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Decentralized Control Based on the Value of Information in Large Vehicle Arrays

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***Abstract*— An approach for determining the value of information in a dynamic network is formulated as a data rate optimization which reduces multi-criteria linear-quadratic-Gaussian (LQG) controllers with centralized information to controllers with distributed or decentralized information. The dynamic network can be an array of vehicles, but the theory is more general. Two essential notions are used to decompose a centralized controller for each vehicle. First, given a quadratic cost criterion for each vehicle, a LQG Nash equilibrium is established where each controller has access to *all* the information in the network. The resulting optimal deterministic cost criteria become functions of the system parameters and statistics. Secondly, by augmenting these criteria with costs associated with transmitting the measurement and control values within the network, data rate parameters, bounded between zero and one, are determined through a deterministic multi-criteria optimization. If the data rate associated with a measurement is zero, then that measurement is not transmitted and no longer used in the local state estimator. If the control used by a vehicle is *not* transmitted, then that control can only be constructed from the local estimates. To simplify the computations, a static optimization problem is also suggested to obtain suboptimal solutions. Examples composed of a string of vehicles are presented. For simplification only data rates of zero or one are considered. The results of the optimization show that the control values should not be transmitted when a minimal number of measurements are used. It appears that with this minimal number of measurements, *a priori* knowledge of the control structure produces observable and detectable systems when no control is transmitted, whereas the knowledge of the control value and the minimal measurement set leads to an unobservable system and therefore, system instability.**

***Index Terms*— Decentralized control, limiting information**

I. Introduction

Consider a formation of a large number of vehicles. Absolute and relative position, velocity and attitude information of all the vehicles in the formation is useful to maintain the formation within the closest tolerances, but all these measurements may not be needed to meet the system requirements. For example, it may be useful to have the absolute and relative position, velocity and attitude information of the vehicles near by to be transmitted at high data rates, but not necessarily for those vehicles that are remote where the transmission data rates could be significantly reduced. However, it has been shown in [1] that string stability of a platoon of vehicles requires only that each individual vehicle know the relative position of the vehicle just in front of it and the absolute velocity of the lead vehicle. Although string stability was shown, no formal method was developed to indicate that this is the minimal amount of information in some sense. Our objective is to develop a methodology applicable to more complex formations for determining the minimal information structure required in a swarm of vehicles.

Techniques for determining decentralized controllers for large systems are found in [2]. However, to determine the minimal information structure required in a swarm of vehicles, a methodology is sought. Our procedure assumes that the controllers are affine. It is known [3] that even for the LQG problem the denial of information leads to optimal controllers that are nonlinear. Our notion is to determine a procedure that only removes information which modestly degrades performance. Another scheme that assumes linear controllers puts an explicit constraint on the information allowed [4], but obtains explicit gains under a condition called quadratic invariance. Although determining the controller gains could be part of the optimization process, in this paper we only limit the transmission of the measurement and control values. Therefore, if only subsets of the information are required, then this methodology allows a decomposition of the centralized control system to a set of decentralized systems, possibly overlapping. This decomposition appears possible for swarms because of the structure of the dynamic system and the measurements, which are essential in determining a decomposition. The dynamics for each vehicle in the swarm are usually uncoupled from all the other vehicles. The syntheses of the controller, based on the measurement structure and the performance criterion, form the interconnection or coupling among all the vehicles. The design of controllers based on all information in the formation leads to complex communication and costly instrumentation systems. Based on the particular characteristics of the formation, a procedure that forms a minimal information design is to be developed.

In Section II the system dynamics, measurement, and LQG cost criterion for each of the vehicles are formulated. In Section III a multi-criteria LQG Nash equilibrium solution is obtained assuming that all the measurements and controls are transmitted over the array. In Section IV the notion of measurement and control data rates are introduced to represent the information flow in the dynamic network. By optimizing the cost of each vehicle with respect to the data rates of the measurements and controls, the measurement and control transmissions are determined. In Section V a static optimization methodology is proposed and in Section VI two simple examples illustrate the approach. We conclude

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in Section VII.

II. Formulation

In the following a control problem for vehicle arrays based on linear dynamics, an expectation of a quadratic performance index, and additive Gaussian noise for each vehicle is formulated. Consider a dynamic system composed of M vehicles

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^{M} B_j \tilde{u}_j + \Gamma w \\ y &= Cx + v \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$ is the state vector of the M vehicle system, $\tilde{u}_j \in \mathcal{R}^{m_j}$, $j = 1 \cdots M$ is the control vector of the j^{th} vehicle, $w \in \mathcal{R}^p$ is the zero mean Gaussian white process noise with power spectral density W, $y \in \mathcal{R}^q$ is the measurement vector, and $v \in \mathcal{R}^q$ is the zero mean Gaussian white measurement noise with power spectral density V, assumed to be independent between measurements. Let Y_t be the measurement history

$$Y_t \stackrel{def}{=} \{y(s) : 0 \leq s \leq t\}$$

In order to structure the problem, we note that the matrices A, B_j, Γ may decompose into M distinct systems. The elements A may represent a coupling among the vehicles. However, here we have only assumed that the control through B_j affects only one vehicle.

The following cost criterion J_i, developed for the i^{th} vehicle, is given as

$$\begin{aligned} J_i &= E\left[E\left[\frac{1}{2}\int_0^{\mathrm{T}} \{x^{\mathrm{T}} Q_i x + \tilde{u}_i^{\mathrm{T}} R_i \tilde{u}_i\} dt | Y_t\right]\right] \\ &= E\left[\frac{1}{2}\int_0^{\mathrm{T}} \{\hat{x}^{\mathrm{T}} Q_i \hat{x} + \tilde{u}_i^{\mathrm{T}} R_i \tilde{u}_i + \mathrm{tr}(Q_i P)\} dt\right] \end{aligned} \tag{2}$$

where $E[\cdot]$ is the expectation operator, $E[\cdot|Y_t]$ is the conditional expectation operator, $\hat{x} \stackrel{def}{=} E[x|Y_t] \in \mathcal{R}^n$ is the conditional mean, the error variance is $P \stackrel{def}{=} E[(x-\hat{x})(x-\hat{x})^{\mathrm{T}}|Y_t] \in \mathcal{R}^{n \times n}$, $Q_i \stackrel{def}{=} D_i^{\mathrm{T}} D_i$ and $R_i > 0$ are the weighting on the state estimate and control, respectively. The dynamic system may be simplified by using the minimal realization of the triple (D_i, A, B_i). Although our approach applies to vehicle systems with block diagonal A and controls that effect one vehicle so that the system coupling comes only from the cost criteria and the measurements, all that needs to be assumed is that (A, B, C) be a minimal realization.

