2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Modified Anti-windup Compensators for Stable Linear Systems

Solmaz Sajjadi-Kia and Faryar Jabbari

Abstract— **In this paper, we use the Small Gain Theorem to study the stability, and develop appropriate convex L_2 gain LMIs for synthesis, of static anti-windup gains. The main contribution of this paper is investigating the effects of delaying the activation of anti-windup by allowing actuators to remain in the saturated regime longer, on their own. The basic idea is to apply anti-windup when the performance of saturated system faces substantial degradation. For this, we present a modified anti-windup scheme along with the appropriate convex LMIs to obtain the gains. For two well-known examples, we show that the modified anti-windup scheme renders better performance than the case of immediate application of anti-windup.**

I. INTRODUCTION

Input saturation is a persistent concern in applications. Designing high performance feedback algorithms for linear systems with bounded actuators has been one of the major problems in control for decades, where ad-hoc but intuitive techniques had been used to address this problem. However, over the last decade several groups have started to obtain rigorous stability and performance results for linear systems with input saturation. Roughly speaking, two main approaches have been proposed to address saturation. One approach is a class of methods that are aimed at avoiding saturation by considering it a constraint from the very beginning of control design (see e.g., [1]-[3]). Although this concept is appealing, such designs often address worse case behavior which can be needlessly conservative, since in many cases saturation is encountered rarely or briefly. The second approach is the so-called *Anti-Windup* (AW) compensation method (see e.g., [4]-[8]). AW is a two step procedure, in which the original controller is designed without considering the input saturation. Since this controller is likely to saturate, the system is augmented with an AW protection loop to handle saturation. In this paper, we are concerned with providing a modified form of traditional AW for application of linear systems subject to saturation.

In recent years, the structure shown in Fig. 1, where sat(·) represents the saturation nonlinearity, has been adapted by researchers for most AW augmentation schemes. Traditionally, properties of the AW schemes, such as stability guarantees and graceful degradation of performance of systems were addressed through extensive simulations. More recently, by relying on numerical solvers for Linear Matrix Inequalities (LMIs), rigorous techniques, with stability and performance guarantees have been developed for cases where the augmentation considered is static, or dynamic with an order matching that of the plant (see e.g., [5], [6]).

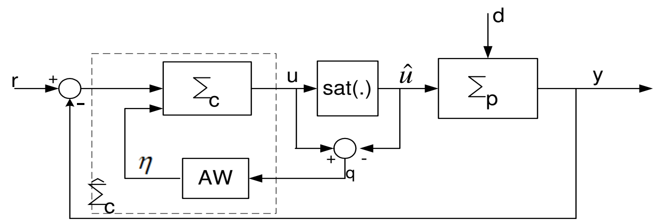

Fig. 1. Standard anti-windup augmentation scheme

In this paper, we consider the static AW augmentation synthesis problem. For this, we use an approach based on the Small Gain Theorem (SGT). The results lead to a variation of LMIs typically used for AW synthesis, though the two approaches are essentially equivalent. The main contribution of this paper, introduced and developed in Section III, is investigating the effect of postponing the activation of AW. The rationale behind this intentional delay can be considered as a tradeoff between the two possible modes of operation: in one, AW is active as soon as saturation is encountered resulting in a stable but low performance controller; On the other hand, if the actuator command is ‘slightly’ or moderately above saturation, the nominal controller acts as a high performance controller subjected to a modest amount of parameter uncertainty at the input. The basic idea is not to apply AW action as soon as saturation is encountered, but instead allow saturated actuators act unassisted up to a point, to be made precise below. This is based on the assumption that the desirable nominal controller that is used possesses a reasonable amount of performance robustness. This idea is somewhat related to the over-saturation (see e.g., [9]) or high-gain (see e.g., [10], [11]) approaches that have been used in the direct and explicit approach to saturation, though unlike these references, the controller used in small signal regime is the high performance nominal one (similar to AW schemes). The results are studied through two well-known examples. Notations are standard. As the need arises, new notations are defined or discussed.

II. STATIC ANTI-WINDUP SYNTHESIS

A. Problem Definition

Consider a system with a *nominal* controller designed to fulfill a specific task, such as tracking or disturbance

This work was supported by NSF Grant CMS-0510874.

