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Harmonic Potential Field Path Planning for High Speed Vehicles

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Abstract—**This paper presents a method to represent complex shaped obstacles in harmonic potential fields used for vehicle path planning. The proposed method involves calculating the potential field for a series of circular obstacles inserted into the unobstructed potential field. The potential field for the total obstacle is a weighted average of the circular obstacle potential fields. This method explicitly calculates a stream function for the potential field. The need for the stream function is explained for situations involving controlling a dynamic system such as a high speed ground vehicle. The traditional potential field controller is also augmented to take the stream function into account. Simulation results are presented to show the effectiveness of the potential field generation technique and the augmented vehicle controller.**

I. Introduction

POTENTIAL fields are one of the more common techniques to plan paths to a goal while avoiding obstacles. The idea was first suggested by Khatib who proposed controlling robotic manipulators and mobile robots using a goal with an attractive potential and obstacles with repulsive potentials [1]. The control effort was the gradient of the combination of these two potentials. A problem with Khatib's potential was the robot being attracted to local minima instead of the goal. A solution to this problem is to utilize potential fields that are solutions to the Laplace equation (harmonic functions) [2]–[4]; these functions do not have local minima. Within the realm of harmonic potential fields, the effect of the obstacle boundary conditions on the field has been studied [5]. Potential field control has also been applied to high speed ground vehicles [6]–[8], although none of these methods utilize potential fields in the traditional goal/obstacle configuration.

One area of potential field control that has not received much attention is arbitrarily shaped obstacles in analytically (as opposed to numerically) generated fields. This paper discusses the need for analytic potential fields when dealing with high speed vehicles and examines possibilities for using multiple circular obstacles to create obstacles of arbitrary shape. The next section describes methods of generating potential field solutions to the Laplace equation. Section III investigates the potential field behavior when multiple obstacles interfere with each other. Section IV presents an augmented potential field controller which allows for a dynamic response in the vehicle. Finally, conclusions are drawn and future directions for the research are proposed.

II. Potential Field Development

One technique to develop a potential field for use in vehicle control is to solve the Laplace equation. Over an East, North (E, N) plane the Laplace equation is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial E^2} + \frac{\partial^2\phi}{\partial N^2} = 0\ . \tag{1}$$

Solutions to this equation are known as harmonic functions. For potential fields, the solution, ϕ, is known as the velocity potential because the gradient of the potential is the velocity vector field (u, v).

$$u = \frac{\partial\phi}{\partial E} = \frac{\partial\theta}{\partial N} \quad ; \quad v = \frac{\partial\phi}{\partial N} = -\frac{\partial\theta}{\partial E} \tag{2}$$

Another function that is useful to examine is the stream function, θ. The contours of the stream function are known as streamlines and represent the trajectories particles following the velocity vector field would take. The stream function associated with a given velocity potential also satisfies the Laplace equation; the two are represented together as a complex potential [9]

$$F = \phi + i\theta \tag{3}$$

along with a complex position

$$z = E + iN\ . \tag{4}$$

As (2) shows, the velocity vector field can also be computed from the stream function.

Most potential field control deals only with the velocity potential, specifically the velocity defined by its gradient. As long as the system is able to exactly follow the velocity vector field, the path followed will be a streamline. However, as is discussed below, for systems with a dynamic response (i.e. ones that do not respond instantaneously to inputs) it is useful to also explicitly consider the stream function and account for the fact that exact tracking of the potential gradient is not always possible.

The Laplace equation can be solved in one of two ways, numerically or analytically. Numeric solutions rely on setting boundary conditions on the velocity potential at the start and goal locations, obstacle boundaries, and solution perimeter. Typically the potential would be low at the start location and high at the goal location, so that the gradient of the field points toward the goal. Obstacle representation is discussed below. The solution perimeter is treated as an obstacle so that the vehicle does not leave the solved

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potential. Once the velocity potential is known, the streamlines can be determined by choosing starting locations and numerically integrating the velocity vector. An explicit stream function is never calculated. The main benefit of this method is that any shape obstacle can be easily defined. However, its disadvantages are there is no explicit stream function and the time required to numerically solve the Laplace equation can be prohibitively long.