The conditional mean $\hat{x}$ and the error variance P are associated with the centralized estimator where each vehicle has access to all the information and the controls used in the vehicle array. The centralized estimator and associate error variance are

$$\dot{\hat{x}} = A\hat{x} + \sum_{j=1}^{M} B_j \tilde{u}_j + PC^{\mathrm{T}} V^{-1}(y - C\hat{x}), \tag{3}$$

$$\dot{P} = AP + PA^{\mathrm{T}} + \Gamma W \Gamma^{\mathrm{T}} - PC^{\mathrm{T}} V^{-1} CP. \tag{4}$$

Our approach is to first solve the centralized stochastic multiple criteria LQG problem (Section III). The cost criteria can then be written in terms of the statistics of the centralized estimator. However, each vehicle does not need access to all the data to perform almost optimally. Therefore, in Section IV by augmenting the cost criterion (2) of each vehicle by a cost imposed on the data rate among vehicles, a multiple criteria optimization problem is formulated in the system statistics.

III. Multi-Criteria Controller with the Centralized Estimator

We suggest below the following multi-person optimization with respect to $\tilde{u}_i(Y_t)$, but with centralized information, i.e. all measurement and control values are transmitted. For this centralized information problem, the dynamic programming approach to the deterministic LQG Nash equilibrium control problem of [5] is extended to include the stochastic LQG Nash equilibrium control problem. In Section IV the notion of data rates is introduced which are used to limit the transmission of information. Once the controller is determined for the centralized information problem, the data rates are allowed to vary and become control variables for a new matrix deterministic Nash equilibrium problem.

Let the optimal value of $\tilde{u}_i$ be $\tilde{u}_i^0$ and the i^{th} cost criterion be defined in (2). Then, the Nash equilibrium is determined from the following person-by-person optimization:

$$\begin{aligned} &\min\nolimits_{\tilde{u}_1(Y_t)} J_1(\tilde{u}_1,\, \tilde{u}_2^0 \cdots, \tilde{u}_M^0) \\ &\qquad\vdots \\ &\min\nolimits_{\tilde{u}_M(Y_t)} J_M(\tilde{u}_1^0, \cdots \tilde{u}_{M-1}^0,\, \tilde{u}_M) \end{aligned} \tag{5}$$

subject to

$$\dot{\hat{x}} = A\hat{x} + \sum_{j=1}^{M} B_j \tilde{u}_j + PC^{\mathrm{T}} V^{-1}(y - C\hat{x}) \tag{6}$$

$$\dot{P} = AP + PA^{\mathrm{T}} + \Gamma W \Gamma^{\mathrm{T}} - PC^{\mathrm{T}} V^{-1} CP \tag{7}$$

Note that all the controls are required in the estimator, but it is only each local control $\tilde{u}_i$ that enters into the local optimization.

The Nash equilibrium strategy set is defined [5] by satisfying (5), which implies person by person optimality for a cooperative game. For the LQG stochastic dynamic programming extension of [5] the value function for the i^{th} player is defined as the piecewise differentiable function $V_i(\hat{x}, t) = \frac{1}{2}\hat{x}^{\mathrm{T}} S_i \hat{x} + a_i(t)$ which satisfies the generalized Hamilton-Jacobi equation for the LQG problem

$$\begin{aligned} -V_{it} = \min_{\tilde{u}_i} [&\frac{1}{2}\hat{x}^{\mathrm{T}} Q_i \hat{x} + \frac{1}{2} u_i^{\mathrm{T}} R_i u_i + \\ &V_{i\hat{x}}(A\hat{x} + \sum_{j=1}^{M} B_i u_j) + \\ &\mathrm{tr}(V_{i\hat{x}\hat{x}} PC^{\mathrm{T}} V^{-1} CP)],\ i = 1, \cdots, M \end{aligned} \tag{8}$$

After substituting $V_i(\hat{x}, t) = \frac{1}{2}\hat{x}^{\mathrm{T}} S_i \hat{x} + a_i(t)$ into (8) and minimizing with respect to $\tilde{u}_i$, the optimal i^{th} controller is

$$\tilde{u}_i^0 = -R_i^{-1} B_i^{\mathrm{T}} S_i \hat{x} \stackrel{def}{=} -G_i \hat{x},\ i = 1, \cdots, M \tag{9}$$

where S_i is the i^{th} solution to the coupled Riccati equation determined by substitution of (9) into (8) as

$$\begin{aligned}-\dot{S}_i =& Q_i + S_i B_i R_i^{-1} B_i^{\mathrm{T}} S_i + \\ & S_i A + A^{\mathrm{T}} S_i - \sum_{j=1}^{M} (S_j B_j R_j^{-1} B_j^{\mathrm{T}} S_i + \\ & S_i B_j R_j^{-1} B_j^{\mathrm{T}} S_j),\ i = 1, \cdots, \end{aligned} \tag{10}$$

$$-\dot{a}_i = \mathrm{tr}(S_i P C^{\mathrm{T}} V^{-1} C P),\ i = 1, \cdots, M \tag{11}$$

Substituting (9) into the cost criterion (2) and taking the expectation produces optimal deterministic cost criterion in the variances of x, $\hat{x}$, and $e = x - \hat{x}$ as $X = E[xx^{\mathrm{T}}]$, $\hat{X} = E[\hat{x}\hat{x}^{\mathrm{T}}]$ and the estimation error variance P, respectively, as

$$\begin{aligned} J_i^o &= \frac{1}{2} \int_0^{\mathrm{T}} \mathrm{tr}\{(Q_i + S_i B_i R_i^{-1} B_i^{\mathrm{T}} S_i)\hat{X} + Q_i P\} dt, \\ &= \frac{1}{2} \int_0^{\mathrm{T}} \mathrm{tr}\{Q_i X + S_i B_i R_i^{-1} B_i^{\mathrm{T}} S_i \hat{X}\} dt, \\ i &= 1, \cdots, M \end{aligned} \tag{12}$$

