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Convex analysis and synthesis for uncertain discrete-time systems with time-varying state delay

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Abstract— **The class of polytopic uncertain discrete-time systems with time-varying delays is investigated in this paper. Firstly, some convex conditions are presented in the form of linear matrix inequalities (LMIs) to test robust stability. Then, LMI conditions to design robust state feedback gains are proposed. Both, delay-dependent and delay-independent conditions are formulated as convex feasibility problems employing extra matrices that yield less conservative results. Some numerical examples are presented to illustrate the application of the conditions for analysis and synthesis of robust state feedback gains.**

I. INTRODUCTION

Systems with delayed states have been received lots of attention in the last years, for instance, see [11], [13], [8] and [14]. The investigation of robust stability analysis as well as robust state feedback gain design for the class of uncertain discrete-time system with state delay (DTSSD) are relevant issues, since the study of an augmented system presents strong restrictions, for example, in case of time-varying delays. Most of the used techniques for robust stability analysis and robust control design are based on Lyapunov-Krasovskii approach [14], [15]. This approach is used to obtain convex formulation problems in terms of linear matrix inequalities (LMIs) which allow to treat time-varying delays.

For uncertain DTSSD systems, it is possible to find approaches based on LMIs, but most of them are based on the quadratic stability (QS), i.e., the matrices of the Lyapunov-Krasovskii functional are constant and independent of the uncertain parameters. In the context of QS, non-convex formulations of delay-independent type have been proposed in [16] where the delay is considered time-invariant. In this case, the uncertainties are supposed to be of the norm-bounded type and the design conditions are nonconvex. In [20] this same type of system is investigated, assuming time-varying delay, but using nonconvex conditions. In [3] and [4] delay-dependent conditions have been proposed as convex problems for the analysis and nonconvex problems for the synthesis. In [3] the uncertain DTSSD is described in a polytopic framework while in [4] it is described in a norm-bounded framework. Recently, in [10], the results of [7] were improved, but the approach is based on QS and the design conditions are nonconvex depending directly on the Lyapunov-Krasovskii functional matrices.

There are a few results using parameter dependent Lyapunov-Krasovskii functionals for DTSSD. Convex delay-independent conditions for robust stabilization of polytopic systems [9] and with $\mathcal{H}_\infty$ guaranteed performance of norm-bounded systems [19] have been proposed, but the delay is supposed to be time-invariant. In [1], convex conditions were proposed for precisely known DTSSD with time-varying delay. Those conditions depend on the size of delay variation, but not on the delay value. In [6] several conditions for access robust stability of DTSSD are discussed and some over bounds usually employed in the literature are discussed in details.

In the present paper, delay-dependent and delay-independent convex conditions for both robust stability analysis and robust synthesis of state feedback gains for DTSSD with time-varying delay are presented. The conditions proposed here employ parameter dependent Lyapunov-Krasovskii functionals, that yield less conservative results. The presented conditions can be seen as an improvement on the results of [1] and [10]. The main advantage of the present proposal is the existence of a convex formulation for the synthesis of robust state feedback gains. Examples are presented to illustrate the efficiency of the proposed conditions.

Notation: The notation used is quite standard: x_k is the state at time k. $\mathbb{R}$ is the set of real numbers and $\mathbb{N}^*$ stands for the set of the natural numbers excluded the 0. $\mathbf{I}$ and $\mathbf{0}$ are the identity and the null matrices of appropriate dimensions, respectively. $M = \text{block-diag}\{M_1,\ M_2\}$ stands for the block-diagonal matrix M made up by the matrices M_1 and M_2 at the main diagonal. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that M is positive (negative) definite. M' stands for the transpose of M. $\star$ is used to indicate diagonally symmetric blocks in the LMIs.

II. PRELIMINARIES

Consider the DTSSD given by

$$x_{k+1} = A(\alpha)x_k + A_d(\alpha)x_{k-d(k)} + B(\alpha)u_k \qquad (1)$$

$$x_k = \phi(k), \qquad k \in [-\bar{d},\ 0] \qquad (2)$$

where k is the sample time, $x_k \equiv x(k) \in \mathbb{R}^n$ is the state vector, $u_k \equiv u(k) \in \mathbb{R}^\ell$ is the control input and $\phi(k)$ is the initial condition, given by the sequence $k = -\bar{d}, \ldots, 0$. The delay, $d(k)$, is considered time-varying and bounded as follows

$$\underline{d} \leq d(k) \leq \bar{d}, \quad (\underline{d},\ \bar{d}) \in \mathbb{N}^* \times \mathbb{N}^* \qquad (3)$$

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with $\underline{d}$ and $\bar{d}$ being, respectively, the minimum and maximum delay value. Observe that, if the delay is uncertainty but time-invariant, then $\underline{d} = \bar{d}$.