Analytic solutions involve summing harmonic primitives representing the start, goal, and obstacles; analytic solutions are typically global so no solution perimeter is present. The superposition principle applies to solutions of the Laplace equation so that as long as the primitives are harmonic, the overall potential field will also be harmonic. Analytic start locations are normally represented by a source

$$F = C \ln z \tag{5}$$

and goals by a sink (source of negative strength, C). Note that the source equation is complex and thus contains information about both the velocity potential and the stream function. This is true for each of the primitives; they explicitly define the stream function. The primitives for obstacles are described in the next section. The main benefits of analytic solutions are the explicit stream function and the fast solution for the potential field (superposition of relatively simple primitive functions). The shortcoming is the limited numbers of primitives available to represent obstacles.

Regardless of the solution method, there are two types of boundary conditions to represent obstacles, Dirichlet and Neumann. Dirichlet boundary conditions create electrostatic potential fields: obstacles repel the robot directly away from them. This condition closely resembles the potential field Khatib proposed and is typically the representation used when controlling robots. With numeric solutions the Dirichlet boundary condition involves setting the velocity potential of obstacles to a constant value higher than the surrounding potential. For analytic solutions an obstacle is represented by one or more sources (the gradient of the source points radially outward).

Neumann boundary conditions represent fluid flow: the velocity vector cannot penetrate the obstacle, but can flow along it. The paths created by Neumann boundary conditions are typically smoother than those with Dirichlet boundary conditions because the flow simply goes around the obstacle instead of directly away from it [4]. To represent a Neumann boundary condition numerically the velocity potential at each point on the obstacle boundary is set to the potential directly outward from the obstacle boundary (the gradient is thus along the obstacle boundary). This boundary condition greatly increases the solution time. The method to represent circular obstacles with Neumann boundary conditions is described in the next section.

For high speed vehicles, an analytic solution is required. As Section IV demonstrates, the stream function is a necessary component of the controller. Therefore, a technique for representing arbitrarily shaped obstacles (the main disadvantage of the analytic technique) is developed in the following section.

III. Analytic Obstacle Representation

To represent a single circular obstacle in a generic potential flow, the circle theorem is employed [4], [9]. This theorem takes the base, undisturbed flow, F_u, and modifies it by "mirroring" it about the boundary of the circle [10]. The total potential is the sum of the unmodified and mirrored potentials.

$$F = F_u(z) + \bar{F}_u\left(\frac{a^2}{z}\right) \tag{6}$$

where a is the circle radius and the conjugate analytic function is defined as

$$\bar{F}(z) = \overline{F(\bar{z})}. \tag{7}$$

Note that on the circle boundary the stream function is identically zero indicating that the boundary is a stream line.

$$\begin{aligned} F\Big|_{|z|=a} &= F_u(z) + \bar{F}_u(\bar{z}) = F_u(z) + \overline{F_u(z)} = 2\phi \\ &\Rightarrow \theta = 0 \end{aligned} \tag{8}$$

The boundary of the obstacle being a streamline implies that the velocity field flows around the obstacle without penetrating it. Inside the obstacle a feature known as a doublet is formed.

The overall concept for representing arbitrarily shaped obstacles is to utilize a combination of circular obstacles to form the desired shape. To this end, different methods of adding multiple obstacles are examined; specifically, the behavior when circular obstacles touch is studied. Ideally, the streamlines will pass around the combined obstacle thus providing the basis for building complex shaped obstacles.

A. Unmodified Obstacle Addition

The first method for combining obstacles is to simply apply the Circle Theorem recursively. The first circle is added to the undisturbed potential. The "undisturbed" potential for the second circle is the resulting potential after the first circle is added, and so on. It is apparent for this method that the order the circles are added is critical. The edge of the last circle added is guaranteed to be a streamline. However, the streamlines near any earlier added circles will be distorted by the addition of later circles. This distortion is apparent in Fig. 1.