The cost criterion (12) can be manipulated into a somewhat simpler form by adding to it the identically zero quantity (Note that $S_i(\mathrm{T}) = 0$)

$$-\mathrm{tr}(S_i(0)X(0)) = \int_0^{\mathrm{T}} \mathrm{tr}(\dot{S}_i X + S_i \dot{X}) dt \tag{13}$$

where the state covariance is propagated as

$$\begin{aligned} \dot{X} = & (A - \sum_{j=1}^{M} B_j R_j^{-1} B_j^{\mathrm{T}} S_j)\hat{X} + \\ & \hat{X}(A - \sum_{j=1}^{M} B_j R_j^{-1} B_j^{\mathrm{T}} S_j)^{\mathrm{T}} + \\ & AP + PA^{\mathrm{T}} + \Gamma W \Gamma^{\mathrm{T}} \end{aligned} \tag{14}$$

The deterministic cost criterion (12) for centralized information becomes

$$\bar{J}_i^o = \mathrm{tr}(S_i(0)X(0)) + \int_0^{\mathrm{T}} \bar{L}_i^o dt,\ i = 1, \cdots, M \tag{15}$$

where the Lagrangian is

$$\begin{aligned} \bar{L}_i^o = \mathrm{tr}[& \sum_{j=1}^{M} (P S_j B_j R_j^{-1} B_j^{\mathrm{T}} S_i + \\ & S_i B_j R_j^{-1} B_j^{\mathrm{T}} S_j P) - \\ & S_i B_i R_i^{-1} B_i^{\mathrm{T}} S_i P + S_i \Gamma W \Gamma^{\mathrm{T}}] \end{aligned} \tag{16}$$

It should be emphasized that a Nash equilibrium is obtained as the solution, which is a person-by-person optimum. However, the choice of Q_i can produce either a very coordinated array or relaxed coordination.

IV. Optimal Data Rates for Measurement and Control Information Flow

In this section we first define the notion of data rates for the transmission of the measurements and controls. Using these definitions, a multiple person-by-person optimization is formulated in the statistics of the system dynamics. The controllers using centralized information are decomposed into local controllers whose information has been restricted to that which is the most valuable. Since we retain the centralized controller structure, this solution will not produce the optimal decentralized controllers. For the problem described below, the data rates are assumed to be functions of time and not a function of the estimated states as in the centralized information LQG Nash equilibrium control problem. Finally, necessary conditions for optimality, which can lead to a numerical method for obtaining the optimal data rates, are given.

A. Measurement Data Rates

To determine the value of the transmission of a measurement, a new measurement for the i^{th} vehicle is defined as a summation of transmitted and estimated data

$$y_i \stackrel{def}{=} \mu_i y + (I - \mu_i)\hat{y}_i \tag{17}$$

where $\mu_i, i = 1 \cdots M,$ is a $q \times q$ diagonal matrix and $\hat{y}_i$ is the estimated measurements ($\hat{y}_i = C\hat{x}_i$). The essential innovation is the introduction of the time-varying parameters μ_i in which each diagonal element j of μ_i associated with all the measurements is bounded as $0 \le \mu_i^j \le 1$.

- If $\mu_i^j = 1$, then the associate measurement is important to the performance of controller i.
- If $\mu_i^j = 0$, then the associated measurement is *not* important to the performance of controller i.
- If μ_i^j takes on interior values, then that measurement is taken intermittently, depending on the value of μ_i^j and can be associated with the data rate.

A cost is imposed on the data rate among vehicles by augmenting the cost criterion of each vehicle, as will be detailed in Section IV-C.

B. Control Data Rates

In a manner similar to (17), the transmission of the controls can be structured as

$$\tilde{u}_j^i \stackrel{def}{=} \alpha_i^j \tilde{u}_j(Y_j) + (I - \alpha_i^j)\tilde{u}_j(Y_i) \tag{18}$$

where α_i^j is a $m_j \times m_j$ diagonal matrix and Y_i is the information at the i^{th} vehicle used to construct its best estimate of the control used by vehicle j. The diagonal elements of α_i^j are to be determined within the interval [0, 1]. If a diagonal element of α_i^j is one, we use the control from agent j in the estimation of $\hat{x}_i$. On the other hand if a diagonal element of α_i^j is zero, then the control for agent j is not transmitted and we estimate it using the state estimate from vehicle i. In particular, in (18) $\tilde{u}_j(Y_j) = -G_j\hat{x}_j$ and $\tilde{u}_j(Y_i) = -G_j\hat{x}_i$ are used where G_j is the control gain defined in (9).

C. Formulation of the Data Rate Optimization Problem

To reduce the amount of information shared among agents, we formulate an optimization problem that simultaneously optimizes both the measurement and the control transmissions among agents.

Consider the following system dynamics

$$\begin{aligned}
\dot{x} &= Ax + \sum_{j=1}^{M} B_j \tilde{u}_j(Y_j) + \Gamma w \\
\dot{\hat{x}}_i &= A\hat{x}_i + \sum_{j=1}^{M} B_j \tilde{u}^i_j + K_i(y_i - \hat{y}_i) \\
&= A\hat{x}_i + \sum_{j=1}^{M} B_j \tilde{u}^i_j + K_i\{\mu_i y + (1-\mu_i)\hat{y}_i \\
&- \mu_i \hat{y}_i - (1-\mu_i)\hat{y}_i\} \\
&= A\hat{x}_i + \sum_{j=1}^{M} B_j \tilde{u}^i_j + K_i\mu_i(y - \hat{y}_i) \\
&= A\hat{x}_i + \sum_{j=1}^{M} B_j \tilde{u}^i_j + K_i\mu_i(Cx + v - C\hat{x}_i)
\end{aligned}$$

where Y_j is the information available at the j^{th} vehicle and used to construct its local control and $\tilde{u}^i_j$ is the estimate of the control in the j^{th} vehicle used in the estimator of the i^{th} vehicle and is defined in (18). K_i is the approximate Kalman gain, $K_i = P_{ii}C^{\mathrm{T}}V^{-1}$, and $\hat{x}_i = \hat{x}_i(Y_i)$ is the state estimate with respect to the measurements history of vehicle i. P_{ii} is computed from an error variance associated with the enlarged system with restricted information flow, defined below.

