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Infinite-Horizon Soft-Constrained Stochastic Nash Games with State-Dependent Noise in Weakly Coupled Large-Scale Systems

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Abstract— **In this paper, we discuss infinite-horizon soft-constrained stochastic Nash games involving state-dependent noise in weakly coupled large-scale systems. First, linear quadratic differential games are formulated in which robustness is attained against model uncertainty. In particular, conditions for the existence of robust equilibria have been derived from the solutions of the sets of cross-coupled stochastic algebraic Riccati equations (CSAREs) for the first time. After establishing an asymptotic structure along with positive semidefiniteness for CSARE solutions, we derive a new algorithm based on Lyapunov iterations for solving the CSAREs. Consequently, we show that the proposed algorithm attains linear convergence and reduced-order computations for a sufficiently small value of ε. Finally, numerical example is provided to verify the efficiency of the proposed algorithm.**

I. INTRODUCTION

The stochastic control problems governed by Itô's differential equation have become a popular research topic during the past decade. Particularly, a practical example of the flexible structure comprising a mass-spring system has been demonstrated [1]. It has been shown that the so-called 'Langevin equation' driven by Gaussian white noise has two different uncertainties involving deterministic and stochastic portions. Moreover, the stochastic H_∞ control problem with state- and control-dependent noise have been investigated. It has attracted much attention and has been widely applied in various fields. In particular, the stochastic H_2/H_∞ control with state-dependent noise has been addressed [3].

Linear quadratic Nash games and their applications have been widely investigated in many literatures (see e.g. [11]). Recently, robust equilibria in indefinite linear quadratic differential games under a deterministic disturbance input affecting the systems have been discussed [5], [6]. These results are based on the steady-state feedback saddle-point solution for soft-constrained Nash games [7]. Although the results in [5], [6] are very elegant in theory and it is easy to obtain a strategy pair by solving the cross-coupled algebraic Riccati equations, stochastic uncertainty has not been considered.

In this paper, we discuss a theoretical and a numerical aspect by extending the results of [5], [6] in the deterministic case to the soft-constrained stochastic Nash games governed by Itô's differential equations with state-dependent noise. It should be noted that earlier studies on weakly coupled stochastic Nash games [8] have not taken the state-dependent noise into consideration. Further, in [13], even the deterministic disturbance input has not been considered. On the other hand, although the stochastic H_2/H_∞ control has been considered, stochastic noise and unknown deterministic disturbance [3] involving multiple players have not been addressed. Hence, the concept of Nash equilibrium cannot be ascertained. The main contributions of this paper are as follows. First, linear quadratic differential games are investigated with respect to an infinite horizon. It should be noted that systems governed by Ito's differential equations are disturbed by deterministic noise and strategy spaces involve the linear feedback strategy with a memory-less perfect-state information structure [9]. After formulating the soft-constrained problem for the one-player case, a set of sufficient conditions is given as the saddle-point solution. Moreover, in order to guarantee the existence of strategy pairs, sets of cross-coupled stochastic algebraic Riccati equations (CSAREs) are introduced for the first time. As a result, these strategy pairs can be obtained by solving the CSAREs. Second, the soft-constrained stochastic Nash games for weakly coupled large-scale systems are investigated from the numerical viewpoint. Since the proposed numerical computation is based on the Lyapunov iterations, linear convergence is guaranteed for a sufficiently small parameter ε. Finally, in order to demonstrate the efficiency of the proposed algorithm, numerical example is included.

Notation: The notations used in this paper are fairly standard. Superscript T denotes the matrix transpose. $M = (m_{ij})$ denotes standard notation of a matrix (m_{ij} are the elements of M). I_n denotes an $n \times n$ identity matrix. $\mathbf{block\ diag}$ denotes a block diagonal matrix. $\|\cdot\|$ denotes the Euclidean norm of a matrix. E denotes the expectation. $\otimes$ denotes the Kronecker product. The space of the $\mathbf{R}^k$-valued functions that are quadratically integrable on $(0,\ \infty)$ are denoted by $L_2^k(0,\ \infty)$.

II. SOFT-CONSTRAINED STOCHASTIC NASH GAMES

Consider stochastic linear time-invariant weakly coupled large-scale systems.

$$dx(t) = [A_\varepsilon x(t) + B_{1\varepsilon} u_1(t) + B_{2\varepsilon} u_2(t) + E_\varepsilon v(t)]dt + A_{1\varepsilon} x(t) dw(t),\ x(0) = x^0, \tag{1}$$

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where

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ v(t) := \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix},$$

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{bmatrix},\ A_{1\varepsilon} := \begin{bmatrix} A_{111} & \varepsilon A_{112} \\ \varepsilon A_{121} & A_{122} \end{bmatrix},$$

$$B_{1\varepsilon} := \begin{bmatrix} B_{11} \\ \varepsilon B_{21} \end{bmatrix},\ B_{2\varepsilon} := \begin{bmatrix} \varepsilon B_{12} \\ B_{22} \end{bmatrix},\ E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} \\ \varepsilon E_{21} & E_{22} \end{bmatrix}.$$

$x_i(t) \in \mathbf{R}^{n_i},\ i = 1,\ 2$ represent the i-th state vectors. $u_i(t) \in \mathbf{R}^{m_i},\ i = 1,\ 2$ represent the i-th control inputs. $v_i(t) \in \mathbf{R}^{l_i},\ i = 1,\ 2$ represent the i-th disturbance. $w(t) \in \mathbf{R}$ is a one-dimensional standard Wiener process defined in the filtered probability space [1], [2], [3], [4]. Moreover, $v_i(t) \in L_2^{l_i}(0,\ \infty)$ is considered to be an unknown finite-energy stochastic disturbance that adversely affects the to-be-controlled output (whose desired value is represented by 0) [2]. Here, ε denotes a relatively small positive coupling parameter that relates the linear system with the other subsystems.

