2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Optimal Filtering for Incompletely Measured Polynomial States with Multiplicative Noise

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Abstract— **In this paper, the optimal filtering problem for polynomial system states with polynomial multiplicative noise over linear observations with an arbitrary, not necessarily invertible, observation matrix is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance. As a result, the Ito differentials for the optimal estimate and error variance corresponding to the stated filtering problem are first derived. A transformation of the observation equation is introduced to reduce the original problem to the previously solved one with an invertible observation matrix. The procedure for obtaining a closed system of the filtering equations for any polynomial state with polynomial multiplicative noise over linear observations is then established, which yields the explicit closed form of the filtering equations in the particular cases of linear and bilinear state equations. In the example, performance of the designed optimal filter is verified against the optimal filter for a quadratic state with a state-independent noise and a conventional extended Kalman-Bucy filter.**

I. Introduction

Although the general optimal solution of the filtering problem for nonlinear state and observation equations confused with white Gaussian noises is given by the equation for the conditional density of an unobserved state with respect to observations [1], there are a very few known examples of nonlinear systems where that equation can be reduced to a finite-dimensional closed system of filtering equations for a certain number of lower conditional moments. The most famous result, the Kalman-Bucy filter [2], is related to the case of linear state and observation equations, where only two moments, the estimate itself and its variance, form a closed system of filtering equations. However, the optimal nonlinear finite-dimensional filter can be obtained in some other cases, if, for example, the state vector can take only a finite number of admissible states [3] or if the observation equation is linear and the drift term in the state equation satisfies the Riccati equation $df/dx + f^2 = x^2$ (see [4]). The complete classification of the "general situation" cases (this means that there are no special assumptions on the structure of state and observation equations and the initial conditions), where the optimal nonlinear finite-dimensional filter exists, is given in [5]. The last two papers actually deal with specific types of polynomial filtering systems. There also exists a considerable bibliography on robust filtering for the "general situation" systems (see, for example, [6]–[11]). Apart form the "general situation," the optimal finite-dimensional filters have recently been designed ([12]–[15]) for certain classes of polynomial system states with Gaussian initial conditions over linear observations with an invertible observation matrix.

This paper presents the optimal finite-dimensional filter for incompletely measured polynomial system states with polynomial multiplicative noise over linear observations with an arbitrary, not necessarily invertible, observation matrix, thus generalizing the results of ([12], [13], [14]). Designing the optimal filter for polynomial systems with polynomial multiplicative noise over observations with a non-invertible observation matrix presents a significant advantage in the filtering theory and practice, since it enables one to address the optimal filtering problems for incompletely measured polynomial states with polynomial observation nonlinearities, such as the optimal cubic sensor problem (see [16]) in the presence of unmeasured states. The optimal filtering problem is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance [17]. As the first result, the Ito differentials for the optimal estimate and error variance corresponding to the stated filtering problem are derived. Next, a transformation of the observation equation is introduced to reduce the original problem to the previously solved one with an invertible observation matrix [14]. It is then proved, using the technique of representing the superior moments of a Gaussian random variables as functions of its expectation and variance, that a closed finite-dimensional system of the optimal filtering equations with respect to a finite number of filtering variables can be obtained for a polynomial state equation with polynomial multiplicative noise and linear observations with an arbitrary observation matrix. In this case, the corresponding procedure for designing the optimal filtering equations is established. Finally, the closed system of the optimal filtering equations with respect to two variables, the optimal estimate and the error variance, is derived in the explicit form in the particular cases of linear and bilinear state equations.

In the illustrative example, performance of the designed optimal filter is verified for a quadratic bi-dimensional state over linear observations against the optimal filter for a quadratic state with a state-independent noise and a conventional extended Kalman-Bucy filter. The simulation results show a definite advantage in favor of the designed optimal filter. Indeed, it can be observed that the estimation error produced by the optimal filter rapidly reaches and then maintains the zero mean value even in a close vicinity of

The authors thank the Mexican National Science and Technology Council (CONACyT) for financial support under Grants no. 55584 and 52953.

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the asymptotic time point, where the reference state goes to infinity for a finite time. On the contrary, the estimation errors given by the other two applied filters diverge to infinity near the asymptotic time point.