Defining $e_i \overset{def}{=} x - \hat{x}_i$, we get:

$$\begin{aligned}
\dot{e}_i =& Ae_i + \sum_{j=1}^{M} B_j \tilde{u}_j(Y_j) \\
&- \sum_{j=1}^{M} B_j \tilde{u}^i_j + \Gamma w - K_i\mu_i C e_i - K_i \mu_i v
\end{aligned} \tag{19}$$

Construct the control at vehicle j to be used in the filter of vehicle i as given in (18). This yields:

$$\begin{aligned}
\dot{e}_i =& Ae_i + \sum_{j=1}^{M} B_j[\tilde{u}_j(Y_j) - \alpha^j_i \tilde{u}_j(Y_j) \\
&- (I - \alpha^j_i)\tilde{u}_j(Y_i)] + \Gamma w - K_i\mu_i C e_i - K_i\mu_i v \\
=& Ae_i + \sum_{j=1}^{M} B_j \left(I - \alpha^j_i\right)(\tilde{u}_j(Y_j) - \tilde{u}_j(Y_i)) \\
&+ \Gamma w - K_i\mu_i C e_i - K_i\mu_i v\,,\ i = 1, \cdots, M
\end{aligned}$$

Since $\tilde{u}_j(Y_j) = -G_j\hat{x}_j$ and $\tilde{u}_j(Y_i) = -G_j\hat{x}_i$ where G_j is the control gain defined in (9),

$$\begin{aligned}
\dot{e}_i =& Ae_i + \sum_{j=1}^{M} B_j \left(I - \alpha^j_i\right)(-G_j\hat{x}_j + G_j\hat{x}_i) \\
&+ \Gamma w - K_i\mu_i C e_i - K_i\mu_i v \\
=& Ae_i + \sum_{j=1}^{M} B_j \left(I - \alpha^j_i\right) G_j(e_j - e_i) \\
&+ \Gamma w - K_i\mu_i C e_i - K_i\mu_i v
\end{aligned} \tag{20}$$

To get the equation that propagates the $M \cdot n \times M \cdot n$ covariance, $\bar{P}$, we define:

$$\Psi = [\bar{e}\bar{e}^{\mathrm{T}}] = \begin{bmatrix} e_1e_1^{\mathrm{T}} & e_1e_2^{\mathrm{T}} & \dots & e_1e_N^{\mathrm{T}} \\ e_2e_1^{\mathrm{T}} & \dots & \dots & e_2e_N^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \vdots \\ e_Ne_1^{\mathrm{T}} & \dots & \dots & e_Ne_N^{\mathrm{T}} \end{bmatrix} \tag{21}$$

Applying the Itô stochastic differential on an element-by-element basis and collecting all the individual terms:

$$\dot{\Psi} = \dot{\bar{e}}\bar{e}^{\mathrm{T}} + \bar{e}\dot{\bar{e}}^{\mathrm{T}} + \bar{\Gamma}\bar{W}\bar{\Gamma}^{\mathrm{T}} + \bar{K}\bar{V}\bar{K}^{\mathrm{T}} \tag{22}$$

Using the results from (20) and keeping in mind that $\bar{P} = E[\bar{e}\bar{e}^{\mathrm{T}}]$, we get the covariance propagation equation

$$\dot{\bar{P}} = (\bar{A} - \bar{K}\bar{C})\bar{P} + \bar{P}(\bar{A} - \bar{K}\bar{C})^{\mathrm{T}} + \bar{\Gamma}\bar{W}\bar{\Gamma}^{\mathrm{T}} + \bar{K}\bar{V}\bar{K}^{\mathrm{T}} \tag{23}$$

where:

$$\bar{\Gamma}\bar{W}\bar{\Gamma}^{\mathrm{T}} = \text{ones}(M) \otimes \Gamma W \Gamma^{\mathrm{T}},$$

ones(M) denotes an $M \times M$ matrix having all elements as 1,

$$\begin{gathered}
\otimes \text{ denotes the Kroencker product,} \\
\bar{C} = \text{Blockdiag}(C)\,, \quad \bar{V} = \text{Blockdiag}(V), \\
\bar{K} = \text{Blockdiag}(K_1\mu_1, \cdots, K_N\mu_N), \\
K_i\mu_i V \mu_j K_j^T = P_{ii}C^T\mu_i V^{-1}\mu_j C P_{jj}, \\
\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \dots & \bar{A}_{1M} \\ \bar{A}_{21} & \bar{A}_{22} & \dots & \bar{A}_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{M1} & \bar{A}_{M2} & \dots & \bar{A}_{MM} \end{bmatrix}
\end{gathered}$$

with

$$\bar{A}_{aa} = A - \sum_{\substack{j=1 \\ j \neq a}}^{M} B_j(I - \alpha^j_a)G_j$$

$$\bar{A}_{ab} = B_b(I - \alpha^b_a)G_b$$

and

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1M} \\ P_{21} & P_{22} & \dots & P_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ P_{M1} & P_{M2} & \dots & P_{MM} \end{bmatrix}$$

Note that if the centralized filter gain K is used rather than K_i, then (23) is a Lyapunov equation.

The associated information cost is chosen as

$$L^i_y = \sum_{j=1}^{q} \mu^j_i b^j_{y_i} \tag{24}$$

where $\{b_{y_i}^j\}$ are constant weightings for the cost of transmitting measurement information. Similarly, the associated control cost is chosen as

$$L_u^i = \sum_{j=1}^{M} \alpha_i^j b_{u_i}^j \tag{25}$$

where $\{b_{u_i}^j\}$ are constant weightings for the cost of transmitting control information.

