained in Theorems 1 and 2 are. To answer this question, an elegant result on the tightness of this bound will be presented next.

Theorem 4: In the case when A_h and A_c are identical, Theorems 1 and 2 both arrive at the exact solution $\mu = 0$.

Proof: For the case when $A_h = A_c$, it can be easily verified that $R_1 = R_2 = -I$; consequently, $m_1 = m_2 = 1$ and $m_3 = 2$. Now, Theorem 2 states (after some simplifications) that μ is equal to the infimum of $k_1 + 2k_2 + k_3$ under the inequality constraints $k_1 > 1$ and:

$$k_1k_3 - k_1 - k_3 - 2k_2 - k_2^2 > 0 \quad (37)$$

The latter inequality is equivalent to:

$$(k_1 - 1)(k_3 - 1) \geq (k_2 + 1)^2 \tag{38}$$

Hence, μ is equal to 0, and is attained when $k_1 = k_3 \to 1^+$ and $k_2 = -1$. ■

Remark 5: It can be inferred from Theorem 4 and the continuity, that if A_h is sufficiently close to A_c, then the upper bounds proposed in this paper will be close to zero. As can be noticed from the proof of this theorem, the result is not trivial at all. In other words, it is not straightforward to conclude from Theorems 1 and 2 that if $A_h = A_c$, then the corresponding upper bounds will be equal to the exact value, i.e. $\mu = 0$.

IV. Numerical Examples

Example 1: Consider an interconnected system $\mathcal{S}$ with nine SISO subsystems of order 1, and assume that the interconnections from subsystem i to subsystem j, $\forall i, j \in \{1, 2, ..., 9\}$, $i < j$, are in general "weaker" than the ones in the opposite direction. Hence, to design a decentralized controller for the system with nine local controllers, one can eliminate these weak interconnections and design a decentralized controller for the resultant hierarchical model using any arbitrary method. For simplicity, assume that a static decentralized controller has been designed for the hierarchical model. For the performance analysis of the pseudo-hierarchical system under the designed controller, two different choices will be considered for the closed-loop matrix A_{nh} in the sequel.

Consider first a matrix A_{nh} of the following form:

$$\frac{1}{4.35}\begin{bmatrix} 1 & 0.5 & 2 & 0.1 & 0.5 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 1 & 1.5 & 0.5 & 1 & 0 & 1 & 0.2 & 0.25 \\ 1 & 0.3 & 1 & 0 & 0.2 & 1 & 0.2 & 0.5 & 0.31 \\ 0 & 0 & 0.3 & 1 & 3 & 1 & 0.05 & 0.1 & 0.01 \\ 0.3 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0.8 & 0 & 1 \\ 0 & 0 & .04 & 0.5 & 0.6 & 0 & 0.5 & 1 & 1 \\ 0.01 & 0 & 0 & 0.1 & 0.1 & 0 & 0.5 & 1 & 2 \\ 0.4 & 0.9 & 0.04 & 0.03 & 0 & 0.3 & 0.05 & 1 & 0 \end{bmatrix}$$

It can be observed that the lower-diagonal entries of this matrix have smaller magnitudes compared to the upper-diagonal ones in general (which introduce "weak" interconnections in the digraph of the system). The hierarchical matrix (A_h) obtained by neglecting the lower-diagonal entries of the above matrix is given by:

$$\frac{1}{4.35}\begin{bmatrix} 1 & 0.5 & 2 & 0.1 & 0.5 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 1 & 1.5 & 0.5 & 1 & 0 & 1 & 0.2 & 0.25 \\ 0 & 0 & 1 & 0 & 0.2 & 1 & 0.2 & 0.5 & 0.31 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0.05 & 0.1 & 0.01 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It can be verified that J_d and J_h for this example are equal to 67.9336 and 38.6145, respectively, resulting in $\frac{J_d}{J_h} = 1.7593$. On the other hand, an upper bound on $\frac{J_d}{J_h}$ can be attained from Theorem 1 by solving Problem 1, which leads to:

$$k_1 = 2.131, \quad k_2 = -1.7529, \quad k_3 = 5.7894$$

Due to the relation $\mu = \min(k_1 + 2k_2 + k_3)$, the upper bound μ on the ratio $\frac{J_d}{J_h}$ is equal to 4.4145. Note that although the obtained upper bound is approximately 2.5 times greater than the exact value, it has been attained through a quite simple procedure, which is very desirable for large-scale systems. This relatively large difference between $\frac{J_d}{J_h}$ and the corresponding upper bound μ is due to the fact that the neglected interconnections are not "weak" enough to be ignored. For instance, there are some large lower-diagonal entries (such as $\frac{0.9}{4.35}$), which are comparable and greater than some of the upper-diagonal entries.

Since Problem 1 involves matrix variables, its handling may be formidable for typical large-scale systems. Thus, let Theorems 2 and 3 be utilized here to obtain an upper bound μ. In this case, the variables k_1, k_2 and k_3 are obtained to be equal to 1.5206, -0.9144 and 5.2813, respectively, which correspond to the upper bound limit $\mu = 4.973$. Even though the matrix constraint in Problem 1 may seem to be oversimplified when introducing Problem 2, the results obtained for this example show that this is not necessarily the case. In other words, the bound $\mu = 4.973$ obtained by using Theorem 3 is relatively close to the bound $\mu = 4.4145$ resulted from Theorem 1.

Example 2: Consider the previous example, but with a new matrix A_{nh} given below:"

$$\frac{1}{4.35}\begin{bmatrix} 1 & 0.5 & 2 & 0.1 & 0.5 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 1 & 1.5 & 0.5 & 1 & 0 & 1 & 0.2 & 0.25 \\ 0 & 0 & 1 & 0 & 0.2 & 1 & 0.2 & 0.5 & 0.31 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0.05 & 0.1 & 0.01 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0.2 \\ 0.05 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0.001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case, the "weak" interconnections of the previous system have been further weakened in order for the pseudo-hierarchical system to become closer to its reference hierarchical model. Notice that the reference hierarchical model of this system is still given by the matrix A_h presented for the previous case. It can be verified that J_d and J_h in this case are equal to 0.0117 and 38.6145, respectively. The sizeable drop in the magnitude of J_d compared to the previous case manifestly confirms that the closeness of the pseudo-hierarchical and the corresponding reference hierarchical models has a significant impact on the accuracy of the proposed performance evaluation. The upper bound limit μ obtained by using Theorems 1 and 2 are equal to 0.0041 and 0.0671, respectively. These results are in accordance with the statement of Theorem 4.

V. Conclusions

This paper deals with the performance analysis of large-scale systems with pseudo-hierarchical structures; i.e., those systems that are "close" to being hierarchical, due to certain weak interconnections between their subsystems. It is assumed that a stabilizing decentralized controller is available for the system, which is basically designed to achieve some control objectives for the reference hierarchical model; i.e., a hierarchical model which is obtained by eliminating some weak interconnections in the original system. Since this indirect design technique may not result in a good performance for the original pseudo-hierarchical system, this work aims to quantitatively measure the closeness of the system and the corresponding reference hierarchical model under the designed controller. For this purpose, a LQ cost function is properly defined to measure the discrepancy between the original pseudo-hierarchical system and its hierarchical counterpart under the designed controller. Since computing the exact value of this cost function involves solving a large-scale Lyapunov equation, it is desired instead to obtain an upper bound on it. To this end, a simple LMI optimization problem with only three variables is proposed here to attain said upper bound. To further simplify the procedure of obtaining a proper bound, the matrix optimization problem is reduced to a scalar optimization one for which an explicit solution is obtained. In addition, it is shown that the closer the pseudo-hierarchical system to the reference hierarchical model is, the smaller these bounds are, and when the models are identical, these bounds are both equal to zero. This demonstrates that the bounds obtained through the proposed simple optimization problems are not too conservative. The ideas developed here are illustrated in two numerical examples.

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