The cost function for each strategy subset is defined by

$$\begin{aligned} &J_i(u_1,\ u_2,\ v,\ x(0)) \\ &= E\int_0^\infty \Big[x^T(t)Q_{i\varepsilon}x(t) + u_i^T(t)R_{ii}u_i(t) \\ &\quad +\varepsilon u_j^T(t)R_{ij}u_j(t) - v^T(t)V_{i\varepsilon}v(t)\Big]dt, \end{aligned} \tag{2}$$

where $i,\ j = 1,\ 2,\ i \neq j,\ Q_{i\varepsilon} = Q_{i\varepsilon}^T \geq 0,$

$$Q_{1\varepsilon} = \begin{bmatrix} Q_{111} & \varepsilon Q_{112} \\ \varepsilon Q_{112}^T & \varepsilon Q_{122} \end{bmatrix},\ Q_{2\varepsilon} = \begin{bmatrix} \varepsilon Q_{211} & \varepsilon Q_{212} \\ \varepsilon Q_{212}^T & Q_{222} \end{bmatrix},$$

$R_{ii} = R_{ii}^T > 0 \in \mathbf{R}^{m_i \times m_i},\ R_{ij} = R_{ij}^T \geq 0 \in \mathbf{R}^{m_j \times m_j},$

$V_{1\varepsilon} = \textbf{block diag}\begin{pmatrix} V_{11} & \varepsilon^{-1}V_{12} \end{pmatrix} > 0,$

$V_{2\varepsilon} = \textbf{block diag}\begin{pmatrix} \varepsilon^{-1}V_{21} & V_{22} \end{pmatrix} > 0.$

Stabilizability-an essential assumption in this paper-has been introduced [3].

Definition 1: The stochastically controlled system governed by the Itô's equation $dx = (Fx + Gu)dt + G_1 x dw_1$, $x(0) = x_0$ is called stabilizable in the mean-square sense if there exists a feedback law $u = Kx$ such that for any x_0, the closed-loop system $dx = (F + GK)xdt + G_1 x dw_1$, $x(0) = x_0$ is asymptotically mean-square stable, i.e., $\lim_{t\to\infty} Ex^T(t)x(t) = 0$, where K is a constant matrix.

For the matrices A_ε, $B_{j\varepsilon}$, $j = 1,\ \dots,N$ and $A_{1\varepsilon}$, the set $\mathbf{F}_N$ is defined as $\mathbf{F}_N := \Big\{(F_{1\varepsilon},\ \dots,F_{N\varepsilon}) \mid$ The closed-loop system $dx(t) = [A_\varepsilon + \sum_{p=1}^N B_{p\varepsilon}F_{p\varepsilon}]x(t)dt + A_{1\varepsilon}x(t)dw(t)$ is asymptotically mean-square stable. $\Big\}$.

The soft-constrained stochastic Nash equilibrium strategy pair $(u_1^*,\ u_2^*)$, $u_i^*(t) := F_{i\varepsilon}^* x(t)$ is defined such that it satisfies the following conditions [5], [6].

$$\bar{J}_1(F_{1\varepsilon}^*x,\ F_{2\varepsilon}^*x,\ x(0)) \leq \bar{J}_1(F_{1\varepsilon}x,\ F_{2\varepsilon}^*x,\ x(0)), \tag{3a}$$
$$\bar{J}_2(F_{1\varepsilon}^*x,\ F_{2\varepsilon}^*x,\ x(0)) \leq \bar{J}_2(F_{1\varepsilon}^*x,\ F_{2\varepsilon}x,\ x(0)), \tag{3b}$$

where

$$\bar{J}_i(F_{1\varepsilon}x,\ F_{2\varepsilon}x,\ x(0)) := \sup_{v\in L_2^{\bar{k}}(0,\ \infty)} J_i(F_{1\varepsilon}x,\ F_{2\varepsilon}x,\ v,\ x(0)),$$

$$\begin{aligned} &J_i(F_{1\varepsilon}x,\ F_{2\varepsilon}x,\ v,\ x(0)) \\ &= E\int_0^\infty \Big[x^T(t)(Q_{i\varepsilon} + F_{i\varepsilon}^T R_{ii}F_{i\varepsilon} \\ &\quad +\varepsilon F_{j\varepsilon}^T R_{ij}F_{j\varepsilon})x(t) - v^T(t)V_{i\varepsilon}v(t)\Big]dt, \end{aligned}$$

for all $x(0)$ and for all $(F_{1\varepsilon},\ F_{2\varepsilon})$ that satisfy $(F_{1\varepsilon}^*,\ F_{2\varepsilon}) \in \mathcal{F}_2$, $(F_{1\varepsilon},\ F_{2\varepsilon}^*) \in \mathcal{F}_2$, and $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*) \in \mathcal{F}_2$.

It should be noted that in this study, the strategies u_i^* are restricted as the linear feedback strategies [9]. The weighting matrix $V_{i\mu}$ is always symmetric and positive definite for all $i = 1,\ 2$. Since J_i is negative, this matrix constrains the disturbance vector v in an indirect way; therefore, it can be used to describe the aversion to the model risk of player i [5]. In particular, if the quantity $v^T V_{i\mu} v$ is large for vector v, this means that player i does not expect large deviations in the nominal dynamics in the direction of $E_\varepsilon v$. In a previous result [5] and numerous existing results, this so-called soft-constrained formulation has been used.

A. ONE-PLAYER CASE

First, a one-player case is discussed. The result obtained for that particular case will be used as the basis for the derivation of the results for the general 2-player case.