The uncertain matrices $[A(\alpha)|A_d(\alpha)|B(\alpha)] \equiv [A|A_d|B](\alpha) \in \mathbb{R}^{n\times 2n+\ell}$ are time-invariant and belong to the polytope

$$\mathcal{P} \equiv \Big\{[A|A_d|B](\alpha) \; : \; [A|A_d|B](\alpha) = \sum_{i=1}^{N}[A|A_d|B]_i\alpha_i, \quad \alpha\in\Omega\Big\} \quad (4)$$

$$\Omega \equiv \Big\{\alpha \; : \; \alpha\in\mathbb{R}^N, \; \sum_{i=1}^{N}\alpha_i = 1, \alpha_i \geq 0\Big\} \quad (5)$$

with known vertices $[A_i|A_{di}|B_i] \equiv [A|A_d|B]_i$, $i = 1,\ldots,N$. It is considered the following control law for system (1)

$$u(k) = Kx_k + K_d x_{k-d(k)} \quad (6)$$

where $[K|K_d] \in \mathbb{R}^{\ell\times 2n}$ are static feedback gains. For system (1), if the delay $d(k)$ can be obtained at each sample time, it is possible to use both gains, K and K_d to improve the closed-loop performance. For instance, this occurs when some kind of time-stamped is employed for measurements or state estimatimation [17]. On the other hand, if the delay $d(k)$ is not available, it is enough to make $K_d = \mathbf{0}$ in (6). Using (6) in (1), one gets

$$x_{k+1} = \tilde{A}(\alpha)x_k + \tilde{A}_d(\alpha)x_{k-d(k)} \quad (7)$$

where $\tilde{A}(\alpha) \equiv A(\alpha) + B(\alpha)K$ and $\tilde{A}_d(\alpha) \equiv A_d(\alpha) + B(\alpha)K_d$ are in the polytope

$$\tilde{\mathcal{P}} \equiv \Big\{[\tilde{A}|\tilde{A}_d](\alpha) \; : \; [\tilde{A}|\tilde{A}_d](\alpha) = \sum_{i=1}^{N}[\tilde{A}|\tilde{A}_d]_i\alpha_i, \quad \alpha\in\Omega\Big\} \quad (8)$$

Lemma 1 (Finsler's Lemma): Let $x \in \mathbb{R}^n$, $\mathcal{Q}(\alpha) = \mathcal{Q}(\alpha)' \in \mathbb{R}^n$ and $\mathcal{B}(\alpha) \in \mathbb{R}^{m\times n}$ be, such that $\mathrm{rank}(\mathcal{B}(\alpha)) < n$. The following statements are equivalent:

i) $x'\mathcal{Q}(\alpha)x < \mathbf{0}, \quad \forall x \; : \; \mathcal{B}(\alpha)x = \mathbf{0}, \quad x \neq \mathbf{0}$

ii) $\exists \; \mathcal{X}(\alpha) \in \mathbb{R}^{n\times m} \; : \; \mathcal{Q}(\alpha) + \mathcal{X}(\alpha)\mathcal{B}(\alpha) + \mathcal{B}(\alpha)'\mathcal{X}(\alpha)' < \mathbf{0}$

The proof follows the directions given in [2].

Definition 1: System (7) subject to (3), (5) and (8) is called *robustly stable* if the trivial solution for the correspondent difference equation is globally and asymptotically stable $\forall \; \alpha\in\Omega$.

The main objective of this work is to present convex conditions that can be used to solve the two problems:

Problem 1: Given $\underline{d}$ and $\bar{d}$ subject to (3), determine if system (7) subject to (5) and (8) is robustly stable.

Problem 2: Determine, if possible, a pair of static gains, $[K|K_d]$ such that (1)-(5), controlled by (6) is robustly stable.

III. Main Results

In this section, it is presented the conditions for the robust stability analysis of (7) and for the design of robust state feedback gains used in (6).

A. Robust stability analysis

Theorem 1: If there exists matrices $P_i = P_i' > 0$, $Q_i = Q_i' > 0$, $Z_i = Z_i' > 0$, $i = 1,\ldots,N$, and matrices G_0, H_0, S_0, F_1, G_1, H_1, M_1, N_1, R_1, F_2, G_2, H_2, M_2, N_2, R_2, with appropriate dimensions, such that LMIs (9) can be verified, with $\beta = \bar{d} - \underline{d} + 1$, $\underline{d}$ and $\bar{d}$ known, then the system (7) subject to (3), (5) and (8) is robustly stable, characterizing a solution to Problem 1. Besides this,

$$V(\alpha,k) = \sum_{v=1}^{5} V_v(\alpha,k) > 0 \quad (10)$$

with

$$V_1(\alpha,k) = x_k'P(\alpha)x_k, \quad (11)$$

$$V_2(\alpha,k) = \sum_{j=k-d(k)}^{k-1} x_j'Q(\alpha)x_j, \quad (12)$$

$$V_3(\alpha,k) = \sum_{\ell=2-\bar{d}}^{1-\underline{d}}\sum_{j=k+\ell-1}^{k-1} x_j'Q(\alpha)x_j, \quad (13)$$