II. Filtering Problem for Polynomial State over Linear Observations

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq t_0$, and let $(W_1(t), F_t, t \geq t_0)$ and $(W_2(t), F_t, t \geq t_0)$ be independent Wiener processes. The F_t-measurable random process $(x(t), y(t)$ is described by a nonlinear differential equation with both polynomial drift and diffusion terms for the system state

$$dx(t) = f(x,t)dt + g(x,t)dW_1(t), \quad x(t_0) = x_0, \qquad (1)$$

and a linear differential equation for the observation process

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t). \qquad (2)$$

Here, $x(t) \in R^n$ is the state vector and $y(t) \in R^m$ is the linear observation vector, $m \leq n$. The initial condition $x_0 \in R^n$ is a Gaussian vector such that x_0, $W_1(t) \in R^p$, and $W_2(t) \in R^q$ are independent. In contrast to the previously obtained results (see [12], [13], [14]), the observation matrix $A(t) \in R^{m \times n}$ is not supposed to be invertible or even square. It is assumed that $B(t)B^T(t)$ is a positive definite matrix, therefore, $m \leq q$. All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.

The nonlinear functions $f(x,t)$ and $g(x,t)$ are considered polynomials of n variables, components of the state vector $x(t) \in R^n$, with time-dependent coefficients. Since $x(t) \in R^n$ is a vector, this requires a special definition of the polynomial for $n > 1$. In accordance with [14], a p-degree polynomial of a vector $x(t) \in R^n$ is regarded as a p-linear form of n components of $x(t)$

$$f(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T + \ldots + a_p(t)x \ldots_{p \ times} \ldots x, \qquad (3)$$

where $a_0(t)$ is a vector of dimension n, a_1 is a matrix of dimension $n \times n$, a_2 is a 3D tensor of dimension $n \times n \times n$, a_p is an $(p+1)$D tensor of dimension $n \times \ldots_{(p+1) \ times} \ldots \times n$, and $x \times \ldots_{p \ times} \ldots \times x$ is a pD tensor of dimension $n \times \ldots_{p \ times} \ldots \times n$ obtained by p times spatial multiplication of the vector $x(t)$ by itself. Such a polynomial can also be expressed in the summation form

$$f_k(x,t) = a_{0\ k}(t) + \sum_i a_{1\ ki}(t)x_i(t) + \sum_{ij} a_{2\ kij}(t)x_i(t)x_j(t) + \ldots$$
$$+ \sum_{i_1 \ldots i_p} a_{p\ ki_1 \ldots i_p}(t)x_{i_1}(t) \ldots x_{i_p}(t), \quad k, i, j, i_1 \ldots\ i_p = 1, \ldots, n.$$

The estimation problem is to find the optimal estimate $\hat{x}(t)$ of the system state $x(t)$, based on the observation process $Y(t) = \{y(s), t_0 \leq s \leq t\}$, that minimizes the Euclidean 2-norm $J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t)) \mid F_t^Y]$ at every time moment t. Here, $E[z(t) \mid F_t^Y]$ means the conditional expectation of a stochastic process $z(t) = (x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0, t]$. As known [17], this optimal estimate is given by the conditional expectation $\hat{x}(t) = m(t) = E(x(t) \mid F_t^Y)$ of the system state $x(t)$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0, t]$. As usual, the matrix function $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y]$ is the estimation error variance.

The proposed solution to this optimal filtering problem is based on the formulas for the Ito differential of the conditional expectation $E(x(t) \mid F_t^Y)$ and its variance $P(t)$ (cited after [17]) and given in the following section.

III. Optimal Filter for Polynomial State over Linear Observations

The optimal filtering equations could be obtained using the formula for the Ito differential of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$ (see [17])

$$dm(t) = E(f(x,t) \mid F_t^Y)dt +$$
$$E(x[\varphi_1(x) - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \times$$
$$\left(B(t)B^T(t)\right)^{-1} (dy(t) - E(\varphi_1(x) \mid F_t^Y)dt),$$

where $f(x,t)$ is the polynomial drift term in the state equation, and $\varphi_1(x)$ is the linear drift term in the observation equation equal to $\varphi_1(x,t) = A_0(t) + A(t)x(t)$. Upon performing substitution, the estimate equation takes the form

$$dm(t) = E(f(x,t) \mid F_t^Y)dt + E(x(t)[A(t)(x(t) - m(t))]^T \mid F_t^Y) \times$$
$$(B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t)) =$$
$$E(f(x,t) \mid F_t^Y)dt + P(t)A^T(t)(B(t)B^T(t))^{-1} \times$$
$$(dy(t) - (A_0(t) + A(t)m(t))dt). \qquad (4)$$

The equation (4) should be complemented with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$.