S. Sajjadi-Kia is Graduate student in the Department of Mechanical and Aerospace Engineering, University of California, Irvine, Irvine, CA 92697, USA ssajjadi@uci.edu

F. Jabbri is with the Department of Mechanical and Aerospace Engineering, University of California, Irvine, Irvine, CA 92697, USA fjabbari@uci.edu

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

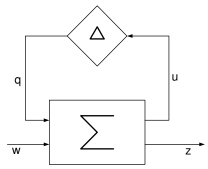

Fig. 2. Standard interconnection for closed-loop system

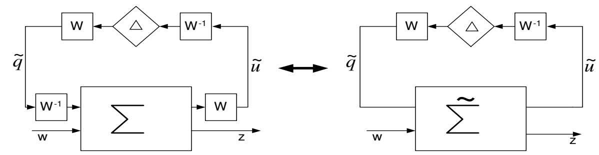

Fig. 3. Standard interconnection for closed-loop system with scale matrix

regulation. Due to possible saturation, an AW block (see Fig. 1) is needed. To formulate the problem, consider Fig. 1, in which

$$\Sigma_p \quad \sim \quad \left\{ \begin{array}{l} \dot{x}_p = A_p x_p + B_1 d + B_2 \hat{u} \\ y = C_2 x_p + D_{21} d + D_{22} \hat{u} \\ z = C_1 x_p + D_{11} d + D_{12} \hat{u} + D_{zr} r \end{array} \right. \tag{1}$$

is the the plant dynamics. And, nominal controller is described as

$$\Sigma_c \quad \sim \quad \left\{ \begin{array}{l} \dot{x}_c = A_c x_c + B_{cy} y + B_{cr} r \\ y_c = u = C_c x_c + D_{cy} y + D_{cr} r \end{array} \right. . \tag{2}$$

The saturation is assumed decentralized with the saturation limit u_{lim} for each u_i $(i = 1, 2, ..., n_u)$. Therefore,

$$\hat{u}_i = sat(u_i) = sgn(u_i)\, min\{|u_i|, u_{lim}\}$$

The purpose of AW block in Fig. 1 is to introduce correction terms in controller to counteract adverse effect of saturation on system stability and performance. However these correction terms should not affect the control loop as long as system actuators do not saturate. Therefore, a common choice for AW block is the static constant gain. In this case, AW block in Fig. 1 is assumed to be

$$\eta = \left[\begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right] = -\Lambda q = -\left[\begin{array}{c} \Lambda_1 \\ \Lambda_2 \end{array} \right] q$$

where $q = u - \hat{u}$, and under this augmentation the controller turns into

$$\widehat{\Sigma}_c \quad \sim \quad \left\{ \begin{array}{l} \dot{x}_c = A_c x_c + B_{cy} w + B_{cr} r - \Lambda_1 q \\ y_c = u = C_c x_c + D_{cy} y + D_{cr} r - \Lambda_2 q \end{array} \right. . \tag{3}$$

Then, as shown in [5], the closed-loop system under a static AW can be depicted in equivalent form as in Fig. 2, where

$$\Sigma \quad \sim \quad \left\{ \begin{array}{l} \dot{x} = Ax + B_w w + (B_q - B_\eta \Lambda) q \\ z = C_z x + D_{zw} w + (D_{zq} - D_{z\eta} \Lambda) q \\ u = C_u x + D_{uw} w + (D_{uq} - D_{u\eta} \Lambda) q \end{array} \right. \tag{4}$$

in which $x = [x_p^T \quad x_c^T]^T$, $w = [r^T \quad d^T]^T$ and system matrices are in terms of plant and the controller matrices. In Fig.2, Δ is a diagonal matrix with each entry $1(\cdot) - sat(\cdot)$, where 1 is the identity. The goal is to find a static gain Λ which makes the closed-loop system stable and also provides desirable L_2 gain while the actuator(s) saturate.

B. LMI-Based Anti-windup Synthesis

To study the stability of the closed-loop system in Fig. 1, often saturation function (or deadzone function) is treated as a sector nonlinearity and the problem is cast in general framework of absolute stability. Then tools such as circle criterion, mostly with LMI characterization developed with quadratic Lyapunove/storage functions, is used to design stabilizing anti-windup gains (see [5] and [6] as two well known representatives). Here, we use Scaled SGT approach which results in a variation of the the LMIs obtained in [5] and [6]. Originally, use of SGT was motivated by the potential for incorporating a measure of scheduling in the basic anti-windup scheme. For the purposes of this paper however, either approach would have sufficed and provided essentially the same results. In all these methods including this work, since we ensure stability for all the possible saturation levels, we are restricting ourself to open-loop stable plants, Σ_p.

To reduce conservatism, we introduce the diagonal $n_u \times n_u$ scaling matrix $W > 0$, as shown in Fig. 3, where W $(M = W^{-2})$ becomes a search variable. Then

$$\tilde{\Sigma} \quad \sim \quad \left\{ \begin{array}{l} \dot{x} = Ax + B_w w + (B_q - B_\eta \Lambda) W^{-1} \tilde{q} \\ z = C_z x + D_{zw} w + (D_{zq} - D_{z\eta} \Lambda) W^{-1} \tilde{q} \\ \tilde{u} = W C_u x + W D_{uw} w + W (D_{uq} - D_{u\eta} \Lambda) W^{-1} \tilde{q} \end{array} \right. \tag{5}$$