$$V_4(\alpha,k) = \sum_{\ell=-\bar{d}}^{-1}\sum_{m=k+\ell}^{k-1} y_m'Z(\alpha)y_m, \quad (14)$$

$$V_5(\alpha,k) = \sum_{j=k-d(k)}^{k-1} y_j'Z(\alpha)y_j, \quad (15)$$

$$P(\alpha) = \sum_{i=1}^{N}\alpha_iP_i; \quad Q(\alpha) = \sum_{i=1}^{N}\alpha_iQ_i; \quad (16)$$

$$Z(\alpha) = \sum_{i=1}^{N}\alpha_iZ_i \quad (17)$$

with $\alpha\in\Omega$, is a Lyapunov-Krasovskii functional for system (7) and

$$y_j = x_{j+1} - x_j \quad (18)$$

Proof: The positivity of the functional (10) is clearly assured by $P_i = P_i' > 0$, $Q_i = Q_i' > 0$, $Z_i = Z_i' > 0$. For (10) be a Lyapunov-Krasovskii it is also necessary that

$$\Delta V(\alpha,k) = V(\alpha,k+1) - V(\alpha,k) < 0 \quad (19)$$

$\forall \; [x_k' \;\; x_{k-d(k)}']' \neq \mathbf{0}$ e $\forall \; \alpha\in\Omega$. Hereafter, $V_v(\alpha,k)$, $v = 1,\ldots,5$, are denoted as $V_v(k)$, $v = 1,\ldots,5$. Equation (19) is calculated taking into account the following terms

$$\Delta V_1(k) = x_{k+1}'P(\alpha)x_{k+1} - x_k'P(\alpha)x_k \quad (20)$$

$$\Delta V_2(k) = x_k'Q(\alpha)x_k - x_{k-d(k)}'Q(\alpha)x_{k-d(k)} + \sum_{i=k+1-d(k+1)}^{k-1} x_i'Q(\alpha)x_i - \sum_{i=k+1-d(k)}^{k-1} x_i'Q(\alpha)x_i \quad (21)$$

$$\Delta V_2(k) \leq x_k'Q(\alpha)x_k - x_{k-d(k)}'Q(\alpha)x_{k-d(k)} + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i'Q(\alpha)x_i \quad (22)$$

$$\Delta V_3(k) = (\bar{d}-\underline{d})x_k'Q(\alpha)x_k - \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i'Q(\alpha)x_i \quad (23)$$

$$\Lambda_i \equiv \begin{bmatrix} P_i + F_1' + F_1 - F_2 - F_2' & G_1' - G_2' - F_1\tilde{A}_i + F_2 & H_1' - F_1\tilde{A}_{di} - H_2' & F_2 + M_1' - M_2' & N_1' - N_2' & R_1' - R_2' & \mathbf{0} \\ \star & \begin{pmatrix} G_2 + G_2' - \tilde{A}_i'G_1' - G_1\tilde{A}_i \\ +\beta Q_i - P_i + G_0 + G_0' \end{pmatrix} & H_0' - G_0 - \tilde{A}_i'H_1' + H_2' - G_1\tilde{A}_{di} & G_2 - \tilde{A}_i'M_1' + M_2' & N_2' - \tilde{A}_i'N_1' & R_2' - \tilde{A}_i'R_1' & S_0' - G_0 \\ \star & \star & -(Q_i + H_1\tilde{A}_{di} + \tilde{A}_{di}'H_1' + H_0 + H_0') & H_2 - \tilde{A}_{di}'M_1' & -\tilde{A}_{di}'N_1' & -\tilde{A}_{di}'R_1' & -(S_0' + H_0) \\ \star & \star & \star & M_2 + M_2' + (\bar{d}+1)Z_i & N_2' & R_2' & \mathbf{0} \\ \star & \star & \star & \star & -Z_i & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -Z_i & \mathbf{0} \\ \star & \star & \star & \star & \star & \mathbf{0} & -(S_0' + S_0) \end{bmatrix} < 0; \quad i = 1, \ldots, N. \quad (9)$$

$$\Delta V_4(k)(y_k)\bar{d}y_k'Z(\alpha)y_k - \sum_{j=-\bar{d}}^{-1} y_{k+j}'Z(\alpha)y_{k+j}$$
$$\leq \bar{d}y_k'Z(\alpha)y_k - y_{k-\bar{d}}'Z(\alpha)y_{k-\bar{d}} - \sum_{i=k+1-\bar{d}}^{k-\underline{d}} y_i'Z(\alpha)y_i \quad (24)$$

$$\Delta V_5(k) = y_k'Z(\alpha)y_k - y_{k-d(k)}'Z(\alpha)y_{k-d(k)} + \sum_{i=k+1-d(k+1)}^{k-1} y_i'Z(\alpha)y_i - \sum_{i=k+1-d(k)}^{k-1} y_i'Z(\alpha)y_i \quad (25)$$

$$\Delta V_5 \leq y_k'Z(\alpha)y_k - y_{k-d(k)}'Z(\alpha)y_{k-d(k)} + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} y_i'Z(\alpha)y_i \quad (26)$$