The original potential in this study represents uniform flow from South to North. Although this is not a typical path planning potential, it does effectively illustrate the flow behavior from the obstacle distortions. The left obstacle is added first and the right second. Fig. 1(a) shows the obstacles moderately spaced (1/2 diameter between them). The gray represents the intended obstacle while the black lines are the actual streamlines. At this spacing it is apparent that the first obstacle is becoming distorted.

Each circle has a streamline that connects to it at two points. These points are known as stagnation or saddle points. The streamline connecting the points defines the actual shape of the obstacle. Note that the obstacle formed by these streamlines does not match the first intended obstacle. This is more apparent in Fig. 1(b) with the touching obstacles; the first obstacle is distorted so that streamlines still pass between the obstacles. In both cases note that the second obstacle is matched exactly by a streamline as predicted by (8). The partial streamlines inside the obstacle show the doublet. Near the center they become dense enough to be indistinguishable and are therefore not plotted. For path planning, streamlines inside obstacles have no effect. They are simply plotted in order to better see how the streamlines are distorted.

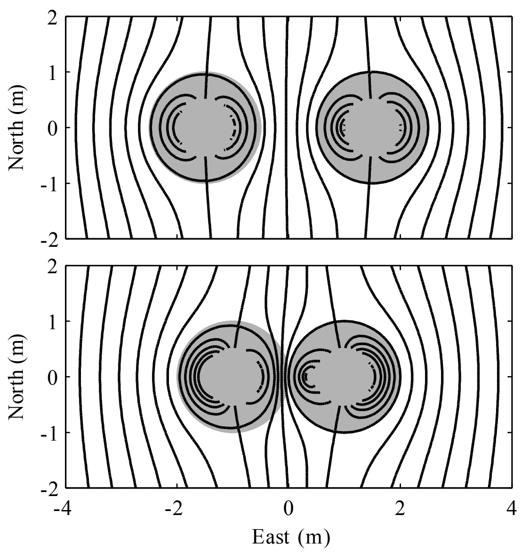

Fig. 1. Obstacles added via the circle theorem; (a) shows moderately spaced circles and (b) shows touching circles.

Clearly this method is not satisfactory to represent a combined obstacle. Any streamline between the stagnation points of the two obstacles will flow between the obstacles even if the intended obstacles are touching.

B. Averaging Velocity

Waydo and Murray suggested a method to remove the distortions created by closely spaced (but not touching) obstacles [4]. Their method consists of first calculating the potential created by placing each obstacle in the original, undisturbed potential (as opposed to placing obstacles in the potential resulting from the previously placed obstacle). The total velocity vector field is a weighted average of the individual obstacle velocity vector fields. The weight for the individual velocities, w_i, depends on the distance to each of the obstacles.

$$w_i = \prod_{j \neq i} \frac{d_j}{d_i + d_j} \tag{9}$$

where d_i is the distance to the obstacle under consideration and d_j the distances to the other obstacles. This weight varies between zero (at another obstacle's boundary) and one (at the obstacle under consideration's boundary). At an obstacle's boundary its weight will be one and all others will be zero. Therefore, the velocity vector at the obstacle edge matches the unmodified flow meaning the obstacle edge is still a streamline. This is true of every obstacle as opposed to only the last obstacle added. It should be noted that because of the weighting factor, the solution is no longer harmonic. However, Waydo and Murray prove that there are still no local minima in the resulting flow.

The work in this paper modifies the weighting function slightly to allow for the case when two obstacles touch. In this case, the weighting for each of the touching obstacles becomes the inverse of the number of obstacles touching (two, unless obstacles overlap).

One downside of this method is that it does not calculate a stream function analytically. As with the numeric solution from Section II, the streamlines must be numerically integrated from the velocity vectors.