Consider a linear time-invariant stochastic stabilizable system

$$\begin{aligned} dx(t) &= [A_\varepsilon x(t) + B_{1\varepsilon}u_1(t) + E_\varepsilon v(t)]dt \\ &\quad +A_{1\varepsilon}x(t)dw(t),\ x(0) = x^0, \end{aligned} \tag{4}$$

where $u_1(t,\ x) := F_{1\varepsilon}x(t)$, $F_{1\varepsilon} \in \mathbf{F}_1$. The cost function is given below.

$$\begin{aligned} &J(u_1,\ v,\ x(0)) \\ &= E\int_0^\infty \Big[x^T(t)Q_{1\varepsilon}x(t) + u_1^T(t)R_{11}u_1(t) \\ &\quad -v^T(t)V_{1\varepsilon}v(t)\Big]dt. \end{aligned} \tag{5}$$

Let us define the strategies spaces $\Gamma_u := \{u_1(t,\ x) := F_{1\varepsilon}x(t) \mid F_{1\varepsilon} \in \mathbf{F}_1\}$ and $\Gamma_v := \{v(t) \mid v(t) \in L^{\bar{l}}(0,\ \infty)\}$.

Definition 2: A strategy pair $(u_1^*,\ v^*) \in \Gamma_u \times \Gamma_v$ is in saddle-point equilibrium if

$$J(u_1^*,\ v,\ x(0)) \leq J(u_1^*,\ v^*,\ x(0)) \leq J(u_1,\ v^*,\ x(0)) \tag{6}$$

for all $(u_1^*,\ v) \in \Gamma_u \times \Gamma_v$ and $(u_1,\ v^*) \in \Gamma_u \times \Gamma_v$.

The following theorem generalizes the existing results of [5], [6], which is a very important result in deterministic soft-constrained Nash games, to a stochastic case.

Theorem 1: Assume that for any $u_1(t)$ and $v(t)$, the stochastic system is stabilizable. Suppose that the following stochastic algebraic Riccati equation (SARE) has the solution $P_\varepsilon \geq 0$.

$$\begin{aligned} &P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon + A_{1\varepsilon}^T P_\varepsilon A_{1\varepsilon} \\ &\quad -P_\varepsilon(S_{1\varepsilon} - M_{1\varepsilon})P_\varepsilon + Q_{1\varepsilon} = 0, \end{aligned} \tag{7}$$

where $S_{1\varepsilon} := B_{1\varepsilon} R_{11}^{-1} B_{1\varepsilon}^T$, $M_{1\varepsilon} := E_\varepsilon V_{1\varepsilon}^{-1} E_\varepsilon^T$.

The strategy pair

$$u_1^*(t,\ x) = F_{1\varepsilon}^* x(t) = -R_{11}^{-1} B_{1\varepsilon}^T P_\varepsilon x(t), \tag{8a}$$
$$v^*(t) = V_{1\varepsilon}^{-1} E_\varepsilon^T P_\varepsilon \tilde{x}(t), \tag{8b}$$
$$d\tilde{x}(t) = [A_\varepsilon - (S_{1\varepsilon} - M_{1\varepsilon})P_\varepsilon]\tilde{x}(t)dt + A_{1\varepsilon}\tilde{x}(t)dw(t),\ \tilde{x}(0) = x^0 \tag{8c}$$

is in saddle-point equilibrium if it is asymptotically mean-square stable and $F_{1\varepsilon}^* \in \mathbf{F}_1$. That is, inequality (6) related to the cost function $J(u_1,\ v,\ x(0))$ is satisfied. Moreover, $J(u_1^*,\ v^*,\ x(0)) = x^T(0)P_\varepsilon x(0)$.

Proof: Since we assume that for any $u_1(t)$ and $v(t)$, the stochastic system is stabilizable, there exists a feedback law $u = Kx$ such that $\lim_{t\to\infty} Ex^T(t)x(t) = 0$. Thus, by applying the Itô's formula to (4) and considering (8a), we get

$$\begin{aligned} &J(u_1,\ v,\ x(0)) \\ &= x^T(0)P_\varepsilon x(0) + E\int_0^\infty \Big[\|u_1(t) - u_1^*(t)\|_{R_{11}}^2 \\ &\quad -\|v(t) - V_{1\varepsilon}^{-1} E_\varepsilon^T P_\varepsilon x(t)\|_{V_{1\varepsilon}}^2\Big] dt. \end{aligned} \tag{9}$$

From this, it follows that

$$\begin{aligned} &J(u_1^*,\ v,\ x(0)) \\ &= x^T(0)P_\varepsilon x(0) - E\int_0^\infty \|v(t) - V_{1\varepsilon}^{-1}E_\varepsilon^T P_\varepsilon \tilde{x}(t)\|_{V_{1\varepsilon}}^2 dt \\ &\le x^T(0)P_\varepsilon x(0), \end{aligned} \tag{10}$$

where $\check{x}(t)$ is governed by

$$d\tilde{x}(t) = \Big[(A_\varepsilon - S_{1\varepsilon}P_\varepsilon)\tilde{x}(t) + E_\varepsilon v(t)\Big]dt + A_{1\varepsilon}\tilde{x}(t)dw(t),\ \tilde{x}(0) = x^0.$$

Furthermore, if $J(u_1^*,\ v,\ x(0)) = x^T(0)P_\varepsilon x(0)$, then $v(t) = v^*(t)$. Hence, $J(u_1^*,\ v,\ x(0)) < x^T(0)P_\varepsilon x(0)$, for all $v(t) \ne v^*(t)$ and $J(u_1^*,\ v^*,\ x(0)) = x^T(0)P_\varepsilon x(0)$.