Therefore, using (20), (22)-(24) and (26), equation (19) can be bounded as

$$\Delta V(k) \leq x_{k+1}'P(\alpha)x_{k+1} + x_k'[\beta Q(\alpha) - P(\alpha)]x_k - x_{k-d(k)}'Q(\alpha)x_{k-d(k)} + y_k'(\bar{d}+1)Z(\alpha)y_k - y_{k-\bar{d}}'Z(\alpha)y_{k-\bar{d}} - y_{k-d(k)}'Z(\alpha)y_{k-d(k)} < 0 \quad (27)$$

with $\beta = \bar{d} - \underline{d} + 1$. By applying Lemma 1 in (27), with $\omega = [x_{k+1}' \ x_k' \ x_{k-d(k)}' \ y_k' \ y_{k-\bar{d}}' \ y_{k-d(k)}']'$, $\mathcal{Q}(\alpha) =$ block-diag$\{P(\alpha),\ \beta Q(\alpha) - P(\alpha),\ -Q(\alpha),\ (\bar{d}+1)Z(\alpha),\ -Z(\alpha),\ -Z(\alpha)\}$,

$$\mathcal{B}(\alpha) = \begin{bmatrix} \mathbf{I} & -\tilde{A}(\alpha) & -\tilde{A}_d(\alpha) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and}$$

$$\mathcal{X}(\alpha) = \begin{bmatrix} F_1(\alpha) & F_2(\alpha) \\ G_1(\alpha) & G_2(\alpha) \\ H_1(\alpha) & H_2(\alpha) \\ M_1(\alpha) & M_2(\alpha) \\ N_1(\alpha) & N_2(\alpha) \\ R_1(\alpha) & R_2(\alpha) \end{bmatrix},$$

it is possible to write an equivalent condition to (27), defined by $\Phi(\alpha) < 0$, where $\Phi(\alpha) = \mathcal{Q}(\alpha) + \mathcal{X}(\alpha)\mathcal{B}(\alpha) + \mathcal{B}(\alpha)'\mathcal{X}(\alpha)'$ is given in (28). Observe that, if $\Phi(\alpha) < 0$ is verified then $\Lambda(\alpha) \equiv \omega'\Phi(\alpha)\omega + \mathcal{N}(\alpha) < 0$ it is also verified $\forall\ \mathcal{N}(\alpha) \equiv 0$.

Then, if we choose

$$\mathcal{N}(\alpha) = 2\big[x_k'G_0(\alpha) + x_{k-d(k)}'H_0(\alpha) + \eta_k'S_0(\alpha)\big] \times \big[x_k - x_{k-d(k)} - \eta_k\big] \quad (29)$$

where $\eta_k \equiv \sum_{j=k-d(k)}^{k-1} y_j$, with y_j defined in (18), $P(\alpha)$, $Q(\alpha)$ and $Z(\alpha)$ given in (16)-(17), $F_1(\alpha) = F_1$, $G_1(\alpha) = G_1$, $H_1(\alpha) = H_1$, $M_1(\alpha) = M_1$, $N_1(\alpha) = N_1$, $R_1(\alpha) = R_1$, $F_2(\alpha) = F_2$, $G_2(\alpha) = G_2$, $H_2(\alpha) = H_2$, $M_2(\alpha) = M_2$, $N_2(\alpha) = N_2$, $R_2(\alpha) = R_2$, $G_0(\alpha) = G_0$, $H_0(\alpha) = H_0$ e $S_0(\alpha) = S_0$, all the matrices having dimensons $n \times n$, can be verified that $\Lambda(\alpha) = \tilde{\omega}'\big(\sum_{i=1}^{N} \Lambda_i\big)\tilde{\omega} < 0$, with $\tilde{\omega} = [\omega',\ \eta_k']'$, $\alpha \in \Omega$ and $\Lambda_i < \mathbf{0}$, with Λ_i given in (9), completing the proof. ■

Note that the addition of the *null term*, given by (29), does not introduce any additional dynamics to the system, but the obtained conditions yields less conservative results. This has been verified numerically by the authors. Also observe that LMIs proposed in Theorem 1 contain slack variables which avoid products between the Lyapunov-Krasovskii matrices and the system matrices. In the following corollary a QS condition is obtained from Theorem 1. In this case, the system matrices can be time-varying.