We assume that closed-loop augmented system is well-posed (see the appendix). For decentralized saturation, Δ is a diagonal dead-zone matrix, thus $\|\Delta\|_2 \leq 1$. This along with diagonal $W > 0$ imply $\|W \Delta W^{-1}\|_2 \leq 1$. Therefore, to have the feedback system in Fig. 3 stable, SGT leads to

$$\|\tilde{\Sigma}_{\tilde{u}\tilde{q}}\|_{2,i} < 1 \tag{6}$$

as a sufficient condition for stability, which can be expressed in the following convex LMI form:

Theorem 1: (synthesis) The closed loop system shown in Fig. 3 is stable if there exist diagonal matrix $M > 0$ $(M = W^{-2})$, Lyapunov matrix $Q > 0$ and matrix X satisfying

$$\left(\begin{array}{ccc} AQ + QA^T & B_q M - B_\eta X & QC_u^T \\ \star & -M & MD_{uq}^T - X^T D_{u\eta}^T \\ \star & \star & -M \end{array} \right) < 0 \tag{7}$$

If this LMI is feasible, the anti-windup gain can be obtained from $\Lambda = XM^{-1}$.

To establish a performance bound for the AW, L_2 gain from w to z is typically considered. The L_2 gain from w to z, γ, can be obtained using the standard inequality ([12])

$$\frac{d}{dt}(x^T Q^{-1} x) + \gamma^{-1} z^T z - \gamma w^T w < 0 \tag{8}$$

Since $\|W \Delta W^{-1}\|_2 \leq 1$, we have $\tilde{q}^T \tilde{q} - \tilde{u}^T \tilde{u} \leq 0$. Then

S-procedure implies that, for some $\tau > 0$

$$\frac{d}{dt}(x^T Q^{-1} x) + \gamma^{-1} z^T z - \gamma w^T w + \tau(\tilde{q}^T \tilde{q} - \tilde{u}^T \tilde{u}) < 0 \tag{9}$$

is the sufficient condition for inequality (8). Expanding this inequality and preforming standard congruent transformations, inequality (9) can be written in the LMI form of the theorem below with $M = \frac{1}{\tau} W^{-2}$ and $X = \Lambda M$.

Theorem 2: (synthesis and performance) The closed loop system shown in Fig. 3 is stable and the L_2 gain from w to z is less than γ, if there exist diagonal matrix $M > 0$, Lyapunov matrix $Q > 0$ and matrix X satisfying

$$\begin{pmatrix} AQ + QA^T & \star & \star & \star & \star \\ B_w^T & -\gamma I & \star & \star & \star \\ C_z Q & D_{zw} & -\gamma I & \star & \star \\ \Pi_1^T & 0 & \Pi_2^T & -M & \star \\ C_u Q & D_{uw} & 0 & D_{uq}M - D_{u\eta}X & -M \end{pmatrix} < 0 \tag{10}$$

where $\Pi_1 = B_q M - B_\eta X,\ \Pi_2 = D_{zq} M - D_{z\eta} X,\ .$ As before, the anti-windup gain can be obtained as $\Lambda = XM^{-1}$.

C. Numerical Example

Consider the following system taken from [7] with input bound $u_{lim} = 1$.

$$\left[\begin{array}{c|c|c} A_p & B_2 & B_1 \\ \hline C_2 & D_{22} & D_{21} \end{array}\right] = \left[\begin{array}{ccc|c|c} -10.6 & -6.09 & -0.9 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline -1 & -11 & -30 & 0 & 0 \end{array}\right]$$

with $z = y - r$ and nominal controller

$$\left[\begin{array}{c|c|c} A_c & B_c y & B_{cr} \\ \hline C_c & D_{cy} & D_{cr} \end{array}\right] = \left[\begin{array}{cc|c|c} -80 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ \hline 20.25 & 1600 & 80 & -80 \end{array}\right].$$

Nominal γ is 1. Using Theorem 2, the following AW gain is obtained

$$\Lambda = [-0.1968 \quad 0.0025 \quad -0.9860]^T$$

which leads to a performance level of $\gamma = 85.78$, very close to the result obtained in [7]: $\gamma = 86.07$. The slight difference probably is the result of different options used in the LMI solver. Applying the same reference signal, $r(t)$, used in [7], the response of the system with and without AW augmentation are shown in Fig. 4. The response is essentially the same as the one obtained in [7].

III. A Modified Anti-windup Scheme

Fig. 4 suggests that for this example, constrained nominal closed-loop system (dotted) shows better tracking behavior than the system with AW (dash-dotted) for much of the simulation time, especially the first 10 seconds. A possible explanation might be the tradeoff between the two possible modes of operation: in one, AW is active as soon as saturation is encountered resulting in a safe (i.e., stable) but low performance controller. On the other hand, if the actuator command is 'slightly' or moderately above saturation, the nominal controller acts as a high performance controller subjected to a modest amount of parameter uncertainty at