Let $\hat{x}(t)$ and $\bar{x}(t)$ be governed by

$$d\hat{x}(t) = [A_\varepsilon \hat{x}(t) + B_{1\varepsilon}F_{1\varepsilon}\hat{x}(t) + E_\varepsilon v^*(t)]dt + A_{1\varepsilon}\hat{x}(t)dw(t),\ \hat{x}(0) = x^0, \tag{11a}$$
$$d\bar{x}(t) = [A_\varepsilon \bar{x}(t) + B_{1\varepsilon}F_{1\varepsilon}^*\bar{x}(t) + E_\varepsilon v^*(t)]dt + A_{1\varepsilon}\bar{x}(t)dw(t),\ \bar{x}(0) = x^0, \tag{11b}$$

respectively. Furthermore, define $\nu(t) := (F_{1\varepsilon}^* - F_{1\varepsilon})\hat{x}(t)$ and $\eta(t) := v^*(t) - V_{1\mu}^{-1}E_\varepsilon^T P_\varepsilon \hat{x}(t)$.

Then,

$$\begin{aligned} &J(u_1,\ v^*,\ x(0)) - J(u_1^*,\ v^*,\ x(0)) \\ &= E\int_0^\infty \Big[\|\nu(t)\|_{R_{11}}^2 - \|\eta(t)\|_{V_{1\varepsilon}}^2\Big]dt. \end{aligned} \tag{12}$$

Introducing $\xi(t) := \bar{x}(t) - \hat{x}(t)$, the following equation is satisfied.

$$\begin{aligned} d\xi(t) &= [(A_\varepsilon - S_{1\varepsilon}P_\varepsilon)\xi(t) + B_{1\varepsilon}\nu(t)]dt \\ &\quad + A_{1\varepsilon}\xi(t)dw(t),\ \xi(0) = 0,\ \eta(t) = V_{1\varepsilon}^{-1}E_\varepsilon^T \xi(t). \end{aligned} \tag{13}$$

Hence, taking into account the fact that for any $\nu(t)$, the closed-loop system is asymptotically mean-square stable,

$$E\int_0^\infty \frac{d}{dt}\xi^T(t)P_\varepsilon \xi(t)dt = 0. \tag{14}$$

Thus, the following inequality holds.

$$\begin{aligned} &J(u_1,\ v^*,\ x(0)) - J(u_1^*,\ v^*,\ x(0)) \\ &= E\int_0^\infty \Big[\|\nu(t)\|_{R_{11}}^2 - \|\eta(t)\|_{V_{1\varepsilon}}^2 - \frac{d}{dt}\xi^T(t)P_\varepsilon\xi(t)\Big]dt \\ &= E\int_0^\infty \Big[\|\nu(t) + F_{1\varepsilon}^*\xi(t)\|_{R_{11}}^2 + \xi^T(t)Q_{1\varepsilon}\xi(t)\Big]dt \ge 0. \end{aligned} \tag{15}$$

This is the desired result. ■

B. SOFT-CONSTRAINED STOCHASTIC NASH EQUILIBRIUM

The soft-constrained stochastic Nash games are given below.

Theorem 2: Suppose there exists positive semidefinite symmetric matrices $P_{i\varepsilon}$.

$$\begin{aligned} &\mathcal{G}_i(\varepsilon,\ P_{1\varepsilon},\ P_{2\varepsilon}) \\ &= P_{i\varepsilon}(A_\varepsilon - S_{j\varepsilon}P_{j\varepsilon}) + (A_\varepsilon - S_{j\varepsilon}P_{j\varepsilon})^T P_{i\varepsilon} + A_{1\varepsilon}^T P_{i\varepsilon} A_{1\varepsilon} \\ &\quad -P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon} + \varepsilon P_{j\varepsilon}S_{ij\varepsilon}P_{j\varepsilon} + P_{i\varepsilon}M_{i\varepsilon}P_{i\varepsilon} + Q_{i\varepsilon} = 0, \end{aligned} \tag{16}$$

where $i,\ j = 1,\ 2, i \ne j$, $S_{i\varepsilon} := B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^T$, $S_{ij\varepsilon} := B_{j\varepsilon}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{j\varepsilon}^T$, $M_{i\varepsilon} := E_\varepsilon V_{i\varepsilon}^{-1}E_\varepsilon^T$.

The strategy set $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*)$ is defined by

$$u_i^*(t) := F_{i\varepsilon}^* x(t) = -R_{ii}^{-1}B_{i\varepsilon}^T P_{i\varepsilon}x(t),\ i = 1,\ 2. \tag{17}$$

Then, $(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*) \in \mathbf{F}_2$ and this strategy set denote soft-constrained stochastic Nash equilibrium. Furthermore, $\bar{J}_i(F_{1\varepsilon}^*,\ F_{2\varepsilon}^*,\ x(0)) = x^T(0)P_{i\varepsilon}x(0)$.