Trying to compose a closed system of the filtering equations, the equation (4) should be complemented with the equation for the error variance $P(t)$. For this purpose, the formula for the Ito differential of the variance $P(t) = E((x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y)$ could be used (cited again after [17]):

$$dP(t) = (E((x(t) - m(t))(f(x,t))^T \mid F_t^Y) +$$
$$E(f(x,t)(x(t) - m(t))^T) \mid F_t^Y) +$$
$$E(g(x,t)g^T(x,t) \mid F_t^Y) - E(x(t)[\varphi_1(x) - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \times$$
$$\left(B(t)B^T(t)\right)^{-1} E([\varphi_1(x) - E(\varphi_1(x) \mid F_t^Y)]x^T(t) \mid F_t^Y))dt +$$
$$E((x(t) - m(t))(x(t) - m(t))[\varphi_1(x) - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \times$$
$$\left(B(t)B^T(t)\right)^{-1} (dy(t) - E(\varphi_1(x) \mid F_t^Y)dt),$$

where $g(x,t)$ is the polynomial diffusion term in the state equation, and the last term should be understood as a 3D tensor (under the expectation sign) convoluted with a vector, which yields a matrix. Upon substituting the expressions for $f(x,t)$ and $\varphi_1(x,t)$, using the variance formula $P(t) =$

$E((x(t)-m(t))x^T(t)) \mid F_t^Y)$, and taking into account that the last addendum is equal to zero, the variance equation can be represented as

$$dP(t) = (E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)+$$
$$E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y)+E(g(x,t)g^T(x,t) \mid F_t^Y)-$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt. \quad (5)$$

The equation (5) should be complemented with the initial condition $P(t_0)=E[(x(t_0)-m(t_0)(x(t_0)-m(t_0)^T \mid F_{t_0}^Y]$.

In [15], a system of the filtering equations obtained for incompletely measured polynomial system states (1), without multiplicative noise, over linear observations. Using the same technique, the optimal filtering equations (4),(5) are finally derived. The details are omitted here due to space shortage.

The equations (4) and (5) for the optimal estimate $m(t)$ and the error variance $P(t)$ form a non-closed system of the filtering equations for the nonlinear state (1) over linear observations (2). The non-closeness means that the system (4),(5) includes terms depending on x, such as $E(f(x,t) \mid F_t^Y)$, $E((x(t)-m(t))f^T(x,t)) \mid F_t^Y)$, and $E(g(x,t)g^T(x,t) \mid F_t^Y)$, which are not expressed yet as functions of the system variables, $m(t)$ and $P(t)$.

In the next subsections, a closed form of the filtering equations will be obtained from (4) and (5) for linear and bilinear functions $f(x,t)$ and $g(x,t)$ in the equation (1). It should be noted, however, that application of the same procedure would result in designing a closed system of the filtering equations for any polynomial functions $f(x,t)$ and $g(x,t)$ in (1).

A. Optimal Filter for Linear State with Linear Multiplicative Noise

In a particular case, if the functions $f(x,t)=a_0(t)+a_1(t)x(t)$ and $g(x,t)=b_0(t)+b_1(t)x(t)$ are linear, where b_1 is a 3D tensor of dimension $n\times n\times n$, the representations for $E(f(x,t) \mid F_t^Y)$, $E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)$, and $E(g(x,t)g^T(x,t) \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$ are derived as follows

$$E(f(x,t) \mid F_t^Y)=a_0(t)+a_1(t)m(t), \quad (12)$$

$$E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y)+ \quad (13)$$
$$E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)=a_1(t)P(t)+P(t)a_1^T(t).$$

$$E(g(x,t)g^T(x,t) \mid F_t^Y)=b_0(t)b_0^T(t)+b_0(t)(b_1(t)m(t))^T+ \quad (14)$$
$$(b_1(t)m(t))b_0^T(t)+b_1(t)P(t)b_1^T(t)+b_1(t)m(t)m^T(t)b_1^T(t),$$

where $b_1^T(t)$ denotes the tensor obtained from $b_1(t)$ by transposing its two rightmost indices.