The augmented cost function for this problem then becomes:

$$\check{J}_i = \text{tr}(S_i(0)X(0)) + \int_0^{\text{T}} \check{L}_i dt \tag{26}$$

where:

$$\check{L}_i = \bar{L}_i^o + L_y^i + L_u^i \tag{27}$$

and where $\bar{L}_i^o$ is given in (16), but with P replaced by P_{ii}.

This problem needs to be solved by simultaneously minimizing for the cost of sharing measurement information and the cost of sharing control information as

$$\begin{gathered} \underset{\mu_1,\alpha_1}{min}\, \check{J}_1\left(\mu_1, \mu_2^*, \cdots, \mu_M^*, \alpha_1, \alpha_2^*, \cdots, \alpha_M^*\right) \\ \vdots \\ \underset{\mu_M,\alpha_M}{min}\, \check{J}_M\left(\mu_1^*, \cdots, \mu_{M-1}^*, \mu_M, \alpha_1^*, \cdots, \alpha_{M-1}^*, \alpha_M\right) \end{gathered} \tag{28}$$

subject to the dynamic constraint (23). The diagonal matrices $\mu_i \in \mathcal{R}^{q\times q}$ and $\alpha_i^j \in \mathcal{R}^{m_j\times m_j}$, $j = 1, \cdots, M$ are considered variables whose elements have values in the interval [0, 1]. The μ_i are used to determine the measurements that need to be shared among vehicles while the

$$\alpha_i \triangleq \left[\alpha_i^1, \cdots, \alpha_i^{i-1}, \alpha_i^{i+1}, \ldots, \alpha_i^M\right], \, i = 1, \cdots, M$$

are used to determine the controls information that needs to be shared among the vehicles. This person-by-person optimization produces the measurement history Y_i, $i = 1, \cdots, M$ and the control vectors that are to be transmitted to the i^{th} vehicle from all the others. Although the cost criterion (26) for each vehicle i only contains its own μ_i and α_i and not those of the other vehicles, the dynamic constraint (23) does include the other vehicles' μ_j and α_j and therefore, the above person-by-person optimal problem does *not* reduce to M independent optimal control problems. Note that using the costs (24) and (25), the optimization problem (28) is bilinear in μ_i and α_i.

D. Necessary Conditions for Optimality

The optimization problem for solving the Nash solution for $\check{J}_i$'s with respect to the (μ_i, α_i $\forall i$) is difficult. Here, we present a set of necessary conditions for optimality. The difficulty is finding an efficient numerical scheme to satisfy these necessary conditions. The anticipated result will be a set of controllers, each choosing the information it needs for good performance.

The necessary conditions for a Nash equilibrium are obtained in [5] by an extension of the variational methods used in optimal control theory. We are seeking optimal solutions for (μ_i, α_i $\forall i$) as a function of time. In our extension we state the necessary conditions for a Nash equilibrium subject to a matrix differential equation [6]. Therefore, we define the variational Hamiltonian as

$$\begin{aligned} &H_k(\bar{P}, t, \mu_1^*, \cdots, \mu_{k-1}^*, \mu_k, \mu_{k+1}^*, \cdots, \mu_M^*, \alpha_1^*, \cdots, \\ &\alpha_{k-1}^*, \alpha_k, \alpha_{k+1}^*, \cdots, \alpha_M^*, \Lambda_k) \\ &= \check{L}_k + \text{tr}\Lambda_k[(\bar{A} - \bar{K}\bar{C})\bar{P} + \bar{P}(\bar{A} - \bar{K}\bar{C})^{\text{T}} \\ &+ \bar{\Gamma}\bar{W}\bar{\Gamma}^{\text{T}} + \bar{K}\bar{V}\bar{K}^{\text{T}}] \end{aligned} \tag{29}$$

where $\Lambda_k \in \mathcal{R}^{Mn\times Mn}$ is the matrix Lagrange multiplier and μ_i^*, α_i^* are the optimal person by person measurement and control information rate vector values, respectively. A Nash equilibrium trajectory must satisfy, for $k = 1, \cdots M$, the following conditions

$$\begin{gathered} \dot{\bar{P}} = (\bar{A} - \bar{K}\bar{C})\bar{P} + \bar{P}(\bar{A} - \bar{K}\bar{C})^{\text{T}} + \bar{\Gamma}\bar{W}\bar{\Gamma}^{\text{T}} + \bar{K}\bar{V}\bar{K}^{\text{T}} \\ \dot{\Lambda}_k = -\frac{\partial H_k}{\partial \bar{P}}, \quad \Lambda_k(\text{T}) = 0, \\ \min_{\mu_k,\alpha_k} H_k(\mu_1^*, \cdots, \mu_{k-1}^*, \mu_k, \mu_{k+1}^*, \cdots, \mu_M^*, \\ \alpha_1^*, \cdots, \alpha_{k-1}^*, \alpha_k, \alpha_{k+1}^*, \cdots, \alpha_M^*) \end{gathered} \tag{30}$$

The result of this person-by-person optimization process is that every vehicle has its own information set $Y_i \overset{def}{=} \{y_s^i : 0 \le s \le \text{t}\}$ for $i = 1, \cdots, M$ where Y_i is the measurement history available at the i^{th} vehicle by direct measurement or transmission.

V. Static Optimization

Since the variational problem is numerically difficult, a simpler problem to solve first would be the static problem where the Lagrangians $\check{L}_i$, $i = 1, \cdots, M$ (26) are person-by-person minimized subject to the steady state variance algebraic constraints, where the left hand side of (23) is set equal to zero. For the measurement and control data rate, $\mu_i, \alpha_i, i = 1, \cdots, M$, the multi-person static deterministic optimization is

$$\begin{aligned} &\min_{\mu_i,\alpha_i} \check{L}_i(\mu_1^*, \cdots, \mu_{i-1}^*, \mu_i, \mu_{i+1}^*, \cdots, \mu_M^*, \\ &\alpha_1^*, \cdots, \alpha_{i-1}^*, \alpha_i, \alpha_{i+1}^*, \cdots, \alpha_M^*), \, i = 1, \cdots, M \end{aligned} \tag{31}$$

subject to

$$\begin{aligned} &0 = (\bar{A} - \bar{K}\bar{C})\bar{P} + \bar{P}(\bar{A} - \bar{K}\bar{C})^{\text{T}} + \bar{\Gamma}\bar{W}\bar{\Gamma}^{\text{T}} + \bar{K}\bar{V}\bar{K}^{\text{T}} \\ &i = 1, \cdots, M. \end{aligned} \tag{32}$$

An example of this static optimization is given below.