Proof: Now, let us consider the following problem in which the cost function (18) is minimal at $F_{i\varepsilon} = F_{i\varepsilon}^*$.

$$\begin{aligned} \phi(F_\varepsilon) := \sup_{v \in L_2^{\bar{k}}(0,\ \infty)} E\int_0^\infty \Big[&x^T(t)(Q_{i\varepsilon} + F_{i\varepsilon}^T R_{ii}F_{i\varepsilon} \\ &+\varepsilon P_{j\varepsilon}^T S_{ij}P_{j\varepsilon})x(t) - v^T(t)V_{i\varepsilon}v(t)\Big]dt, \end{aligned} \tag{18}$$

where $x(t)$ follows from

$$\begin{aligned} dx(t) &= \big[(A_\varepsilon - S_{j\varepsilon}P_{j\varepsilon} + B_{i\varepsilon}F_{i\varepsilon})x(t) + E_\varepsilon v(t)\big]dt \\ &\quad + A_{1\varepsilon}x(t)dw(t),\ x(0) = x^0. \end{aligned} \tag{19}$$

Note that the function ϕ coincides with function J in Theorem 1. Applying Theorem 1 to this minimization problem as

$$\begin{aligned} &A_\varepsilon - S_{j\varepsilon}P_{j\varepsilon} \Rightarrow A_\varepsilon,\ B_{i\varepsilon} \Rightarrow B_{1\varepsilon}, \\ &Q_{i\varepsilon} + \varepsilon P_{j\varepsilon}^T S_{ij}P_{j\varepsilon} \Rightarrow Q_{1\varepsilon},\ R_{ii} \Rightarrow R_{11},\ V_{i\varepsilon} \Rightarrow V_{1\varepsilon} \end{aligned}$$

yields the fact that the function ϕ is minimal at

$$F_{1\varepsilon} = -R_{ii}^{-1}B_{i\varepsilon}^T P_{i\varepsilon} = F_{i\varepsilon}^*. \tag{20}$$

Moreover, the minimal value is $x^T(0)P_{i\varepsilon}x(0)$. ■

It should be noted that the asymptotically mean-square stable can be proved by using the similar technique used in [1].

III. ASYMPTOTIC STRUCTURE OF CSAREs

Firstly, in order to obtain the strategy set based on numerical solutions, the asymptotic structure of CSARE (16) is established. Since A_ε, $A_{1\varepsilon}$, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $Q_{i\varepsilon}$ and $M_{i\varepsilon}$ include the term of the parameter ε, the solution $P_{i\varepsilon}$ of CSARE (16)-if it exists-should contain the parameter ε. By considering this fact, the solution $P_{i\varepsilon}$ of CSARE (16) is assumed to have the following structure.

$$P_{1\varepsilon} = \begin{bmatrix} P_{111} & \varepsilon P_{112} \\ \varepsilon P_{112}^T & \varepsilon P_{122} \end{bmatrix}, \ P_{2\varepsilon} = \begin{bmatrix} \varepsilon P_{211} & \varepsilon P_{212} \\ \varepsilon P_{212}^T & P_{222} \end{bmatrix}. \quad (21)$$

Substituting the matrices A_ε, $A_{1\varepsilon}$, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $Q_{i\varepsilon}$, $M_{i\varepsilon}$ and $P_{i\varepsilon}$ into CSARE (16), letting $\varepsilon = 0$, and partitioning CSARE (16), the following reduced-order stochastic algebraic Riccati equation (SARE) is obtained, where $\bar{P}_{iii}, \ i = 1, \ 2$ is the 0-order solutions of CSARE (16) as $\varepsilon = 0$.

$$\begin{aligned} &\bar{P}_{iii}A_{ii} + A_{ii}^T\bar{P}_{iii} + A_{1ii}^T\bar{P}_{iii}A_{1ii} \\ &\quad -\bar{P}_{iii}(S_{ii} - M_{ii})\bar{P}_{iii} + Q_{iii} = 0, \end{aligned} \quad (22)$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$ and $M_{ii} := E_{ii}V_{ii}^{-1}E_{ii}^T$.

The following condition is assumed.

Assumption 1: SARE (22) has a positive semidefinite solution.

The asymptotic expansion of CSARE (16) for $\varepsilon = 0$ is described by the following lemma.

Lemma 1: Under Assumption 1, there exists a small constant σ^* such that for all $\varepsilon \in (0, \ \sigma^*)$, CSARE (16) admits a positive semidefinite solution $P_{i\varepsilon}^*$ that can be expressed as

$$P_{i\varepsilon} := P_{i\varepsilon}^* = \bar{P}_i + O(\varepsilon), \quad (23)$$

where

$$\bar{P}_1 = \textbf{block diag}\left(\bar{P}_{111} \ 0\right), \ \bar{P}_2 = \textbf{block diag}\left(0 \ \bar{P}_{222}\right).$$

Proof: Since the result of Lemma 1 can be proved by using a technique similar to that used in [13], the proof is omitted. ■

IV. LYAPUNOV ITERATIONS FOR SOLVING CSAREs

When CSARE (16) is solved, the existence of the cross-coupled term makes it difficult to directly solve this equation. Thus, in order to avoid the cross-coupled term, Lyapunov iterations [12] can be applied. It has been shown that Lyapunov iterations yield the positive semidefinite stabilizing solution for the sign-indefinite cross-coupled algebraic Riccati equation (CARE) [12]. However, there are no results for CSARE (16). It should be noted that CSARE (16) is quite different from the existing CARE because there is the stochastic term $A_{1\varepsilon}^T P_{i\varepsilon} A_{1\varepsilon}$ in CSARE (16). This can be convincing motivation to establish a new Lyapunov iteration for solving CSARE (16). On the other hand, when Newton's method and two fixed-point iterations for solving CSARE [13] are applied, many procedure is needed. By using the Lyapunov iteration, the reduction of the required operation count is attained. As a result, the reduction of the CPU time would be guaranteed.