Substituting the expression (12) in (4) and the expressions (13),(14) in (5), the filtering equations for the optimal estimate $m(t)$ and the error variance $P(t)$ are obtained

$$dm(t)=(a_0(t)+a_1(t)m(t))dt+ \quad (15)$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t)-(A_0(t)+A(t)m(t))dt],$$

$$m(t_0)=E(x(t_0) \mid F_t^Y)),$$

$$dP(t)=(a_1(t)P(t)+P(t)a_1^T(t)+b_0(t)b_0^T(t)+ \quad (16)$$
$$b_0(t)(b_1(t)m(t))^T+(b_1(t)m(t))b_0^T(t)+b_1(t)P(t)b_1^T(t)+$$
$$b_1(t)m(t)m^T(t)b_1^T(t))dt-$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt.$$
$$P(t_0)=E((x(t_0)-m(t_0))(x(t_0)-m(t_0))^T \mid F_t^Y)).$$

Note that the observation matrix $A(t)$ should not even be necessarily invertible to obtain the filtering equations (15)–(16). Indeed, the only used polynomial equality, $E(x(t)x^T(t) \mid F_t^Y)=P(t)+m(t)m^T(t)$, is valid for any random variable with finite second moments, not only Gaussian.

B. Optimal Filter for Bilinear State with Bilinear Multiplicative Noise

Let the functions

$$f(x,t)=a_0(t)+a_1(t)x+a_2(t)xx^T \quad (17)$$

and

$$g(x,t)=b_0(t)+b_1(t)x+b_2(t)xx^T \quad (18)$$

be second degree polynomials, where x is an n-dimensional vector, $a_0(t)$ is an n-dimensional vector, $a_1(t)$ and $b_0(t)$ are $n\times n$ - matrices, $a_2(t)$ and $b_1(t)$ are 3D tensors of dimension $n\times n\times n$, and $b_2(t)$ is a 4D tensor of dimension $n\times n\times n\times n$. In this case, the representations for $E(f(x,t) \mid F_t^Y)$, $E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)$, and $E(g(x,t)g^T(x,t) \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$ are derived as follows (see [12], [13])

$$E(f(x,t) \mid F_t^Y)=a_0(t)+a_1(t)m(t)+$$
$$a_2(t)m(t)m^T(t)+a_2(t)P(t), \quad (19)$$

$$E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y)+$$
$$E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)=a_1(t)P(t)+P(t)a_1^T(t)+$$
$$2a_2(t)m(t)P(t)+2(a_2(t)m(t)P(t))^T. \quad (20)$$

$$E(g(x,t)g^T(x,t) \mid F_t^Y)=b_0(t)b_0^T(t)+b_0(t)(b_1(t)m(t))^T+$$
$$(b_1(t)m(t))b_0^T(t)+b_1(t)P(t)b_1^T(t)+b_1(t)m(t)m^T(t)b_1^T(t)+$$
$$b_0(t)(P(t)+m(t)m^T(t))b_2^T(t)+$$
$$b_2(t)(P(t)+m(t)m^T(t))b_0^T(t)+ \quad (21)$$
$$b_1(t)(3m(t)P(t)+m(t)(m(t)m^T(t)))b_2^T(t)+$$
$$b_2(t)(3P(t)m^T(t)+(m(t)m^T(t))m^T(t))b_1^T(t)+$$
$$3b_2(t)P^2(t)b_2^T(t)+3b_2(t)(P(t)m(t)m^T(t)+$$
$$m(t)m^T(t)P(t))b_2^T(t)+b_2(t)(m(t)m^T(t))^2b_2^T(t).$$

where $b_2^T(t)$ denotes the tensor obtained from $b_2(t)$ by transposing its two rightmost indices. Substituting the expression (19) in (4) and the expressions (20),(21) in (5), the filtering equations for the optimal estimate $m(t)$ and the error variance $P(t)$ are obtained