VI. Simulation Examples

Two examples are presented. The first considers two carts in a platoon and then, four carts in a platoon formation. The objective of these two examples are to reveal some of the special characteristics that occur in this class of optimization problems. To simplify the optimization, only the upper (one) and lower (zero) bounds of μ_i^j and α_i^j are considered in the static optimization problem of Section V.

A. Two Vehicle System

Consider a system composed of two vehicles traveling in a single file. The state vector is $x = [d \quad v_1 \quad v_2]^{\mathrm{T}}$.
d denotes relative position between 1^{st} and 2^{nd} vehicle $(i.e.\ d = p_1 - p_2)$
v_1 is the velocity of the first vehicle
v_2 is the velocity of the second vehicle
Measurements are the entire vector x. Each vehicle controls its own acceleration. Thus, system dynamics in relative position and velocity useful for LQG Nash equilibrium computation are

$$\frac{d}{dt}\begin{bmatrix} d \\ v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} d \\ v_1 \\ v_2 \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$y = \underbrace{I_{3\times 3}}_{C} \begin{bmatrix} d & v_1 & v_2 \end{bmatrix}^T + v$$

The system dynamics for the two vehicle system are

$$\begin{aligned} \dot{x} &= Ax + B_1\tilde{u}_1(y_1) + B_2\tilde{u}_2(y_2) + \Gamma w \\ \dot{\hat{x}}_1 &= A\hat{x}_1 + B_1\tilde{u}_1^1 + B_2\tilde{u}_2^1 + K_1\mu_1(Cx + v - C\hat{x}_1) \\ \dot{\hat{x}}_2 &= A\hat{x}_2 + B_1\tilde{u}_1^2 + B_2\tilde{u}_2^2 + K_2\mu_2(Cx + v - C\hat{x}_2) \end{aligned}$$

The process and measurement noises are Gaussian, $v \sim N(0, V)$, $w \sim N(0, W)$ and the matrices Γ, C, W and V are all taken to be identity matrices of the appropriate dimensions. By definition, $e_i = x - \hat{x}_i, i = 1, 2$, we get:

$$\begin{aligned} \dot{e}_1 &= [A - B_2(1-\alpha_1^2)G_2 - K_1\mu_1 C]e_1 \\ &\quad + B_2(1-\alpha_1^2)G_2 e_2 + \Gamma w - K_1\mu_1 v \\ \dot{e}_2 &= B_1(1-\alpha_2^1)G_1 e_1 + [A - B_1(1-\alpha_2^1)G_1 \\ &\quad - K_2\mu_2 C]e_2 + \Gamma w - K_2\mu_2 v \end{aligned}$$

By applying It$\hat{o}$ stochastic differential, we get the covariance propagation equation

$$\dot{\bar{P}} = (\bar{A} - \bar{K}\bar{C})\bar{P} + \bar{P}(\bar{A} \ - \bar{K}\bar{C})^T + \bar{\Gamma}\bar{W}\bar{\Gamma}^T + \bar{K}\bar{V}\bar{K}^T$$

where,

$$\bar{A} = \begin{bmatrix} A - B_2(1-\alpha_1^2)G_2 & B_2(1-\alpha_1^2)G_2 \\ B_1(1-\alpha_2^1)G_1 & A - B_1(1-\alpha_2^1)G_1 \end{bmatrix}$$

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$\bar{K}\bar{V}\bar{K}^T = \begin{bmatrix} K_1 V {K_1}^T & K_1 V {K_2}^T \\ K_2 V {K_1}^T & K_2 V {K_2}^T \end{bmatrix} =$$

$$\begin{bmatrix} P_{11}C^T V^{-1}\mu_1{\mu_1}^T C P_{11} & P_{11}C^T V^{-1}\mu_1{\mu_2}^T C P_{22} \\ P_{22}C^T V^{-1}\mu_2{\mu_1}^T C P_{11} & P_{22}C^T V^{-1}\mu_2{\mu_2}^T C P_{22} \end{bmatrix}$$

and $\bar{K}\bar{C} = \text{Blockdiag}(K_1\mu_1 C, K_2\mu_2 C)$ and $\bar{\Gamma}\bar{W}\bar{\Gamma}^T = \text{ones}(2) \otimes \Gamma W \Gamma^T$

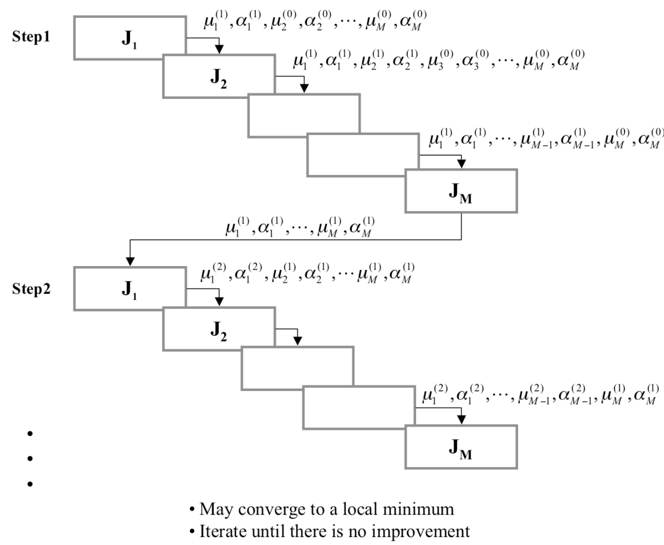

Fig. 1. Measurement and Control Data Rates Optimization Algorithm

The controller gains G_1 and G_2 are

$$G_i = R_i^{-1} B_i^T S_i,\ i = 1, 2$$

where the solution S_i of the coupled Riccati equation of Hamilton-Jacobi equation for the LQG Nash equilibrium, given in (10), are integrated backwards to steady state as

$$S_1 = \begin{bmatrix} 1.2694 & 0.6405 & -0.4603 \\ 0.6405 & 1.4441 & -0.3128 \\ -0.4603 & -0.3128 & 0.6311 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1.2694 & 0.4603 & -0.6405 \\ 0.4603 & 0.6311 & -0.3128 \\ -0.6405 & -0.3128 & 1.4441 \end{bmatrix}$$

where the weights in the cost are chosen as $Q_i = I$ and $R_i = 1$. Estimation gains K_1 and K_2 are computed from a diagonal matrix element P_{ii} of the error variance $\bar{P}$ as

$$K_i = P_{ii} C^T V^{-1} \mu_i, \quad i = 1, 2$$

The procedure for determining the optimal data rates uses the algorithm depicted in Figure 1. The iteration process