Fig. 2 shows two obstacles in vertical flow for both the moderately spaced and touching cases. No streamlines are visible inside the obstacles because the streamlines are generated by integrating the velocity from a sequence of initial locations; obviously none of these traces penetrate an obstacle to make the doublet visible. Note that for both the left and right obstacle the intended obstacle edge is a streamline. The slight deviations near the stagnation points are due to numerical integration errors.

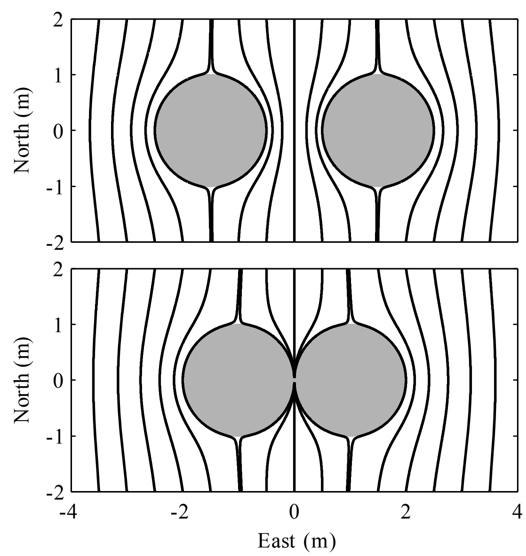

Fig. 2. Obstacles added via averaging the individual circle's velocities; (a) shows moderately spaced circles and (b) shows touching circles.

Even though streamlines do not penetrate the intended obstacles, they do pass between the obstacles even when the obstacles are touching, as seen in Fig. 2(b). As with the previous method, any streamline between the stagnation points will pass between the obstacles.

As Fig. 2(a) shows, this method works well to preserve the exact obstacle edge, but it breaks down when the obstacles touch. Additionally, even though the individual potential functions are computed analytically, the resulting streamlines are generated using numerical integration.

Overall, the method does not create an effective combined obstacle.

C. Averaging Stream Function

The requirement for an explicit stream function motivates the following modification to the averaging method: average the individual stream functions instead of the velocities. The same weighting function, i.e. (9), is used for this purpose. Fig. 3 shows the resulting potential when the stream functions are averaged.

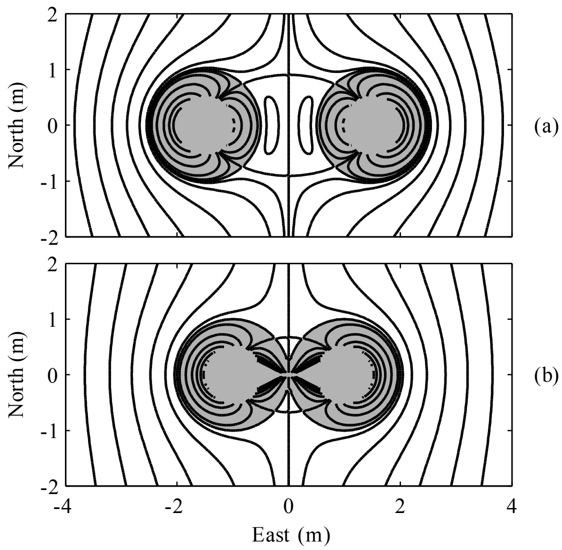

Fig. 3. Obstacles added via averaging the individual circle's stream functions; (a) shows moderately spaced circles and (b) shows touching circles.

The potential in this case is very different from the previous two methods. The obstacle edges are still streamlines as would be expected given that the resultant potential on an obstacle edge is identical to the potential of just that obstacle. However, unlike the other methods the obstacle edge streamline extends to connect the two circles creating a combined obstacle that the streamlines flow around even when the obstacles are not touching. Therefore, no paths pass between the two circles when using a streamline to define the path. The connecting streamlines are due to the averaging. From (8) the stream function on each circle is zero. The connecting streamlines are the contours where the weighted average of the two potentials cancels leaving a net stream function of zero.