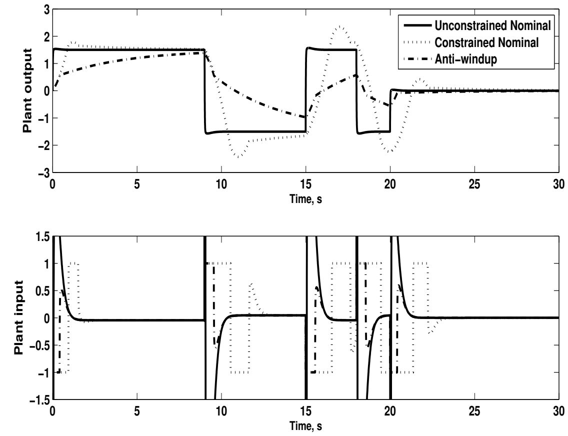

Fig. 4. Plant output and input plots: unconstrained ideal system (solid), saturated nominal closed-loop with no AW (dotted), saturated closed-loop with AW (dash-dotted).

the input, or 'matched' uncertainty, which is a mild form of uncertainty. Recall that γ nominal was 1, while the γ of AW was 85.78.

Based on this observation, the main idea here is to postpone the activation of AW to a point where system really needs AW protection. To investigate further, we preform the following analysis on the nominal constrained closed loop system, i.e. the original system with bounded actuator(s) under nominal controller without AW. Initially, for notational simplicity, we will discuss the single input case. The generalization to multi-input systems will be discussed later. We consider the performance of the saturated constrained system without AW as a function of the maximum value of the command sent to the saturation box.

Consider $u(t)$ and $\hat{u}(t)$ as the input and the output of a saturation box, respectively, with saturation bound u_{lim}. As shown below, the nonlinear saturation element can be replaced by the time varying gain $G(t) \in [g,\, 1]$:

$$\hat{u}(t) = G(t)u(t), \quad G(t) = \begin{cases} 1 & |u(t)| \le u_{lim} \\ sgn(u(t))\frac{u_{lim}}{u(t)} & |u(t)| > u_{lim} \end{cases} \tag{11}$$

It is clear that, when the actuator is not saturated, $G = 1$ and the minimum value attained by $G(t)$, $0 \le g \le 1$, depends on the maximum value of $u(t)$ ([13]).

Considering (11), and assuming $D_{22} = 0$, the nominal constrained closed-loop system (equations (1),(3)) with $x = [x_p^T \quad x_c^T]^T$ can be written as

$$\begin{cases} \dot{x} = A_{cl}(G(t))x + B_{cl}(G(t))w \\ z = C_{cl}(G(t))x + D_{cl}(G(t))w \end{cases} \tag{12}$$

with

$$\left[\begin{array}{c} A_{cl}(G(t)) \\ \hline B_{cl}(G(t)) \end{array}\right] = \left[\begin{array}{cc} A + B_2 G(t) D_{cy} C_2 & B_2 G(t) C_c \\ B_{cy} C_2 & A_c \\ \hline B_2 G(t) D_{cr} & B_2 G(t) D_{cy} D_{21} + B_1 \\ B_{cr} & B_{cy} D_{21} \end{array}\right]$$

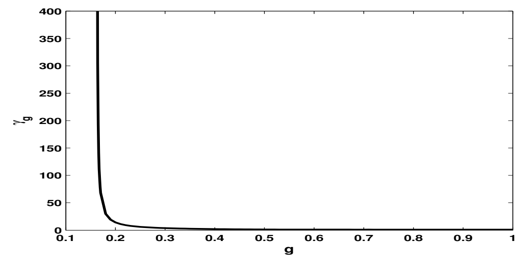

Fig. 5. Performance of the saturated system for $G(t) \in [g \quad 1]$.

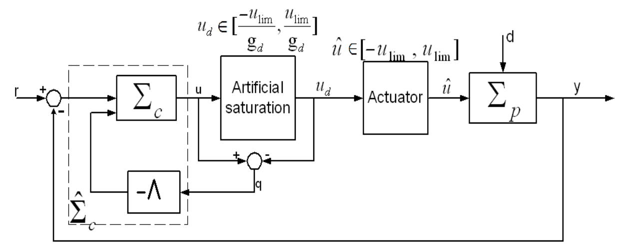

Fig. 6. New modified anti-windup scheme.

$$\left[\frac{C_{cl}(G(t))}{D_{cl}(G(t))}\right] = \left[\frac{C_1 + D_{12}G(t)D_{cy}C_2 \qquad D_{12}G(t)C_c}{D_{zr} + D_{12}G(t)D_{cr} \;\; D_{11} + D_{12}G(t)D_{cy}D_{21}}\right]$$

The matrices above represent a LPV system with variable $G(t) \in [g \quad 1]$ $(0 \le g)$. Therefore, an estimate for the L_2 gain of this closed-loop, assuming that $|u(t)| \le \frac{1}{g}u_{lim}$ for all t, can be obtained from minimizing γ_g subject to

$$\begin{pmatrix} A_{cl}(\bar{g})^T P + PA_{cl}(\bar{g}) & \star & \star \\ B_{cl}(\bar{g})^T P & -\gamma_g I & \star \\ C_{cl}(\bar{g}) & D_{cl}(\bar{g}) & -\gamma_g I \end{pmatrix} < 0 \qquad (13)$$

for $\bar{g} = g$ and $\bar{g} = 1$ (i.e., corners of the hypercube of the parameter set). Matrix $A_{cl}(\bar{g})$ etc, means value of $A_{cl}(G(t))$ evaluated at $G(t) = \bar{g}$.