Weighting	alpha (α_i)	miu (μ_i)	
$b_{y_i}^j = 0.1,\ \ b_{u_i}^j = 0.1$	$[0 \quad 0]$	$diag[1 \quad 1 \quad 1]$	$[1^{st}\ car]$
	$[0 \quad 0]$	$diag[1 \quad 1 \quad 1]$	$[2^{nd}\ car]$
$b_{y_i}^j = 10,\ \ b_{u_i}^j = 10$	$[0 \quad 0]$	$diag[0 \quad 1 \quad 0]$	
	$[0 \quad 0]$	$diag[1 \quad 0 \quad 1]$	
$b_{y_i}^j = 50,\ \ b_{u_i}^j = 50$	$[0 \quad 0]$	$diag[0 \quad 1 \quad 0]$	
	$[0 \quad 0]$	$diag[1 \quad 0 \quad 0]$	

Weighting	alpha (α_i)	miu (μ_i)
$b_{y_i}^j = 1,\ \ b_{u_i}^j = 10$	$[0 \quad 0]$	$diag[1 \quad 1 \quad 0]$
	$[0 \quad 0]$	$diag[1 \quad 0 \quad 1]$
$b_{y_i}^j = 10,\ \ b_{u_i}^j = 1$	$[0 \quad 0]$	$diag[0 \quad 1 \quad 0]$
	$[0 \quad 0]$	$diag[1 \quad 0 \quad 1]$

Fig. 2. Data Rates from the Optimization of the Two Car Platoon

starts with the unity values of ($\mu_i^{(0)}$, $\alpha_i^{(0)}$, $i = 1, \cdots, M$) used in the centralized controllers. Then, we optimize pairwise for $\mu_i^{(j)}$ and $\alpha_i^{(j)}$ in a manner illustrated in Figure 1. Although convergence is not guaranteed, it appears that convergence was obtained for this problem and the results are shown in Figure 2. The $b_{y_i}^j$'s and $b_{u_i}^j$'s are the values of the weights in (24) and (25), respectively. Note that α_i^j is always chosen as zero for all $b_{y_i}^j$'s and $b_{u_i}^j$'s chosen, suggesting that the control values should, in general, never be transmitted. As the values of $b_{y_i}^j$ and $b_{u_i}^j$ increase, fewer measurements are transmitted. It appears that the minimal measurement transmission (note that it can be local information here) is that the 1^{st} car uses its own velocity information and the 2^{nd} car uses only the relative position ($b_{y_i}^j = 50$, $b_{u_i}^j = 50$). In Figure 3 it is shown that for full measurement transmission, transmitting the control values gives the same value of the error variance as when the control values are not transmitted. However, for measurements resulting from $b_{u_i}^j = 50$ and $b_{u_i}^j = 50$, transmitting the control values leads to instability and not transmitting the control values produces a stable system. The difference is that if the control values are not transmitted, then the *a priori* knowledge of the control structure is explicit in the error variance equation for $\bar{P}$, in particular, $\bar{A}$, and for $\alpha_i^j = 0$ is shown to be stable. In contrast, when $\alpha_i^j = 1$ the propagation of $\bar{P}$ uses A which is only neutrally stable. To better understand

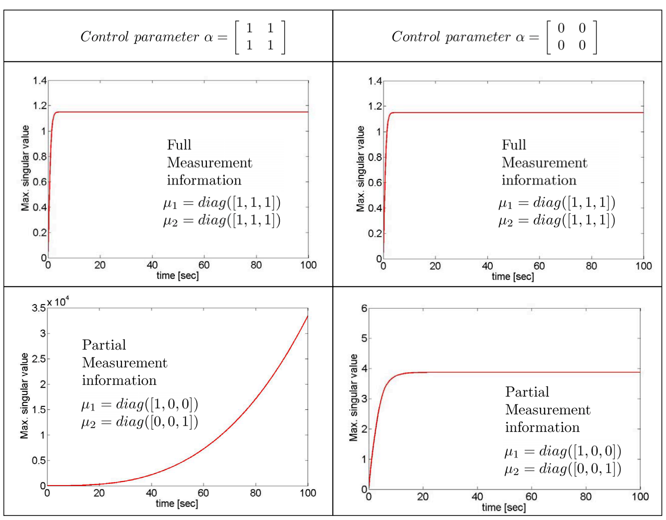

Fig. 3. Two Car Error Variance

the non-intuitive notion that less information (no control transmissions) produces a stable system whereas more information (control transmissions) produces an unstable system, the system observability is analyzed when all the α_i^js are either one or zero for the reduced measurement set of $\mu_1 = \text{diag}[0, 1, 0]$ and $\mu_2 = \text{diag}[1, 0, 0]$. When all the α_i^js are one, then the observability pair $(A, \mu_1 C)$ for car 1 produces an observability matrix of rank 2 and the pair $(A, \mu_2 C)$ for car 2 produces an observability matrix of rank 1. Therefore, the matrix

$$\bar{A} - \bar{K}\bar{C} = \begin{bmatrix} A - K\mu_1 C & 0 \\ 0 & A - K\mu_2 C \end{bmatrix}$$

is unstable for any K. However, if the α_i^js are zero, then the dynamic system for each car is

$$\dot{\tilde{x}}_i = \tilde{A}_i \tilde{x}_i + B_i u_i\,, \quad y_i = \mu_i C x\,,\ i = 1, 2$$

where

$$\tilde{A}_1 = \begin{bmatrix} A & -B_2 G_2 \\ K_2 \mu_2 C & A - B_1 G_1 - B_2 G_2 - K_2 \mu_2 C \end{bmatrix}$$

$$\tilde{A}_2 = \begin{bmatrix} A & -B_1 G_1 \\ K_1 \mu_1 C & A - B_1 G_1 - B_2 G_2 - K_1 \mu_1 C \end{bmatrix}$$

$$\tilde{x}_1 = \begin{bmatrix} x \\ \hat{x}_2 \end{bmatrix}, \quad \tilde{x}_2 = \begin{bmatrix} x \\ \hat{x}_1 \end{bmatrix}$$