$$dm(t)=(a_0(t)+a_1(t)m(t)+a_2(t)m(t)m^T(t)+a_2(t)P(t))dt+$$

$$P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t)-(A_0(t)+A(t)m(t))dt],$$
$$m(t_0)=E(x(t_0)\mid F_t^Y)), \qquad (22)$$
$$dP(t)=(a_1(t)P(t)+P(t)a_1^T(t)+2a_2(t)m(t)P(t)+$$
$$2(a_2(t)m(t)P(t))^T+b_0(t)b_0^T(t)+b_0(t)(b_1(t)m(t))^T+$$
$$(b_1(t)m(t))b_0^T(t)+b_1(t)P(t)b_1^T(t)+$$
$$b_1(t)m(t)m^T(t)b_1^T(t)+b_0(t)(P(t)+m(t)m^T(t))b_2^T(t)+$$
$$b_2(t)(P(t)+m(t)m^T(t))b_0^T(t)+$$
$$b_1(t)(3m(t)P(t)+m(t)(m(t)m^T(t)))b_2^T(t)+$$
$$b_2(t)(3P(t)m^T(t)+(m(t)m^T(t))m^T(t))b_1^T(t)+$$
$$3b_2(t)P^2(t)b_2^T(t)+3b_2(t)(P(t)m(t)m^T(t)+$$
$$m(t)m^T(t)P(t))b_2^T(t)+b_2(t)(m(t)m^T(t))^2b_2^T(t))dt-$$
$$P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt.$$
$$P(t_0)=E((x(t_0)-m(t_0))(x(t_0)-m(t_0))^T\mid F_t^Y)). \qquad (23)$$

By means of the preceding derivation, the following result is proved.

Theorem 1. The optimal finite-dimensional filter for the bilinear state with bilinear multiplicative noise (1), where the bilinear polynomials $f(x,t)$ and $g(x,t)$ are defined by (17),(18), over the linear observations (2), is given by the equation (22) for the optimal estimate $m(t)=E(x(t)\mid F_t^Y)$ and the equation (23) for the estimation error variance $P(t)=E[(x(t)-m(t))(x(t)-m(t))^T\mid F_t^Y]$.

Thus, based on the general non-closed system of the filtering equations (4),(5), it is proved that the closed system of the filtering equations can be obtained for any polynomial state (1) over linear observations (2). Furthermore, the specific form (22),(23) of the closed system of the filtering equations corresponding to a bilinear state is derived.

IV. Example

This section presents an example of designing the optimal filter for a quadratic bi-dimensional state with a quadratic multiplicative noise over linear observations and comparing it to the optimal filter for a quadratic state with a state-independent noise and a conventional extended Kalman-Bucy filter.

Let the bi-dimensional real state $x(t)$ satisfy the quadratic system

$$\dot{x}_1(t)=x_2(t)+0.1x_1^2(t)\psi_1(t), \quad x_1(0)=x_{10}, \qquad (24)$$
$$\dot{x}_2(t)=0.1x_2^2(t), \quad x_2(0)=x_{20},$$

and the scalar observation process be given by the linear equation

$$y(t)=x(t)+\psi_2(t). \qquad (25)$$

where $\psi_1(t)$ and $\psi_2(t)$ are white Gaussian noises, which are the weak mean square derivatives of standard Wiener processes (see [17]). The equations (24),(25) present the conventional form for the equations (1),(2), which is actually used in practice [18].

The filtering problem is to find the optimal estimate for the quadratic bi-dimensional state with quadratic noise (24), using linear observations (25) confused with independent and identically distributed disturbances modeled as white Gaussian noises. Let us set the filtering horizon time to $T=0.92$.

The filtering equations (22),(23) take the following particular form for the system (24),(25)

$$\dot{m}_1(t)=m_2(t)+P_{11}(t)[y(t)-m_1(t)], \qquad (26)$$
$$\dot{m}_2(t)=0.1m_2^2+0.1P_{22}(t)+P_{12}(t)[y(t)-m_1(t)],$$

with the initial condition

$$m(0)=E(x(0)\mid y(0))=m_0,$$

and

$$\dot{P}_{11}(t)=2P_{12}(t)-0.97P_{11}^2(t)+0.03P_{12}^2(t)+ \qquad (27)$$
$$0.06P_{12}(t)m_1(t)m_2(t)+0.01m_1^4+0.01m_1^2m_2^2,$$
$$\dot{P}_{12}(t)=P_{22}(t)+0.2m_2(t)P_{12}(t)-P_{11}(t)P_{12}(t),$$
$$\dot{P}_{22}(t)=0.4m_2(t)P_{22}(t)-P_{12}^2(t),$$