Figure 5 shows the result of analysis for the system of Section II-C. In this figure, each point (g, γ), represents the performance of the system for $G(t) \in [g \quad 1]$, i.e. the actuator is guaranteed to receive – *somehow* – control command with peak value of $\frac{1}{g}u_{lim}$ or less. Figure 5 suggests that the constrained nominal closed-loop system has adequate performance up to about $g = 0.2$, or equivalently $|u(t)| \le \frac{1}{0.2}1 = 5$ $(G(t) \in [0.2 \quad 1])$.

Motivated by this, we propose to combine over-saturation with AW through the scheme shown in Fig. 6. In this new scheme we add to the loop an artificial saturation box with saturation bound $\frac{1}{g_d}u_{lim}$, where g_d is the design point, specified by designer, i.e.,

$$u_d(t) = sgn(u(t)) \; min\{(|u(t)|, \frac{1}{g_d}u_{lim}\}$$

Then we use $q(t) = u(t) - u_d(t)$, as the input signal to the static AW block. This lets the system stay saturated up to the point g_d, i.e., AW activates when $|u(t)| > \frac{1}{g_d}u_{lim}$. One can obtain the design point, g_d, by trial and error, focused around the 'bend' in the γ_g vs. g plots similar to the one in Fig. 5. Note that g_d depends on how much input uncertainty the nominal closed-loop can tolerate. For some problems g_d can be very close to 1 which necessitates the immediate activation of the AW. For g_d less than 1, we use the following method to obtain the gains of over-saturated AW introduced in Fig. 6.

Considering (11) and Fig. 6, since input to the actuator, $|u_d(t)| \le \frac{1}{g_d}u_{lim}$, we have $\hat{u}(t) = G(t)u_d(t)$ where $G(t) \in [g_d, 1]$. Using diagonal scaling matrix $W > 0$ as applied earlier, we have $\tilde{u} = Wu$ and $\tilde{q} = Wq$. Therefore, the weighted closed-loop system with $x = [x_p^T \quad x_c^T]^T$ can be obtained as

$$\begin{cases} \dot{x} = A(G)x + B(G)_w w + (B(G)_q - B(G)_\eta \Lambda)W^{-1}\tilde{q} \\ z = C(G)_z x + D(G)_{zw} w + (D(G)_{zq} - D(G)_{z\eta}\Lambda)W^{-1}\tilde{q} \\ \tilde{u} = WC_u x + WD_{uw} w + W(D_{uq} - D_{u\eta}\Lambda)W^{-1}\tilde{q} \end{cases} \qquad (14)$$

with $A(G)$, etc as obvious matrices, where $G(t)$ appears linearly, e.g.,

$$A(G) = \begin{bmatrix} A_p + B_2G(t)D_{cy}C_2 & B_2G(t)C_c \\ B_{cy}C_2 & A_c \end{bmatrix}.$$

Note that (14) is very much the same as (5) except that now the system matrices are time varying. Thus, we can depict (14) in a same block diagram form of Fig. 3, only with time varying Σ and $\widehat{\Sigma}$.

To design Λ, once again we use L_2 gain from w to z as the performance measure. Next, consider the modified AW scheme in Fig. 6:

• When $|u(t)| \le \frac{1}{g_d}u_{lim}$ we have $q = 0$ $(\tilde{q}(t) = 0)$. Then

$$\frac{d}{dt}(x^T Q^{-1} x) = (A(G)x + B(G)_w w)^T Q^{-1} x +$$
$$x^T Q^{-1}(A(G)x + B(G)_w w)$$

Therefore, inequality (8) can be written in the LMI form

$$\begin{pmatrix} A(G)Q + QA(G)^T & B(G)_w & C(G)_{zw}^T Q \\ \star & -\gamma I & D(G)_{zw}^T \\ \star & \star & -\gamma I \end{pmatrix} < 0 \qquad (15)$$

where LPV variable $G(t) \in [g_d, 1]$, and it is sufficient to check this inequity for $G = g_d$ and $G = 1$.

• When $|u(t)| > \frac{1}{g_d}u_{lim}$, we have $|u_d(t)| = \frac{1}{g_d}u_{lim}$. As a result $q(t) \ne 0$ $(\tilde{q}(t) \ne 0)$, and also $\hat{u}(t) = g_d u_d(t)$. Since $\|W\Delta W^{-1}\|_2 \le 1$, we have $\tilde{q}^T\tilde{q} - \tilde{u}^T\tilde{u} \le 0$. Using the S-procedure, we can arrive at the same inequality as (9). We can case (9) as LMI (17) in Theorem below.