For this case where the α_i^js are zero, a necessary condition for the stability of

$$\begin{aligned} \bar{A} - \bar{K}\bar{C} &= A^{(1)} - \begin{bmatrix} K \\ 0 \end{bmatrix} [\mu_1 C \quad 0] \\ &= A^{(2)} - \begin{bmatrix} 0 \\ K \end{bmatrix} [0 \quad \mu_2 C], \end{aligned}$$

where $A^{(i)}$ are similar to $\tilde{A}_i$, is that the observability pair $(\tilde{A}_i,\ \mu_i C)$ be detectable. The similarity transform is based on using $x - \hat{x}_i$ rather than $\hat{x}_i$ in $\tilde{x}_i$. For this two car problem, the observability pair $(\tilde{A}_1,\ \mu_1 C)$ for car 1 is observable and the observability pair $(\tilde{A}_2,\ \mu_2 C)$ for car 2 is *not* observable, but detectable where the observability matrix is rank 1.

B. Four Vehicle System

Let us consider a system composed of four vehicles traveling in a platoon formation. This system is represented by (1). The state vector for this system is $x = [d_1\, d_2\, d_3\, v_1\, v_2\, v_3\, v_4]^{\mathrm{T}}$, where:

d_1 denotes relative position of 1^{st} and 2^{nd} vehicles.
d_2 denotes relative position of 2^{nd} and 3^{rd} vehicles.
d_3 denotes relative position of 3^{rd} and 4^{th} vehicles.
v_1 is the velocity of the first vehicle
v_2 is the velocity of the second vehicle
v_3 is the velocity of the third vehicle
v_4 is the velocity of the fourth vehicle

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and B is a 7×4 matrix where the upper three rows have zero elements and the lower 4×4 block is the identity matrix. The process and measurement noises are Gaussian, $v \sim N(0, V)$, $w \sim N(0, W)$ and the matrices Γ, C, W and V are all taken to be identity matrices of the appropriate dimensions.

We first design the control system for this problem by solving the set of coupled Riccati equations (10) presented

Weighting	alpha (α_i)	miu (μ_i)	
$b_{y_i}^j = 0.1,\quad b_{u_i}^j = 0.1$	$[0\ 0\ 0\ 0]$	$diag[0\ 0\ 0\ 1\ 0\ 0\ 0]$	$[1^{st}\ car]$
	$[0\ 0\ 0\ 0]$	$diag[1\ 1\ 0\ 1\ 1\ 0\ 0]$	$[2^{nd}\ car]$
	$[0\ 0\ 0\ 0]$	$diag[1\ 1\ 1\ 1\ 1\ 1\ 0]$	$[3^{rd}\ car]$
	$[0\ 0\ 0\ 0]$	$diag[1\ 1\ 1\ 1\ 0\ 1\ 1]$	$[4^{th}\ car]$
$b_{y_i}^j = 1,\quad b_{u_i}^j = 1$	$[0\ 0\ 0\ 0]$	$diag[0\ 0\ 0\ 1\ 0\ 0\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[1\ 1\ 0\ 1\ 1\ 0\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[1\ 1\ 0\ 0\ 0\ 1\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[1\ 1\ 1\ 0\ 0\ 0\ 1]$	
$b_{y_i}^j = 10,\quad b_{u_i}^j = 10$	$[0\ 0\ 0\ 0]$	$diag[0\ 0\ 0\ 1\ 0\ 0\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[1\ 0\ 0\ 0\ 1\ 0\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[0\ 1\ 0\ 0\ 0\ 1\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[0\ 0\ 1\ 0\ 0\ 0\ 1]$	
$b_{y_i}^j = 100,\quad b_{u_i}^j = 100$	$[0\ 0\ 0\ 0]$	$diag[0\ 0\ 0\ 1\ 0\ 0\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[1\ 0\ 0\ 0\ 0\ 0\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[0\ 1\ 0\ 0\ 0\ 0\ 0]$	
	$[0\ 0\ 0\ 0]$	$diag[0\ 0\ 1\ 0\ 0\ 0\ 0]$	

Fig. 4. Data Rates from the Optimization of the four vehicle platoon

in section III for a four player problem where these Riccati equations are integrated backwards to steady state. Unity weights for Q_i and R_i in the above equations are chosen. Similar results occur when each vehicle keeps track of its own velocity and its relative position with respect to the vehicle directly ahead.

The results of the optimization process are given in Figure 4 for increasing values of $b_{y_i}^j$ and $b_{u_i}^j$. Again, for all values of $b_{y_i}^j$ and $b_{u_i}^j$, it is optimal not to transmit the control values. The *a priori* shared control structure induces stability as given by $\bar{A}$ for $\alpha_i^j = 0$

VII. CONCLUSIONS

The above formulation is an approach to the resolution of the information transmission over a distributed vehicle system network for a class of LQG problems. We approach this problem by minimizing the information flow within the formation by considering the information needs of each individual vehicle. Data rate variables are established for both the measurement and control transmission. The essential objective is to minimize local cost criteria with respect to these data rate variables so that the information flow can be reduced without significant deterioration in performance. The numerical characteristics of this class of optimal information transmission problems are established, and a simplification is made based on a static optimization. The approach developed here contributes to the beginning of a formal methodology to establish performance measures and procedures for distributed control schemes.

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