The distorted streamlines near where the obstacles touch in Fig. 3(b) is due to a singularity in the weighting function that occurs when obstacles touch. Note that this singularity only exists in areas internal to the circles and that the weight at the point where the circles touch is still defined. When using this method for path generation, the area interior to the obstacles is not considered so the singularity has no effect on the potential field.

There are other interesting features of this potential. First, two additional stagnation points exist on the connecting streamlines. The stagnation points do not cause concern. They are equilibrium points, but are unstable (saddle points). They form the new combined obstacle's edge; just as with the stagnation points in Fig. 1. Additionally, two "vortices" exist inside the combined obstacle (the closed contour streamlines between the circles). These vortices represent local extrema in the potential field (center equilibrium points). This reinforces the conclusion that weighted averages of harmonic functions are not necessarily harmonic. Because the extrema are contained within the combined obstacle they do not affect the useful portion of the potential function. However, there is no guarantee that the extrema will always occur in non-important portions of the potential.

Overall, this method allows circular obstacles to be combined to create complex shaped obstacles. It also explicitly calculates the stream function. As the next section demonstrates, more than two circles can be combined to form the shape although to avoid problems with unwanted extrema it is better to keep the number of circles low and when possible, make the circles touch.

IV. Controller Development

To validate the potential field generation method described above a representative field is created and a vehicle simulation is employed to evaluate the use of the potential field for controlling a vehicle. To simulate the vehicle, the well known bicycle model (Fig. 4) is used [11]. The vehicle states (slip angle, β, and yaw rate, r) are supplemented by kinematic relationships for heading, ψ, and lateral position error y_{err}. These four states are used in the control effort. North and East states are also included to complete the vehicle simulation. The equations of motion for this system are

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -\dfrac{C_0}{mV} & -\dfrac{C_1}{mV^2}-1 \\ -\dfrac{C_1}{I_z} & -\dfrac{C_2}{I_z V} \end{bmatrix} \begin{bmatrix} \beta \\ r \end{bmatrix} + \begin{bmatrix} \dfrac{C_{\alpha F}}{mV} \\ \dfrac{aC_{\alpha F}}{I_z} \end{bmatrix} \delta. \tag{10}$$

$$\begin{aligned} \dot{\psi} &= r \\ \dot{y}_{err} &= V\psi_{err} \\ \dot{N} &= V\cos(\psi+\beta) \\ \dot{E} &= V\sin(\psi+\beta) \end{aligned} \qquad \begin{aligned} C_0 &= C_{\alpha F} + C_{\alpha R} \\ C_1 &= aC_{\alpha F} - bC_{\alpha R} \\ C_2 &= a^2 C_{\alpha F} + b^2 C_{\alpha R} \end{aligned}$$

δ is the steering angle input, V the vehicle speed (held constant in the simulation), m the vehicle mass, I_z the moment of inertia, a and b the distances from the center of gravity to the front and rear axles respectively, and $C_{\alpha F}$ and $C_{\alpha R}$ the front and rear axle cornering stiffnesses.

The controller used is either

$$\delta = k_\psi \psi_{err} \tag{11}$$

or

$$\delta = k_\psi \psi_{err} + k_y y_{err} \tag{12}$$

depending on whether only the gradient of the potential field is tracked or a streamline is also followed. These are not necessarily the ideal controllers; however, they serve to illustrate the need for following a particular streamline instead of the potential field gradient alone.

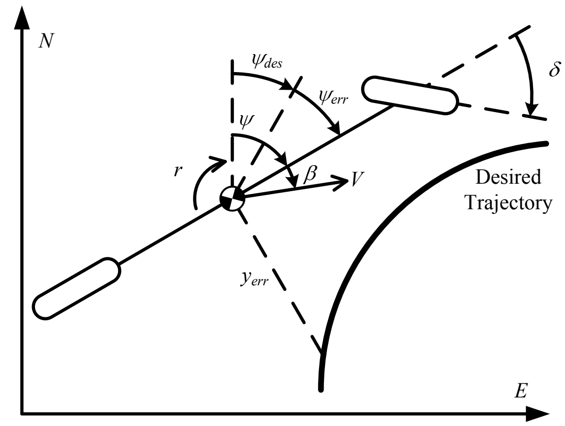

Fig. 4. Bicycle model and relationship to desired trajectory.