Corolary 1: If there exist matrices $P_i = P_i' > 0$, $Q_i = Q_i' > 0$, $i = 1, \ldots, N$, and G_0, H_0, S_0, F_1, G_1, H_1, F_2, G_2, H_2, with appropriate dimensions, such that the LMIs (30) are verified, with known $\beta = \bar{d} - \underline{d} + 1$, $\underline{d}$ and $\bar{d}$, then the system (7) subject to (5) and (8) is robustly stable, independently of the delay value. In this case, the delay can be time-variant, if $|d(k+1) - d(k)| \leq \beta - 1$. Besides, $V(\alpha, k) = \sum_{v=1}^{3} V_v(\alpha, k) > 0$ with V_v, $v = 1, \ldots, 3$, given in (11)-(13) and matrices $P(\alpha)$ and $Q(\alpha)$ given in (16) with $\alpha \in \Omega$, is a Lyapunov-Krasovskii functional to the system (7).

It is important to observe that the conditions of Corollary 1 cannot be obtained from (9) by taking the limit $\bar{d} \to \infty$. Even being delay-independent, the conditions of Corollary 1 allow a time-varying delay with a maximum variation rate of $\bar{d} - \underline{d}$. Obviously, if $\underline{d} = \bar{d}$, then the delay is taken as constant. It is also worth noting that the proposed conditions

$$
\Phi(\alpha) \equiv \begin{bmatrix}
\begin{pmatrix} F_1(\alpha)+F_1(\alpha)' \\ -F_2(\alpha)-F_2(\alpha)'+P(\alpha) \end{pmatrix} & \begin{pmatrix} G_1(\alpha)'+F_2(\alpha) \\ -G_2(\alpha)'-F_1(\alpha)\tilde{A}(\alpha) \end{pmatrix} & H_1(\alpha)'-H_2(\alpha)'-F_1(\alpha)\tilde{A}_d(\alpha) & F_2(\alpha)+M_1(\alpha)'-M_2(\alpha)' & N_1(\alpha)'-N_2(\alpha)' & R_1(\alpha)'-R_2(\alpha)' \\
\star & \begin{pmatrix} \beta Q(\alpha)-P(\alpha)+G_2(\alpha)+G_2(\alpha)' \\ -G_1(\alpha)\tilde{A}(\alpha)-\tilde{A}(\alpha)'G_1(\alpha)' \end{pmatrix} & \begin{pmatrix} H_2(\alpha)'-\tilde{A}(\alpha)'H_1(\alpha)' \\ -Q(\alpha)-G_1(\alpha)\tilde{A}_d(\alpha) \end{pmatrix} & G_2(\alpha)+M_2(\alpha)'-\tilde{A}(\alpha)'M_1(\alpha)' & N_2(\alpha)'-\tilde{A}(\alpha)'N_1(\alpha)' & R_2(\alpha)'-\tilde{A}(\alpha)'R_1(\alpha)' \\
\star & \star & -(H_1(\alpha)\tilde{A}_d(\alpha)+\tilde{A}_d(\alpha)'H_1(\alpha)') & H_2(\alpha)-\tilde{A}_d(\alpha)'M_1(\alpha)' & -\tilde{A}_d(\alpha)'N_1(\alpha)' & -\tilde{A}_d(\alpha)'R_1(\alpha)' \\
\star & \star & \star & (\bar{d}+1)Z(\alpha)+M_2(\alpha)+M_2(\alpha)' & N_2(\alpha)' & R_2(\alpha)' \\
\star & \star & \star & \star & -Z(\alpha) & \mathbf{0} \\
\star & \star & \star & \star & \star & -Z(\alpha)
\end{bmatrix} \quad (28)
$$

$$
\Theta_i \equiv \begin{bmatrix}
\begin{pmatrix} P_i+F_1'+F_1 \\ -F_2-F_2' \end{pmatrix} & G_1'-G_2'-F_1\tilde{A}_i+F_2 & H_1'-F_1\tilde{A}_{di}-H_2' & \mathbf{0} \\
\star & \begin{pmatrix} G_2+G_2'-\tilde{A}_i'G_1'-G_1\tilde{A}_i \\ +\beta Q_i-P_i+G_0+G_0' \end{pmatrix} & \begin{pmatrix} H_0'-G_0-\tilde{A}_i'H_1' \\ +H_2'-G_1\tilde{A}_{di} \end{pmatrix} & S_0'-G_0 \\
\star & \star & -\begin{pmatrix} Q_i+H_1\tilde{A}_{di} \\ +\tilde{A}_{di}'H_1'+H_0+H_0' \end{pmatrix} & -(S_0'+H_0) \\
\star & \star & \star & -(S_0'+S_0)
\end{bmatrix} < 0, \quad i=1,\dots,N \quad (30)
$$

in both Theorem 1 and Corollary 1 can be used to verify the robust stability of the dual system, that is, for the system (7) replacing $\tilde{A}(\alpha)$ and $\tilde{A}_d(\alpha)$ by $\tilde{A}(\alpha)'$ and $\tilde{A}_d(\alpha)'$, respectively.