with the initial condition

$$P(0)=E((x(0)-m(0))(x(0)-m(0))^T\mid y(0))=P_0.$$

The estimates obtained upon solving the equations (26)–(27) are compared first to the estimates satisfying the optimal filtering equations for a quadratic state with a state-independent noise (see [12]), based on the system (24) where the quadratic multiplicative noise $x^2(t)\psi_1(t)$ is replaced by the standard additive noise $\psi_1(t)$. The corresponding filtering equations are given by

$$\dot{m}_{I1}(t)=m_{I2}(t)+P_{I11}(t)[y(t)-m_{I1}(t)], \qquad (28)$$
$$\dot{m}_{I2}(t)=0.1m_{I2}^2+0.1P_{I22}(t)+P_{I12}(t)[y(t)-m_{I1}(t)],$$

with the initial condition

$$m_I(0)=E(x(0)\mid y(0))=m_{I0},$$

and

$$\dot{P}_{I11}(t)=2P_{I12}(t)+0.01-P_{I11}^2(t), \qquad (29)$$
$$\dot{P}_{I12}(t)=P_{I22}(t)+0.2m_{I2}(t)P_{I12}(t)-P_{I11}(t)P_{I12}(t),$$
$$\dot{P}_{I22}(t)=0.4m_{I2}(t)P_{I22}(t)-P_{I12}^2(t),$$

with the initial condition

$$P_I(0)=E((x(0)-m(0))(x(0)-m(0))^T\mid y(0))=P_{I0}.$$

The estimates obtained upon solving the equations (26)–(27) are also compared to the estimates satisfying the following extended Kalman-Bucy filtering equations for the quadratic state (24) over the linear observations (25), which are obtained assuming the standard additive noise term $\psi_1(t)$ in the first component of the state, using the direct copy of the state dynamics (24) in the estimate equation, and assigning the filter gain as the solution of the Riccati equation:

$$\dot{m}_{K1}(t)=m_{K2}(t)+P_{K11}(t)[y(t)-m_{K1}(t)], \qquad (30)$$

$$\dot{m}_{K2}(t) = 0.1m_{K2}^2 + 0.1P_{K22}(t) + P_{K12}(t)[y(t) - m_{K1}(t)],$$

with the initial condition

$$m_K(0) = E(x(0) \mid y(0)) = m_{K0},$$

and

$$\dot{P}_{K11}(t) = 2P_{K12}(t) + 0.01 - P_{K11}^2(t),$$

$$\dot{P}_{K12}(t) = P_{K22}(t) + 0.2P_{K12}(t) - P_{K11}(t)P_{K12}(t), \qquad (31)$$

$$\dot{P}_{K22}(t) = 0.4P_{K22}(t) - P_{K12}^2(t),$$

with the initial condition

$$P_K(0) = E((x(0) - m(0))(x(0) - m(0))^T \mid y(0)) = P_{K0}.$$

Numerical simulation results are obtained solving the systems of filtering equations (26)–(27), (28)–(29), and (30)–(31). The obtained values of the estimates $m_1(t)$, $m_2(t)$, $m_{I1}(t)$, $m_{I2}(t)$, $m_{K1}(t)$ and $m_{K2}(t)$ satisfying the equations (26), (28), and (30), respectively, are compared to the real values of the state variables $x_1(t)$ and $x_2(t)$ in (24).

For each of the three filters (26)–(27), (28)–(29), and (30)–(31), and the reference system (24)–(25) involved in simulation, the following initial values are assigned: $x_{10} = 10.1$, $x_{20} = 10.1$, $m_{10} = 1.1$, $m_{20} = 1.1$, $P_{110} = 10$, $P_{120} = 1$, $P_{220} = 10$. Gaussian disturbances $\psi_1(t)$ in (24) and $\psi_2(t)$ in (25) are realized using the built-in MatLab white noise function.