A. Traditional Potential Field Controller

To illustrate the different control methods, a map (Fig. 5) is created. The map consists of a start location (source) in the southeast corner and a goal location (sink) in the northwest corner. Three obstacles exist, a wall, square, and circular obstacle. The circular obstacle is created by using the Circle Theorem. The square obstacle consists of four circles connected by averaging the stream functions. The wall obstacle is formed by averaging two small circles, one at each end of the line. Note that obstacles consisting of multiple circles (e.g. the wall and square) use the averaging method from Section III.C to connect the circles and generate the desired shape. However, independent obstacles are inserted into the potential without averaging them to other obstacles. This allows streamlines (and thus vehicle paths) to pass between unrelated obstacles but not between circles connected to form a combined obstacle.

The traditional method of using a potential field to control a robot is to move the robot in the direction indicated by the gradient of the velocity potential at the robot's location. Typically, robots controlled via this method can change direction effectively instantaneously and immediately move in the direction of the gradient. If the gradient is tracked exactly, then the path followed by the robot will be a streamline. The equivalent vehicle controller takes the direction of the potential gradient as the desired heading and bases the steer angle on the heading error.

However, as the equations of motion described before indicate a vehicle cannot instantaneously change its heading; there is a lag between the steer angle applied and the heading change. This lag means that the vehicle will not exactly follow a streamline. Usually this is not a problem since the gradient of the potential always points to the goal location. Therefore, even if the vehicle leaves its original streamline it will still reach the goal. However, if an obstacle is present, the lag in the vehicle response may cause the vehicle to intersect the obstacle. The vehicle cannot turn to avoid the obstacle as tightly as the potential gradient turns due to the vehicle dynamic response.

Fig. 5 illustrates this concept. Initially, the vehicle is closely following a streamline that avoids the square obstacle. This streamline is followed purely based on the vehicle's initial heading. Approximately half way to the obstacle the vehicle leaves the streamline. Because that streamline has no particular significance to the controller, nothing pulls the vehicle back to the streamline and away from the obstacle. By the time the potential gradient points far enough away from the obstacle to significantly alter the vehicle's trajectory, it is too late for the vehicle to miss the obstacle.

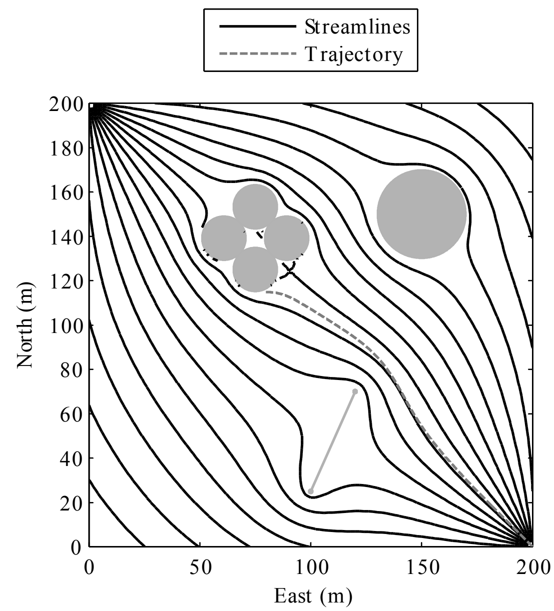

Fig. 5. Traditional potential field controller; vehicle is driving from southeast to northwest.

B. Augmented Potential Field Controller

To guarantee that the trajectory remains a safe distance from all obstacles a particular streamline needs to be followed. The vehicle controller in this case still uses the potential field gradient at the vehicle's location as a desired heading but it is augmented to include the lateral distance to the desired streamline. This lateral error is calculated by performing a search of the stream function along the lateral axis of the vehicle to find the desired stream function value (recall that a streamline is a contour of the stream function at a particular value). For this search to be performed the stream function must be explicitly defined. Therefore, numerically integrating the velocity vector to generate streamline traces is not sufficient.