Theorem 3 (Synthesis and performance): The closed loop system shown in Fig. 6 is stable and the L_2 gain from w to z is less than γ, if there exist diagonal matrix $M > 0$, Lyapunov matrix $Q > 0$ and matrix X satisfying

$$\begin{pmatrix} A(\bar{g})Q + QA(\bar{g})^T & \star & \star \\ B(\bar{g})_w^T & -\gamma I & \star \\ QC(\bar{g})_{zw} & D(\bar{g})_{zw} & -\gamma I \end{pmatrix} < 0 \text{ for } \bar{g} = 1\,,\, g_d \qquad (16)$$

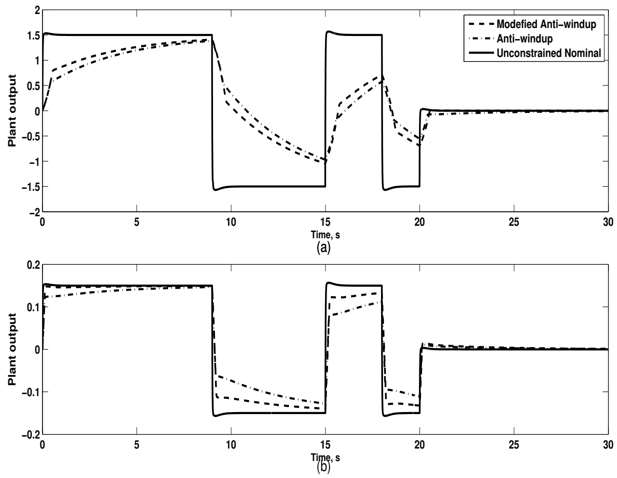

Fig. 7. Plant output for the example of Section II-C: Unconstrained ideal response (solid), immediate activation of AW (dash-dotted) and modified AW (dashed). (a) the same reference signal of Section II-C (peak value 1.5) (b) smaller reference signal (peak value 0.15).

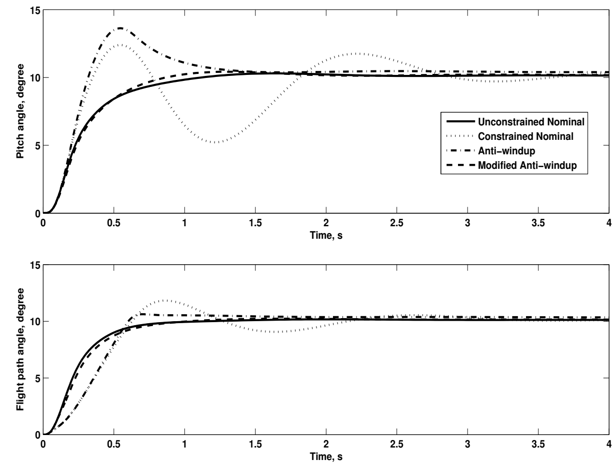

Fig. 8. System response in second example; unconstrained nominal closed-loop (solid), constrained nominal closed-loop (dotted), immediate activation of anti-windup (dash-dotted) and modified anti-windup (dashed)

$$\begin{pmatrix} \Omega & \star & \star & \star & \star \\ B(g_d)_w^T & -\gamma I & \star & \star & \star \\ C(g_d)_z Q & D(g_d)_{zw} & -\gamma I & \star & \star \\ \Pi_1^T & 0 & \Pi_2^T & -M & \star \\ C_u Q & D_{uw} & 0 & \Pi_3 & -M \end{pmatrix} < 0 \quad (17)$$

where

$$\Omega = A(g_d)Q + QA(g_d)^T, \quad \Pi_1 = B(g_d)_q M - B(g_d)_\eta X$$

$$\Pi_2 = D(g_d)_{zq} M - D(g_d)_{z\eta} X, \quad \Pi_3 = D_{uq} M - D_{u\eta} X$$

If this problem is feasible, then anti-windup gain, Λ, can be obtained from $\Lambda = XM^{-1}$.

LMI (16) with $\bar{g} = 1$ and $\bar{g} = g_d$ is sufficient condition for LMI (15). However, note that submatrix (1:3,1:3) of LMI (17) is the repetition of LMI (16) for $\bar{g} = g_d$. Therefore, we can check LMI (16) just for $\bar{g} = 1$. Also, as we have a single actuator, M is a scalar here.

In case of multiple actuators with decentralized saturation, say n_u actuators, $G(t)$ as equivalent form for saturation, is a $n_u \times n_u$ diagonal matrix with entries each as time varying gains defined in (11). As a result, the LPV model of the system (equation (12) in analysis and equation (14) in synthesis) has 2^{n_u} corners. Analysis and synthesis should be conducted for all these 2^{n_u} corners. For brevity, details are omitted.