B. Robust Feedback Gain Design

The robust analysis conditions given in Theorem 1 can be exploited to obtain a convex condition for robust feedback gain design yielding a solution to Problem 2. This is presented in the following theorem.

Theorem 2: If there are matrices $P_i = P_i' > 0$, $Q_i = Q_i' > 0$, $Z_i = Z_i' > 0$, $i = 1, \dots, N$, and G_0, H_0, S_0, F_2, G_2, H_2, M_2, N_2, R_2, F, W, W_d, with compatible dimensions such that the LMIs (32) can be verified with $\beta = \bar{d} - \underline{d} + 1$, $\underline{d}$ and $\bar{d}$ known, then the closed-loop of system (1) subject to (3)-(5) and control law given by (6) with

$$K = W'(F')^{-1} \text{ and } K_d = W_d'(F')^{-1} \quad (31)$$

is robustly stable and is a solution to Problem 2. Besides, (10)-(18) is a Lyapunov-Krasovskii functional that assures the robust stability of the closed-loop system.

Proof: The proof can be obtained from Theorem 1 defining $F_1 = F$, $G_1 = H_1 = M_1 = N_1 = R_1 = 0$, replacing $\tilde{A}_i$ and $\tilde{A}_{di}$ by $(A_i + B_iK)'$ and $(A_{di} + B_iK_d)'$, respectively, and defining $FK' = W$ and $FK_d' = W_d$. ■

For the synthesis, it is also possible to present a delay-independent condition that allows a delay variation $|\Delta d_k| \le \bar{d} - \underline{d}$, $\Delta d_k = d(k+1) - d(k)$.

Corolary 2: If there exist matrices $P_i = P_i' > 0$, $Q_i = Q_i' > 0$, $i = 1, \dots, N$, and G_0, H_0, S_0, F, F_2, G_2, H_2, W and W_d, with compatible dimensions, such that the LMIs (33) can be verified, with known $\beta = \bar{d} - \underline{d} + 1$, $\underline{d}$ and $\bar{d}$, then the system (7) subject to (5) and (8) is robustly stabilizable, for any value of the delay $d(k)$, by static robust feedback gains K and K_d given in (31). In this case, the delay can be time-variant, if $|\Delta d_k| \le \beta - 1$. Besides, $V(\alpha, k) = \sum_{v=1}^{3} V_v(\alpha, k) > 0$ with V_v, $v = 1, \dots, 3$, given in (11)-(13) and matrices $P(\alpha)$ and $Q(\alpha)$ given in (16) with $\alpha \in \Omega$, is a Lyapunov-Krasovskii functional to the system (7).

Remark 1*:* An important aspect of the presented proposal is that the results can be used to deal with decentralized control by imposing block-diagonal structure to matrices $F = F_D = \text{block-diag}\{F^1, \dots, F^\kappa\}$, $W = W_D = \text{block-diag}\{W^1, \dots, W^\kappa\}$, $W_d = W_{dD} = \text{block-diag}\{W_d^1, \dots, W_d^\kappa\}$ where κ denote the number of subsystems defined. In this case, one gets the robust block-diagonal state feedback gains $K_D = W_D'(F_D')^{-1}$ and $K_{dD} = W_{dD}'(F_D')^{-1}$. In this case, the matrices of the Lyapunov-Krasovskii functional, $P(\alpha)$, $Q(\alpha)$ and $Z(\alpha)$, do not have any restrictions in their structures, which results in a design less conservative.

Remark 2*:* Quadratic stability conditions can be obtained from all conditions proposed here by imposing $P_i = P$, $Q_i = Q$ and $Z_i = Z$, $i = 1, \dots, N$. In this case, the system matrices can be time-varying encompassing systems subject to actuators faults.

Remark 3*:* The numerical complexity of the proposed conditions depend on the number of variables, $\mathcal{K}$, and on the number of rows in the LMIs, $\mathcal{L}$. Using the *LMI Control Toolbox*, the number of floating point operations necessary in the solution of the LMIs problems has an order

$$\Psi_i \equiv \begin{bmatrix} P_i + F' + F - (F_2 + F_2)' & F_2 - G_2' - FA_i' - WB_i' & -(FA_{di}' + W_d B_i' + H_2') & F_2 - M_2' & -N_2' & -R_2' & \mathbf{0} \\ \star & G_2 + G_2' + \beta Q_i - P_i + G_0 + G_0' & H_0' - G_0 + H_2' & G_2 + M_2' & N_2' & R_2' & S_0' - G_0 \\ \star & \star & -(Q_i + H_0 + H_0') & H_2 & \mathbf{0} & \mathbf{0} & -(S_0' + H_0) \\ \star & \star & \star & M_2 + M_2' + (\bar{d}+1)Z_i & N_2' & R_2' & \mathbf{0} \\ \star & \star & \star & \star & -Z_i & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -Z_i & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -(S_0' + S_0) \end{bmatrix} < 0, \quad i = 1 \dots, N \tag{32}$$