Fig. 6 shows the trajectory followed by using the augmented potential field controller. The desired streamline is chosen manually to safely avoid the obstacles as well as comply with the dynamic constraints (e.g. maximum lateral acceleration) of the vehicle. Note that following a specific streamline also allows the desired path to be independent of the vehicle's initial heading. For instance, if it were more advantageous to move to the south of the wall to avoid it, one of those streamlines could be chosen even though the vehicle is initially pointing between the obstacles. This scenario is not possible using traditional potential field control.

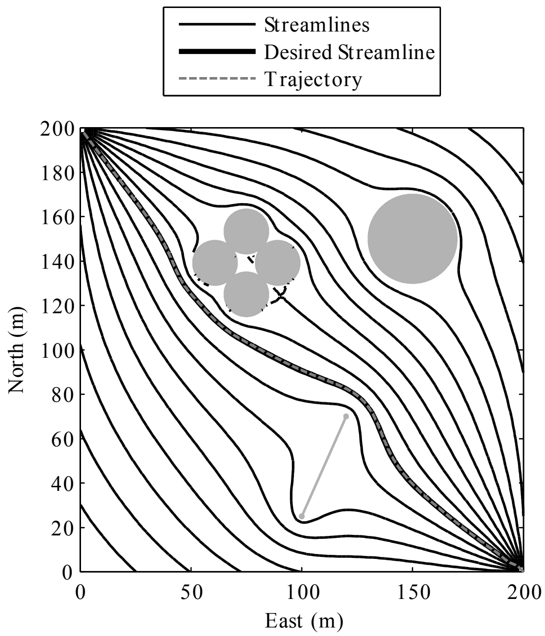

Fig. 6. Augmented potential field controller; vehicle is driving from southeast to northwest.

V. Conclusion

This paper has examined various analytical potential fields which flow around multiple circular obstacles as those obstacles begin to touch. The potential fields are initially developed by solving the Laplace equation with boundary conditions corresponding to obstacles and start/goal locations. Based on these studies a technique which allows complex shaped obstacles to be built from circles inserted into the potential field is developed. The streamlines of the potential field completely avoid the obstacles constructed with this method. An important feature of the method is that an explicit stream function is calculated. Unless a particular streamline is followed, vehicles that have a dynamic (i.e. non-instantaneous) response have the possibility of intersecting obstacles. The need for an explicit stream function also makes numerical solutions of the Laplace equation impractical.

The potential field development technique presented in the paper should lend itself to real-time implementation. The entire potential field does not have to be recomputed when a new obstacle is detected. Instead, the new obstacle is added into the existing potential field. The equations for adding an obstacle to a potential field are relatively simple analytic functions and should not require much computation time.

A. Future Work

The obstacle representation and simple controller presented in this paper fit into the larger objective of adapting harmonic potential field path planning techniques to apply to high speed vehicles (i.e. vehicles with dynamic limitations). There are several other research areas that build on the work in this paper to meet the larger objective. Among them are improving the streamline following controller to use the stream function error directly instead of searching for the lateral distance error; this will require a nonlinear controller. Additionally a method to automatically determine a streamline to follow while still respecting the vehicle's dynamic constraints and remaining a safe distance from all obstacles needs to be developed. Ideally this method would not require a global search of all streamlines (e.g. searching for minimum instantaneous radius). Finally, the control method should be expanded to determine a desired speed for the vehicle. The magnitude of the potential field gradient is not sufficient for the desired speed. Recall that the potential field is based on fluid dynamics concepts. As fluid flows around obstacles the streamlines bunch together. This implies the speed is increased. Obviously the vehicle speed should not increase when it is avoiding an obstacle. However, features that are present in the potential field (such as the distance to each obstacle used in the weighting function) should provide information that can be used to determine a desired speed.

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