A. Numerical Examples

We continue with the example of Section II-C: Using Theorem 3, the following anti-windup gain is obtained for the modified scheme with design point $g_d = 0.17$:

$$\Lambda = [-0.0029 \quad 0 \quad -0.9998]^T$$

This compensator has $\gamma = 87.50$. Recall with $g_d = 1$, i.e, immediate activation of anti-windup, we had $\gamma = 85.78$. The overall γ for the modified scheme is slightly higher than that of traditional scheme, as expected. As Fig. 7 suggests, however, we get better results, using the modified scheme especially when the system is slightly saturated, particulary for small reference (Fig. 7.b).

Example 2: Consider a model of the longitudinal dynamics of the F8 aircraft with an eighth order linear nominal controller taken from [14]. The state equations are

$$\dot{x}(t) = \begin{bmatrix} -0.8 & -0.0006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} x(t)$$

and the controller is

$$K(s) = \tfrac{1}{s}(G\,[sI - A_a - B_a G - HC_a]^{-1}\,H)$$

with matrices A_a, B_a, C_a, H and G as given in [14]. The two inputs are the elevator and flaperon angles, each limited to ± 25 degree. Two outputs of the system are pitch angle and flight path angle. As reported in [14], this system experiences a substantial performance degradation in its tracking maneuvers when system saturates.

By selecting $z = y - r$ where r denotes the reference input, using the traditional scheme of anti-windup (procedure of Section II-B), a static anti-windup compensator can be obtained. The resulting anti-windup compensator gain, Λ, is

$$10^{-4} \begin{bmatrix} 2143 & 17077 & -4061 & -982299 & -28 & 724 & 2251 & 17804 & -9899 & 930 \\ -1068 & 5252 & -1951 & -298896 & -121 & 164 & -1035 & 5505 & 29 & -9495 \end{bmatrix}^T$$

This augmentation guarantees a L_2 gain of $\gamma = 22.15$. This example is also studied in [6] where almost the same value is obtained for γ; $\gamma = 22.19$. We used the procedure of Section III to design a modified anti-windup. Using plots similar to Fig. 5, after some trial and error, the following design point is chosen

$$G_d = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.4 \end{bmatrix}$$

Using this design point, the following gain, Λ, is obtained for the modified scheme.

$$10^{-4} \begin{bmatrix} 25 & -14 & 1 & -1742 & 0 & 1 & 25 & -13 & -10000 & 1 \\ -3 & -10 & -2 & -1307 & 0 & 0 & -3 & -8 & 0 & -9999 \end{bmatrix}^T$$

With this anti-windup gain, the overall L_2 gain of the augmented system is $\gamma = 24.59$. Fig. 8, shows the response of the ideal unconstrained closed-loop system (solid), response of the constrained nominal closed-loop (i.e., no AW) (dotted), the response of the system with the immediate activation of AW (dashed-dotted), and response of system under the modified AW (dashed). As shown, without any AW, both outputs of the system experience large overshoot. Immediate activation of AW tries to improve the performance but still we have significant overshoots. The same response also is reported in [6]. Although γ for modified AW is slightly higher than the traditional case, as expected, system has better response. Overshoot is removed and system shows very close tracking response to the ideal unconstrained system.

IV. Conclusion

In this paper, we developed a variation of L_2 gain convex LMI for the synthesis of static anti-windup gain through the Scaled Small Gain Theorem. We then proposed a new modified anti-windup scheme. The main idea is to apply anti-windup when it is necessary and the performance of saturated system faces substantial degradation. In this scheme, we let the system stay saturated as long as the performance of the un-augmented system is expected to be adequate. Beyond this point, we apply a static anti-windup to ensure the stability and performance of the saturated system. So far, we have obtained design point, g_d, by trial and error, focused around the 'bend' in the γ_g vs. g plots similar to the one in Fig. 5. Obtaining a systematic method is an active area of on-going research. The results of simulation for two standard examples show that new scheme can work better than the traditional anti-windup especially when actuator commands are close to saturation bounds.

V. Appendix: Ill-conditioned Algebraic loop

Consider $u = C_u x + D_{uw} w + (D_{uq} - D_{u\eta}\Lambda)q$. We replace q by its value $q = u - sat(u) = (I - G(t))u$:

$$u = C_u x + D_{uw} w + (D_{uq} - D_{u\eta}\Lambda)(I - G(t))u$$

where $G(t)$ is a $n_u \times n_u$ diagonal matrix with each diagonal element as defined in (11), corresponding to each actuator. To turn this implicit equation into an explicit equation, one takes the implicit terms to the left hand-side

$$(I - (D_{uq} - D_{u\eta}\Lambda)(I - G(t)))u = C_u x + D_{uw} w$$

Thus, for well-posedness, we need $(I - (D_{uq} - D_{u\eta}\Lambda)(I - G(t)))$ invertible. Note that wellposedness does not rules out the occurrence of numerical problems such as ill-conditioned algebraic loops. As mentioned in [15] and [16], in several examples with AW, solution can be close to the ill-posed interconnection. Therefore, often, numerical solvers are not able to determine the corresponding control action. In [15] and [16] different conditions are offered to remove this problem. To ensure wellposedness and avoid ill-conditioned algebraic loops, we use the following equation