$$\Xi_i \equiv \begin{bmatrix} P_i + F' + F - F_2 - F_2' & -G_2' - FA_i' - WB_i' + F_2 & -FA_{di}' - W_d B_i' - H_2' & \mathbf{0} \\ \star & G_2 + G_2' + \beta Q_i - P_i + G_0 + G_0' & H_0' - G_0 + H_2' & S_0' - G_0 \\ \star & \star & -Q_i - H_0 - H_0' & -S_0' - H_0 \\ \star & \star & \star & -S_0' - S_0 \end{bmatrix} < 0, i = 1, \dots, N \tag{33}$$

given by $\mathcal{K}^3\mathcal{L}$ [5]. Using the program SeDuMi [18], the number of floating point operations performed to the same problems has an order given by $\mathcal{K}^2\mathcal{L}^{5/2} + \mathcal{L}^{7/2}$. Then, the conditions present here have $\mathcal{K}_{\text{T1}} = \frac{3n[n(10+N)+N]}{2}$, $\mathcal{K}_{\text{T2}} = \frac{n^2(3N+20)+n(3N+\ell)}{2}$, $\mathcal{K}_{\text{C1}} = n[n(9+N)+N]$, $\mathcal{K}_{\text{C2}} = n^2(N+7) + n(N+\ell)$ scalars variables and $\mathcal{L}_{\text{T1}} = \mathcal{L}_{\text{T2}} = 9Nn$ and $\mathcal{L}_{\text{C1}} = \mathcal{L}_{\text{C2}} = 6Nn$ rows.

IV. Numeric Examples

In all examples it has been used an Intel Core 2 Duo $T7600$, 2.33 GHz processor with 2 Gb of RAM and the *LMI Control Toolbox* [5]. The results achieved are compared with other obtained by conditions available in the literature.

Example 1: Consider the discrete-time system with delay, given by (7), presented in [9] where the polytope vertices $\mathcal{P}$ are given by

$$[\tilde{A}|\tilde{A}_d]_1 = \left[\begin{array}{cc|cc} -0.545 & -0.43 & 0.24 & 0.07 \\ 0.185 & -0.61 & -0.12 & 0.09 \end{array}\right] \tag{34}$$

$$[\tilde{A}|\tilde{A}_d]_2 = \left[\begin{array}{cc|cc} -0.455 & -0.37 & 0.36 & 0.13 \\ 0.215 & -0.59 & -0.08 & 0.11 \end{array}\right] \tag{35}$$

In [9], the delay-independent robust stability is assured for a constant delay. By using conditions of Corollary 1, it is possible to verify that this system has a delay-independent robust stability for a *time-varying* satisfying $|d(k+1) - d(k)| \leq 15$.

Example 2: Consider the precisely known discrete-time system with delayed states studied in [10], that is described by (7) with

$$\left[\tilde{A}|\tilde{A}_d\right] = \left[\begin{array}{cc|cc} 0.6 & 0 & 0.1 & 0 \\ 0.35 & 0.7 & 0.2 & 0.1 \end{array}\right] \tag{36}$$

In [10] this system is identified as stable for $2 \leq d(k) \leq 13$. By using conditions of Theorem 1 it is verified that this system is robustly stable for $2 \leq d(k) \leq 14$. This shows that Theorem 1 can yield less conservative results.

Example 3: Suppose that the system investigated in Example 2 is affected by an uncertain parameter which leads to a 2 vertex polytopic representation. The first polytope is defined by (36) and the other one is given by $1.1[\tilde{A}|\tilde{A}_d]$. Using the conditions of Theorem 1 it is possibly to verify the robust stability for $2 \leq d(k) \leq 5$. Note that, due the presence of the uncertain parameter, some recent results in the literature, such as [10] and [1] cannot be applied in this case.