The following graphs are obtained: graphs of the error between the reference states variables $x_1(t)$ and $x_2(t)$ satisfying the equations (24) and the optimal filter estimates $m_1(t)$ and $m_2(t)$ satisfying the equations (26), are shown in Figure 1; graph of the error between the reference states variables $x_1(t)$ and $x_2(t)$ satisfying the equations (24) and the estimates $m_{I1}(t)$ and $m_{I2}(t)$ satisfying the equations (28), are shown in Figure 3; graph of the error between the reference states variables $x_1(t)$ and $x_2(t)$ satisfying the equations (24) and the estimates $m_{K1}(t)$ and $m_{K2}(t)$ satisfying the equations (30), are shown in Figure 5. The graphs of all estimate errors are shown on the simulation interval from $t_0 = 0$ to $T = 0.92$. Graphs of those estimation errors are also shown closely in the simulation interval from $t = 0.80$ to $T = 0.92$ in Figs. 2, 4, and 6, respectively. It can be observed that the estimation errors given by the optimal filter (26) rapidly reach and then maintain the zero mean value even in a close vicinity of the asymptotic time point $T = 0.99$, where the reference quadratic state variables (24) diverge to infinity. On the contrary, the errors given by the other considered filters reach zero more slowly or do not reach it at all, have systematic (biased) deviations from zero, and clearly diverge to infinity near the asymptotic time point. Note that the optimal filtering error variance $P(t)$ does not converge to zero as time tends to the asymptotic time point, since the polynomials dynamics of fourth order is stronger than the quadratic Riccati terms in the right-hand side of the equation (27).

Thus, it can be concluded that the obtained optimal filter (26)-(27) for a quadratic state with a quadratic multiplicative noise over incomplete linear observations yield definitely better estimates than the optimal filter for a quadratic state with a state-independent noise or a conventional extended Kalman-Bucy filter.

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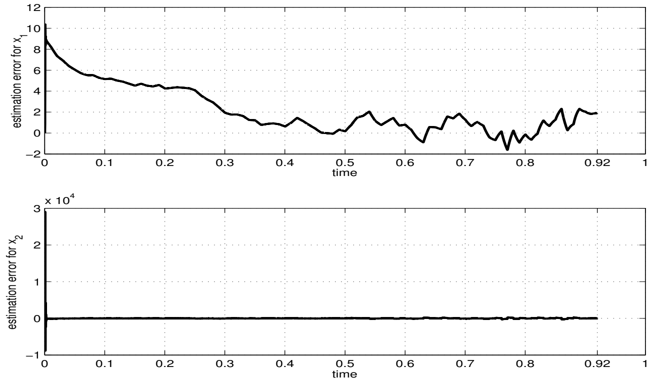

Fig. 1. Graph of the error between the real state $x_1(t)$ satisfying the first equation in (24) and the optimal filter estimate $m_1(t)$ satisfying the first equation in (26), and graph of the error between the real state $x_2(t)$ satisfying the second equation in (24) and the optimal filter estimate $m_2(t)$ satisfying the second equation in (26), on the entire simulation interval $[0, 0.92]$.

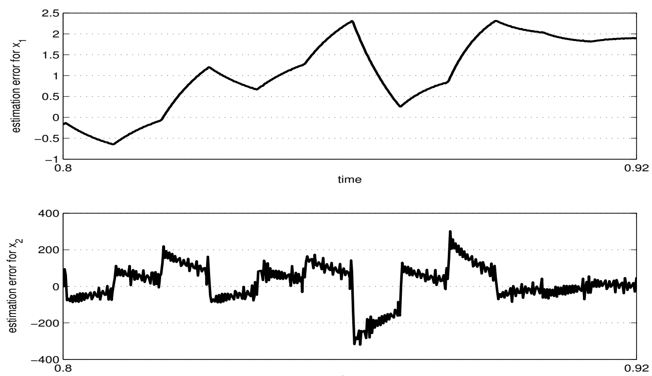

Fig. 2. Graph of the error between the real state $x_1(t)$ satisfying the first equation in (24) and the optimal filter estimate $m_1(t)$ satisfying the first equation in (26), graph of the error between the real state $x_2(t)$ satisfying the second equation in (24) and the optimal filter estimate $m_2(t)$ satisfying the second equation in (26), on the simulation interval $[0.80, 0.92]$.