$$2I - (D_{uq} - D_{u\eta}\Lambda)\mathcal{G} - \mathcal{G}(D_{uq} - D_{u\eta}\Lambda)^T > \epsilon I \tag{18}$$

for all the possible diagonal $\mathcal{G} \in R^{n_u \times n_u}$ whose diagonal elements are either 0 or 1 ($\mathcal{G}$ is the corners of $(I - G(t))$, thus we have 2^{n_u} possibilities). One possible $\mathcal{G}$ is matrix zero which implies $0 \leq \epsilon < 2$. Apparently $\epsilon = 0$ suffices to guarantee the invertibility of $(I - (D_{uq} - D_{u\eta}\Lambda)(I - G(t)))$, though adding $\epsilon > 0$ pushes the determinant of this matrix away from the zero. In case of scaled system, this condition becomes:

$$2M - (D_{uq}M - D_{u\eta}X)\mathcal{G} - \mathcal{G}(D_{uq}M - D_{u\eta}X)^T > \epsilon M$$

The choice of ϵ can be made by trial and error, to find a value that solves the ill-condition algebraic loop and sacrifices less performance of the design (γ).

References

[1] Kose, I.E., Jabbari, F., "Control of Systems with Actuator Amplitude and Rate Constraints", *Proceeding of American Control Conference*, pp. 4915-4919, 2001.

[2] Jabbari, F., Kose, I.E., "Rate and Magnitude-Bounded Actuators: Scheduled Output Feedbcak Design", *Int. J. Robust and Nonlinear Control*, Vol. 14, pp. 1169-1184, 2004.

[3] Kose, I.E., Jabbari, F., "Scheduled Controllers for Linear Systems with Bounded Actuators", *Automatica*, Vol. 39, pp. 1377-1387, 2004.

[4] Teel, A. R. and Kapoor, N. "The L2 Anti-windup Problem: Its Definition and Solution", *European Control Conference*, Brussels, Belgium, 1997.

[5] Mulder E.F., Kothare,M.V., Morari, M., "Multivariable Anti-windup Controller Synthesis Using Linear Matrix Inequalities", *Automatica*, Vol. 37, pp. 1407-1416, 2001.

[6] Grimm, G., Hatfield, I., Teel, A.R., Turner, M.C., and Zaccarian, L., "Antiwindup for Stable Linear Systems with Input Saturation: an LMI-Based Synthesis", *IEEE Trans. on Automatic Control*, Vol. 48, No. 9, 2003.

[7] Grimm, G., Teel, A.R., and Zaccarian, L., "Results on Linear LMI-Based External Anti-windup Design", *IEEE Trans. on Automatic Control*, Vol. 48, No. 9, Sep. 2003.

[8] Hu, T., Teel, A.R., and Zaccarian, L., "Regional Anti-windup Compensation for Linear Systems with Input Saturation", *Proceeding of American Control Conf.*, Portland, USA, June 8-10 2005.

[9] DeDona, J.A., Moheimani, S.O.R., Goodwin, G.C., "Robust combined PLC/LHG controller with oversaturation in input signal, ACC-2000, pp. 750-751.

[10] Lin, Z., Saberi, A., "A semi-global low-and-high gain design technique for linear systems with input saturaiotn," *Int. J. Robust and Nonlinear Control*, 5, pp. 381-398, 1995.

[11] Nguyen. T., Jabbari, F., "Output feedback controllers for disturbacne attenuation with actuator amplitude and rate saturation," *Automatica*, 36(9), pp 1339-1346, 2000.

[12] Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V., "Linear Matrix Inequalities in System and Control Theory", *SIAM studies in Applied Mathematics*, Vol. 15, 1994.

[13] Amato, F., Iervolino, R., Scala, S., and Verde, L., "Category II Pilot in-the-Loop Oscillations Analysis from Robust Stability Methods", *J. Guidance, Control and Dynamics*, Vol. 24, No. 3, 2001.

[14] Kapasouris, P., Athans, M., and Stein, G., "Design of Feedback Control Systems for Stable Plants with Saturating Actuators", *Proceeding of the 27th IEEE Conf. on Decision and Control*, Austin, TX, December 1988.

[15] Grimm, G., Teel, A.R., and Zaccarian, L., "Estabilishing Lipschitz Properities of Multivariable Algebraic Loops with Incremental Sector Nonlinearities", *Proceeding of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, USA, December 2003.

[16] Syaichu-Rahman, A., Middleton, R.H., "On the Robustness of Multivariable Algebraic Loops with Sector Nonlinearities", *Proceeding of the 41st IEEE Conference on Decision and Control*, Las Vegas, NV, USA, December 2002.