In order to obtain the solution of CSAREs (16), the following useful algorithm is given.

$$\begin{aligned} &P_{i\varepsilon}^{(k+1)}\left(A_\varepsilon - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(k)} + M_{i\varepsilon}P_{i\varepsilon}^{(k)}\right) \\ &\quad + \left(A_\varepsilon - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(k)} + M_{i\varepsilon}P_{i\varepsilon}^{(k)}\right)^T P_{i\varepsilon}^{(k+1)} \\ &\quad + A_{1\varepsilon}^T P_{i\varepsilon}^{(k+1)} A_{1\varepsilon} + P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^{(k)} M_{i\varepsilon} P_{i\varepsilon}^{(k)} \\ &\quad + \varepsilon P_{j\varepsilon}^{(k)} S_{ij\varepsilon} P_{j\varepsilon}^{(k)} + Q_{i\varepsilon} = 0, \ k = 0, \ 1, \ \ldots, \end{aligned} \quad (24a)$$

$$P_{1\varepsilon}^{(k)} := \begin{bmatrix} P_{111}^{(k)} & \varepsilon P_{112}^{(k)} \\ \varepsilon P_{112}^{(k)T} & \varepsilon P_{122}^{(k)} \end{bmatrix}, \ P_{2\varepsilon}^{(k)} := \begin{bmatrix} \varepsilon P_{211}^{(k)} & \varepsilon P_{212}^{(k)} \\ \varepsilon P_{212}^{(k)T} & P_{222}^{(k)} \end{bmatrix} \quad (24b)$$

with the initial conditions

$$P_{i\varepsilon}^{(0)} = \bar{P}_i. \quad (25)$$

The following theorem indicates that the proposed algorithm which is based on Lyapunov iterations attain the linear convergence.

Theorem 3: Under Assumption 1, there exists the small constant $\bar{\sigma}$ such that for all $\varepsilon \in (0, \ \bar{\sigma})$, $\bar{\sigma} \le \sigma^*$, the iterative algorithm (24a) converges to the exact solution of $P_{i\varepsilon}^*$ with the rate of the linear convergence, where $P_{i\varepsilon}^{(k)}$ is positive semidefinite matrix and $A_\varepsilon - S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\varepsilon}P_{i\varepsilon}^{(k)}$ is stable matrix. That is, the following conditions are satisfied.

$$\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^*\| = O(\varepsilon^{k+1}), \quad (26a)$$

$$\mathrm{Re}\lambda\left[A_\varepsilon - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(k)} + M_{i\varepsilon}P_{i\varepsilon}^{(k)}\right] < 0, \ k = 0, \ 1, \ \ldots \quad (26b)$$

The following lemma will play an important role in establishing (26a).

Lemma 2: If $dz(t) = Az(t)dt + \sum_{p=1}^{N} A_p z(t) dw_p(t)$ is exponentially mean-square stable and $Q = Q^T$, $z^T(0)Pz(0) = \int_0^\infty z^T(t)Qz(t)dt$ where P satisfies the stochastic algebraic Lyapunov equation (SALE) $A^TP + PA + \sum_{p=1}^{N} A_p^T P A_p + Q = 0$.

Proof: The proof of this theorem can be derived by using the mathematical induction. When $k = 0$, taking (23) into account, it is easy to verify that the first order approximations $P_{i\varepsilon}^*$ corresponding to the small parameter ε are the same as $P_{i\varepsilon}^{(0)}$. Moreover, since

$$\begin{aligned} &A_\varepsilon - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(0)} + M_{1\varepsilon}P_{1\varepsilon}^{(0)} \\ &= \textbf{block diag}\left(\ D_{11} \quad H_{11}\ \right) + O(\varepsilon) := \mathcal{D}_1 + O(\varepsilon), \\ &A_\varepsilon - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(0)} + M_{2\varepsilon}P_{2\varepsilon}^{(0)} \\ &= \textbf{block diag}\left(\ H_{22} \quad D_{22}\ \right) + O(\varepsilon) := \mathcal{D}_2 + O(\varepsilon), \end{aligned}$$

where $D_{ii} := A_{ii} - S_{ii}\bar{P}_{iii} + M_{ii}\bar{P}_{iii}$, $H_{ii} = A_{ii} - S_{ii}\bar{P}_{iii}$, $i = 1, \ 2$, there exists the small perturbation parameter σ_0 such that $A_\varepsilon - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(0)} + M_{i\varepsilon}P_{i\varepsilon}^{(0)}$ is stable because $\mathcal{D}_i$ is stable for sufficiently small ε. When $k = h,\ h \geq 1$, it is assumed that

$$\|P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\| = O(\varepsilon^{h+1}), \tag{27a}$$

$$\mathrm{Re}\lambda\left[A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(h)} + M_{i\varepsilon}P_{i\varepsilon}^{(h)}\right] < 0. \tag{27b}$$

Subtracting (16) from (24a) and setting $k = h$, the following equations are satisfied.

$$\begin{aligned}
&\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\right)\left(A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(h)} + M_{i\varepsilon}P_{i\varepsilon}^{(h)}\right)\\
&+\left(A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(h)} + M_{i\varepsilon}P_{i\varepsilon}^{(h)}\right)^{T}\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\right)\\
&+A_{1\varepsilon}^{T}\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\right)A_{1\varepsilon}\\
&+P_{i\varepsilon}^{*}S_{j\varepsilon}\left(P_{j\varepsilon}^{*} - P_{j\varepsilon}^{(h)}\right) + \left(P_{j\varepsilon}^{*} - P_{j\varepsilon}^{(h)}\right)S_{j\varepsilon}P_{i\varepsilon}^{*}\\
&+\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right)S_{i\varepsilon}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right)\\
&-\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right)M_{i\varepsilon}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right)\\
&+\varepsilon\left[P_{j\varepsilon}^{(h)}S_{ij\varepsilon}P_{j\varepsilon}^{(h)} - P_{j\varepsilon}^{*}S_{ij\varepsilon}P_{j\varepsilon}^{*}\right] = 0.
\end{aligned} \tag{28}$$