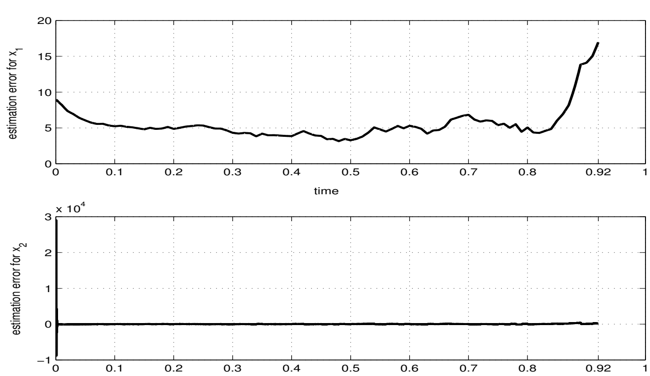

Fig. 3. Graph of the error between the real state $x_1(t)$ satisfying the first equation in (24) and the estimate $m_{I1}(t)$ satisfying the first equation in (28), and graph of the error between the real state $x_2(t)$ satisfying the second equation in (24) and the estimate $m_{I2}(t)$ satisfying the second equation in (28), on the entire simulation interval $[0, 0.92]$.

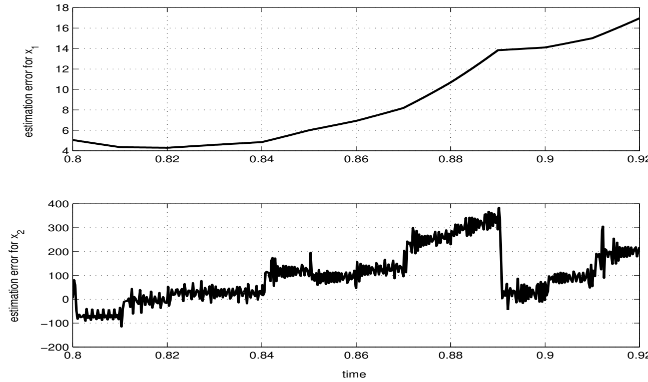

Fig. 4. Graph of the error between the real state $x_1(t)$ satisfying the first equation in (24) and the estimate $m_{I1}(t)$ satisfying the first equation in (28), and graph of the error between the real state $x_2(t)$ satisfying the second equation in (24) and the estimate $m_{I2}(t)$ satisfying the second equation in (28), on the simulation interval $[0.80, 0.92]$.

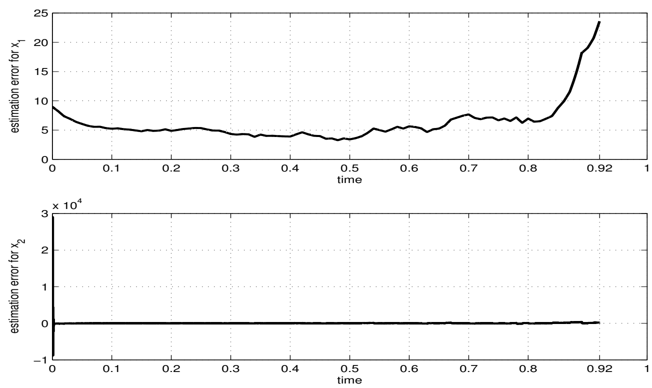

Fig. 5. Graph of the error between the real state $x_1(t)$ satisfying the first equation in (24) and the estimate $m_{K1}(t)$ satisfying the first equation in (30), and graph of the error between the real state $x_2(t)$ satisfying the second equation in (24) and the estimate $m_{K2}(t)$ satisfying the second equation in (30), on the entire simulation interval $[0, 0.92]$.

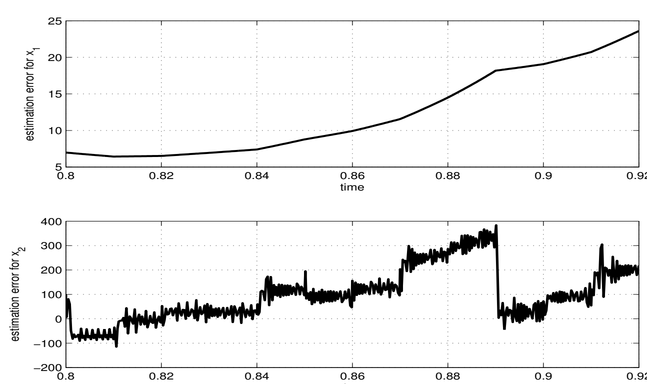

Fig. 6. Graph of the error between the real state $x_1(t)$ satisfying the first equation in (24) and the estimate $m_{K1}(t)$ satisfying the first equation in (30), and graph of the error between the real state $x_2(t)$ satisfying the second equation in (24) and the estimate $m_{K2}(t)$ satisfying the second equation in (30), on the simulation interval $[0.80, 0.92]$.