Example 4: This example use the system studied in [12] which is associated to a combined production and marketing process modeled as $x_{k+1} = A(\alpha)x_k + A_d(\alpha)x_{k-d(k)} + B(\alpha)u_k + B_d(\alpha)u_{k-d(k)}$. In [12] the system matrices are given by

$$[A|A_d]_n = \left[\begin{array}{cccc|cccc} 0.7 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ -0.7 & 0 & 1 & 0 & 0 & -0.2 & 0 & 0 \\ 0 & -0.5 & 0 & 1 & -0.1 & 0 & 0 & 0 \end{array}\right]$$

$$[B|B_d] = \theta \left[\begin{array}{cccc|cccc} 0 & 0 & 8 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 7 & 0 & 0 & 3 & 0 \\ 1 & 0 & -8 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & -7 & 0 & 0 & -3 & 0 \end{array}\right]$$

and θ is assumed to be 0.5 or 1, characterizing a switched system with an uncertain but constant delay. Here, this system is considered as uncertain with $0.5 \leq \theta \leq 1$, and $[A|A_d](\rho) = \rho[A|A_d]_n$, with $0.5 \leq \rho \leq 1$ and having a time-varying delay. The uncertain parameters θ and ρ define a polytope with 4 vertices given by the combination of their extremum values. Note that the additional term $B_d(\alpha)u_{k-d(k)}$ can be taken into account by the conditions presented here replacing B_i by B_{di} in the entries $(1,3)$ and $(3,1)$ of the LMIs (32) and (33). Three different conditions are investigated for this system. In the first one, a linear search is performed on $\bar{d}$ to find the maximum value of this bound such that the system is stabilizable by a full gain K, i.e., with $K_d = \mathbf{0}$. The maximum value achieved by conditions of Theorem 2 is $1 \leq d(k) \leq 31$. In the second condition investigated, a pair of robust state feedback gains K and K_d are design while a linear search on $\bar{d}$ is performed. In this case it has been determined the bounds $1 \leq d(k) \leq 96$. The last investigated condition consists in to find the maximum delay bound $\bar{d}$ such that the considered system is robustly stabilizable with K and K_d, both block-diagonal with two 2×2 blocks. In this case, it is found $1 \leq d(k) \leq 6$, with robust state feedback gains

$$K = \text{block-diag}\left\{ \begin{bmatrix} \frac{168}{661} & \frac{-19}{8538} \\ \frac{-19}{74149} & \frac{2939}{12736} \end{bmatrix}, \begin{bmatrix} \frac{55}{5498} & \frac{2}{170423} \\ \frac{6}{206279} & \frac{157}{21368} \end{bmatrix} \right\} \quad (37)$$

$$K_d = \text{block-diag}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{14}{29705} & \frac{-3}{597601} \\ \frac{-1}{159868} & \frac{62}{59317} \end{bmatrix} \right\} \quad (38)$$

These robust state feedback gains are used to simulated the considered uncertain systems with initial conditions, equation (2), given by $\phi(k) = [1,\ -1,\ -1,\ 1]'$, $-6 \leq k \leq 0$. In Figure 1 it is shown the behaviors of the 4 states of the system, that has been simulated for 4 combinations of α randomly generated and for of α taken at each vertex of the polytope. Figure 2 shows the time varying delay $1 \leq d(k) \leq 6$ used in all simulations.

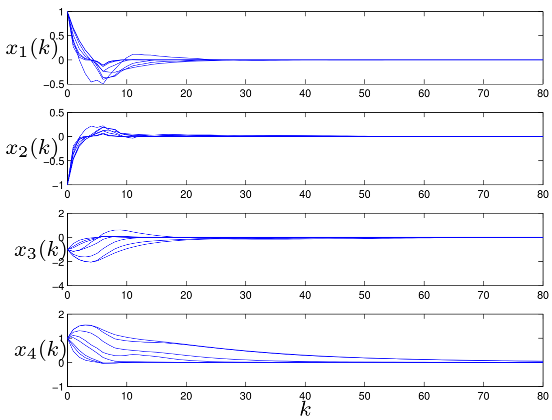

Fig. 1. The behaviors of the states $x_1(k)$ (top) to $x_4(k)$ (bottom), with $1 \leq d(k) \leq 6$ (see Figure 2), with gains given by (37)-(38) and initial conditions $\phi(k) = [1,\ -1,\ -1,\ 1]'$, $k \in [-6,\ 0]$.

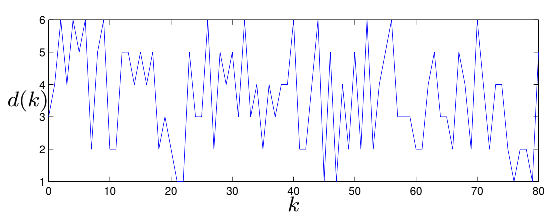

Fig. 2. Time-varying delay used in simulation of Example 4.

V. Conclusions

Delay-dependent and delay-independent convex conditions for both, robust stability and robust stabilizability of uncertain discrete-time systems with time-varying delay were given. The uncertainties are considered in a polytopic description and affecting all system matrices. The proposed conditions employ parameter dependent Lyapunov-Krasovskii functionals and extra-variables yielding less conservative results than other conditions available in the literature. All conditions in this paper encompass quadratic stability based ones as a special cases. It has been shown that robust stabilizability conditions can be easily used to design decentralized state feedback gains, since the design equations are convex. Time simulations are included.

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