Using the fact that the assumption (27a) holds, it is easy to derive that

$$\begin{aligned}
&P_{i\varepsilon}^{*}S_{j\varepsilon}\left(P_{j\varepsilon}^{*} - P_{j\varepsilon}^{(h)}\right) = O(\varepsilon^{h+2}),\\
&\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right)S_{i\varepsilon}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right) = O(\varepsilon^{2h+2}),\\
&\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right)M_{i\varepsilon}\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon}^{*}\right) = O(\varepsilon^{2h+2}),\\
&\varepsilon\left[P_{j\varepsilon}^{(h)}S_{ij\varepsilon}P_{j\varepsilon}^{(h)} - P_{j\varepsilon}^{*}S_{ij\varepsilon}P_{j\varepsilon}^{*}\right] = O(\varepsilon^{h+2}).
\end{aligned}$$

It should be noted that if $i \neq j$, $P_{i\varepsilon}^{*}S_{j\varepsilon} = O(\varepsilon)$ holds. Thus, the following relation is satisfied.

$$\begin{aligned}
&\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\right)\left(A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(h)} + M_{i\varepsilon}P_{i\varepsilon}^{(h)}\right)\\
&+\left(A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(h)} + M_{i\varepsilon}P_{i\varepsilon}^{(h)}\right)^{T}\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\right)\\
&+A_{1\varepsilon}^{T}\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\right)A_{1\varepsilon} + O(\varepsilon^{h+2}) = 0.
\end{aligned} \tag{29}$$

By using Lemma 2 and taking into account the fact that the stability assumption (27b) holds, the following result is satisfied.

$$\|P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}^{*}\| = O(\varepsilon^{h+2}). \tag{30}$$

Furthermore, it is shown that there exists the small positive perturbation parameter σ_{h+1} such that

$$\begin{aligned}
&A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(h+1)} + M_{i\varepsilon}P_{i\varepsilon}^{(h+1)}\\
&= A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{*} + M_{i\varepsilon}P_{i\varepsilon}^{*} + O(\varepsilon^{h+2}) = \mathcal{D}_i + O(\varepsilon)
\end{aligned}$$

is stable. Consequently, choosing $\bar{\sigma} := \min\{\sigma_0,\ \ldots\ ,\sigma_{h+1}\}$, the relation (27b) holds for all $k \in \mathbf{N}$. This completes the proof of Theorem 3 concerned with the Lyapunov iterations. ■

Using the asymptotic structure of the solutions (23), the local uniqueness of the convergence solutions is studied.

Theorem 4: Under Assumption 1, there exist the sufficiently small constant $\hat{\sigma}$ such that for all $\varepsilon \in (0,\ \hat{\sigma})$, $\hat{\sigma} \leq \bar{\sigma} \leq \sigma^{*}$, the convergence solution $P_{i\varepsilon}^{*}$ of the iterative solution $P_{i\varepsilon}^{(k)}$ is unique in the neighborhood of $\varepsilon = 0$.

Proof: First, under Assumption 1, there exists the neighborhood of $\varepsilon = 0$ such that the CSARE (16) admits a unique positive semidefinite solution $P_{i\varepsilon}^{*}$ by means of the implicit function theorem (see the proof of Lemma 1). That is, for sufficiently small ε, the CSARE (16) has a unique positive semidefinite solution $P_{i\varepsilon}^{*}$. Taking into account the fact that the solutions $P_{i\varepsilon}^{*}$ of (23) and (27a) are equivalent, the iterative solution $P_{i\varepsilon}^{(k)}$ converges to $P_{i\varepsilon}^{*}$ and it is a unique solution for sufficiently small ε. ■

It seems to be formidable for solving the stochastic algebraic Lyapunov equation (SALE) (24a) because such equations are not the ordinary algebraic Lyapunov equation (ALE). In fact, The MATLAB function `lyap` cannot be used. Then, using the gradient-based (GI) algorithm [10], it can be easily solved with the same order dimension of the systems.

For example, let us consider the SALE (24a) in the following general form.

$$\Xi^{T}Y + Y\Xi + \Xi_1^{T}Y\Xi_1 + U = 0, \tag{31}$$

where $\Xi,\ \Xi_1,\ Y = Y^{T} \geq 0,\ U = U^{T} \in \mathbf{R}^{n_i \times n_i}$.

It should be noted that for SALE (31),

$$\begin{aligned}
&A_{\varepsilon} - \sum_{p=1}^{2} S_{p\varepsilon}P_{p\varepsilon}^{(k)} + M_{i\varepsilon}P_{i\varepsilon}^{(k)} \Rightarrow \Xi,\\
&A_{1\varepsilon} \Rightarrow \Xi_1,\ P_{i\varepsilon}^{(k+1)} \Rightarrow Y,\\
&P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} - P_{i\varepsilon}^{(k)}M_{i\varepsilon}P_{i\varepsilon}^{(k)} + \varepsilon P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)} + Q_{i\varepsilon} \Rightarrow U.
\end{aligned}$$

The GI algorithm for solving the SALE (31) is given below [10].

$$Y(m) = \frac{Y_{s1}(m) + Y_{s2}(m) + Y_1(m)}{3}, \tag{32}$$

where

$$\begin{aligned}
&Y_{s1}(m) := Y(m-1) + \lambda\Theta(m)\Xi^{T},\\
&Y_{s2}(m) := Y(m-1) + \lambda\Xi\Theta(m),\\
&Y_1(m) := Y(m-1) + \lambda\Xi_1\Theta(m)\Xi_1^{T},\\
&\Theta(m) := U - \Xi^{T}Y(m) - Y(m)\Xi - \Xi_1^{T}Y(m)\Xi_j,\\
&Y(0) = 0,\ \lambda^{-1} := 2\|\Xi\| + \|\Xi_1\|^2.
\end{aligned}$$

It should be noted that the convergence has been proved in [10]. Finally, using the GI algorithm (32), the same order computations of each system dimension can be attained.

TABLE I

ERROR PER ITERATIONS.

k	$\|\mathcal{G}^{(k)}(1.0e-02)\|$	$\|\mathcal{G}^{(k)}(1.0e-03)\|$	$\|\mathcal{G}^{(k)}(1.0e-04)\|$	$\|\mathcal{G}^{(k)}(1.0e-05)\|$
0	$1.9380e-001$	$1.9346e-002$	$1.9342e-003$	$1.9341e-004$
1	$6.1463e-004$	$5.8604e-006$	$5.8569e-008$	$5.8568e-010$
2	$2.9966e-006$	$2.8384e-009$	$1.9861e-011$	$1.9880e-011$
3	$2.3193e-008$	$1.9879e-011$		
4	$1.0336e-010$			
5	$1.9885e-011$			

V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, a numerical example is given. The system matrices are given as follows.

$$
\begin{aligned}
&n_1 = n_2 = 20,\ A_{11} = -I_{20} + \mathcal{R}_1,\ A_{12} = \mathcal{R}_1,\\
&A_{21} = \mathcal{R}_1,\ A_{22} = -1.2I_{20} + \mathcal{R}_1,\\
&\mathcal{R}_1 = (r_{ij}^1),\ r_{ij}^1 := \varepsilon \sin(i-j),\\
&A_{111} = 1.0e-03 \times (I_{20} + \mathcal{R}_2),\ A_{112} = 1.0e-03 \times \mathcal{R}_2,\\
&A_{121} = 1.0e-03 \times \mathcal{R}_2,\ A_{122} = 1.0e-03 \times (I_{20} + \mathcal{R}_2),\\
&\mathcal{R}_2 = (r_{ij}^2),\ r_{ij}^2 := \varepsilon \cos^2(i-j),\\
&B_{11} = (b_{ij}^{11}),\ b_{ij}^{11} := \cos(i-j),\\
&B_{22} = (b_{ij}^{22}),\ b_{ij}^{22} := \sin(i-j),\ B_{ij} = 0,\ i \neq j,\\
&E_{11} = (e_{ij}^{11}),\ e_{ij}^{11} := 0.1 \times \sin^2(i-j),\\
&E_{22} = (e_{ij}^{22}),\ e_{ij}^{22} := 0.1 \times \cos^2(i-j),\ E_{ij} = 0,\ i \neq j,\\
&V_1 = \textbf{block diag}\left(\begin{array}{cc} V_{11} & \mu^{-1}I_{20} \end{array}\right),\\
&V_2 = \textbf{block diag}\left(\begin{array}{cc} \mu^{-1}I_{20} & V_{22} \end{array}\right),\\
&V_{ii} = \textbf{diag}\left(\begin{array}{cccc} 1 & 2 & \cdots & 20 \end{array}\right),\\
&Q_1 = \textbf{block diag}\left(\begin{array}{cc} Q_{11} & \varepsilon I_{20} \end{array}\right),\\
&Q_2 = \textbf{block diag}\left(\begin{array}{cc} \varepsilon I_{20} & Q_{22} \end{array}\right),\\
&Q_{11} = 2I_{20} + \varepsilon \mathcal{R}_3^T \mathcal{R}_3,\ \mathcal{R}_3 := (r_{ij}^3),\ r_{ij}^3 := \sin^2(i-j),\\
&Q_{22} = 2I_{20} + \varepsilon \mathcal{R}_4^T \mathcal{R}_4,\ \mathcal{R}_4 := (r_{ij}^4),\ r_{ij}^4 := \cos^2(i-j),\\
&R_{11} = R_{22} = 1,\ R_{12} = 0,\ R_{21} = 0.
\end{aligned}
$$

Small parameter $\varepsilon = 0.01$ is selected. It should be noted that algorithm (24a) converges to the exact solution with an accuracy of $\|\mathcal{G}^{(k)}(\varepsilon)\| < 1.0e-10$ after five iterations, where $\|\mathcal{G}^{(k)}(\varepsilon)\| := \sum_{p=1}^{2} \|\mathcal{G}_p(\varepsilon,\ P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)})\|$.

In order to verify the exactitude of the solution, the remainder per iteration is calculated by substituting $P_{i\varepsilon}^{(k)}$ into CSARE (16). In Table 1, the results of the error $\|\mathcal{G}^{(k)}(\varepsilon)\|$ per iteration are given for several values of ε. As a result, it can be seen that algorithm (24a) yields linear convergence. Hence, the proposed algorithms of equation (24a) in this paper are very attractive. Furthermore, even if weakly coupled large-scale systems (1) are composed of two 20-dimensional subsystems, the required workspace is 20. This feature is very useful from the practical viewpoint.

VI. CONCLUSIONS

Infinite-horizon soft-constrained stochastic Nash games have been discussed. First, conditions required for the existence of Nash equilibrium have been established by utilizing CSAREs. Second, a numerical algorithm based on Lyapunov iterations for solving the CSAREs that arose in the stochastic Nash games for weakly coupled large-scale systems has been studied. It has been shown that both linear convergence and reduced-order computations can be attained. Thus, the proposed algorithm is expected to be very useful and reliable for a sufficiently small value of ε. Finally, numerical example has yielded excellent results using which linear convergence has been verified and the proposed algorithm has succeeded in reducing the computational workspace.

The stabilizable assumption needs to be relaxed in establishing the conditions for future investigation. Moreover, in order to implement the present control methodology for more practical plants, it will be need to extend the bound of the small parameter ε.

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