2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

γ-Independent H_∞-Discretization of Sampled-Data Systems by Modified Fast-Sample/Fast-Hold Approximation with Fast Lifting

Tomomichi Hagiwara and Koichiro Okada

Department of Electrical Engineering, Kyoto University
Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan.

Abstract— **This paper is concerned with H_∞-discretization for analysis and design of sampled-data control systems and provides a new method with an approximation approach called modified fast-sample/fast-hold approximation. By applying the fast-lifting technique, quasi-finite-rank approximation of an infinite-rank operator and then the loop-shifting technique, this new method can discretize the continuous-time generalized plant in a γ-independent fashion even when the given sampled-data system has a nonzero direct feedthrough term from the disturbance input w to the controlled output z, unlike in the previous study. With this new method, we can obtain both the upper and lower bounds of the H_∞-norm or the frequency response gain of any sampled-data systems regardless of the existence of nonzero D_{11}. Furthermore, the gap between the upper and lower bounds can be bounded with the approximation parameter N and is independent of the discrete-time controller. This feature is significant in applying the new method especially to control system design, and this study indeed has a very close relationship to the recent progress in the study of control system analysis/design via noncausal linear periodically time-varying scaling. We demonstrate the effectiveness of the new method through a numerical example.**

I. Introduction

It is essential for the analysis and design of sampled-data systems that we deal with the intersample behavior of continuous-time signals as it is. There exist studies on the techniques for such treatment, e.g., the lifting technique [3],[12]–[15], the FR-operator technique [2], the parametric transfer function approach [11], and so on. These techniques can be regarded as methods for manipulating infinite-dimensional operators in the definitions of the H_∞-norm and the frequency response gain of sampled-data systems and then reducing the infinite-dimensional analysis or design problems to finite-dimensional ones in an exact fashion. On the other hand, an approximation approach called fast-sample/fast-hold (FSFH) approximation [16] was also proposed, in which the approximation error is assured to converge to zero as the approximation parameter N tends to infinity. This approximation approach can be regarded as a method that reduces the infinite-dimensional problems to finite-dimensional ones in an asymptotically exact fashion. A similar approach called modified FSFH approximation was also proposed in [7]. This latter approach also discretizes the continuous-time generalized plant in a γ-independent fashion as in the former conventional FSFH approximation approach[†] and leads to a discrete-time generalized plant with a similar structure to what is obtained by the former. In contrast to the former, however, the latter allows us to obtain both the upper and lower bounds of the H_∞-norm or the frequency response gain of sampled-data systems, and the gap between these bounds can be evaluated in advance for each fixed approximation parameter N. This is very important particularly in control system design, and thus modified FSFH approximation can be said to provide useful features that are not present in the conventional FSFH approximation.

As opposed to the conventional FSFH approximation, however, the H_∞-discretization by the modified FSFH approximation developed in [7] is based on the assumption that the direct feedthrough term from the disturbance input w to the controlled output z, denoted by D_{11}, in the sampled-data system is zero. To get around this assumption, we apply the well-known loop-shifting technique in this paper, but the arguments are nontrivial. This is because the loop-shifting generally leads to a γ-dependent generalized plant, so that simply applying the loop-shifting technique on the continuous-time generalized plant leads to a loss of one of the most important features of the modified FSFH approximation. Thus we develop a method for circumventing the problem by working on what we call fast-lifted frequency response operators [7] and then carrying out some special factorizations of matrices represented as operator compositions.

The contents of this paper are as follows. Section II reviews the lifting-based transfer operators and frequency response operators of sampled-data systems. In Section III, we introduce a key technique for the modified FSFH approximation called fast-lifting, and give an extension of the H_∞-discretization method by taking nonzero D_{11} into consideration. In Section IV, we give a numerical example and demonstrate the effectiveness of the new method, and Section V concludes the paper.

[†] γ-independent discretization is such a discretization method that is required to be carried out only once independently of the H_∞ performance level γ. On the other hand, γ-dependent discretization (e.g., [3]) is a standard method, which is required to be carried out every time γ changes in the so-called γ-iteration process.

II. Lifting-Based Transfer Operators and Frequency Response Operators

We collect in this section some definitions and fundamental results pertinent to the lifting technique [3],[12]–[15].

Let us consider the sampled-data system Σ shown in Fig. 1, in which P represents the continuous-time linear time-invariant (LTI) generalized plant, while Ψ, $\mathcal{S}$ and $\mathcal{H}$ represent the discrete-time LTI controller, the ideal sampler and the zero-order hold, respectively, all operating at the sampling period h. Suppose that P and Ψ are described by

$$\begin{aligned} \frac{dx}{dt} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x \end{aligned} \tag{1}$$

and

$$\begin{aligned} \psi_{k+1} &= A_\Psi \psi_k + B_\Psi y_k \\ u_k &= C_\Psi \psi_k + D_\Psi y_k \end{aligned} \tag{2}$$

respectively, where $y_k = y(kh)$, $u(t) = u_k$ $(kh \le t < (k+1)h)$. We assume that $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $w(t) \in \mathbf{R}^l$ and $z(t) \in \mathbf{R}^p$, and that Σ is internally stable. Let us define $x_k := x(kh)$ and denote by $\{\widehat{w}_k\}_{k=1}^{\infty}$ and $\{\widehat{z}_k\}_{k=1}^{\infty}$ the lifted representations of $w(t)$ and $z(t)$, respectively, with the sampling period h. Now, let us denote by $\mathcal{K}_\mu$, or sometimes just by $\mathcal{K}$ for simplicity, the Hilbert space $(L_2[0,h))^\mu$ of square integrable μ-dimensional vector functions over the time interval $[0,h)$ with the standard inner product. We assume that $\widehat{w}_k \in \mathcal{K}_l$ and thus $\widehat{z}_k \in \mathcal{K}_p$. The lifted representation of the system Σ is given by

$$\begin{aligned} \xi_{k+1} &= \mathcal{A}\xi_k + \mathcal{B}\widehat{w}_k \\ \widehat{z}_k &= \mathcal{C}\xi_k + \mathcal{D}\widehat{w}_k \end{aligned} \tag{3}$$

with the matrix $\mathcal{A}$ and the operators $\mathcal{B}, \mathcal{C}, \mathcal{D}$ defined appropriately, where $\xi_k := [x_k^T,\ \psi_k^T]^T$. Based on this representation, the lifting-based transfer operator of the sampled-data system Σ is defined by

$$\widehat{G}(\zeta) = \mathcal{C}(\zeta I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} \tag{4}$$

and the frequency response operator is defined as $\widehat{G}(e^{j\varphi h})$, $\varphi \in \mathcal{I}_0 := (-\omega_s/2, \omega_s/2]$, where $\omega_s := 2\pi/h$. Furthermore, the frequency response gain and the H_∞-norm of Σ are defined respectively as

$$\|\widehat{G}(e^{j\varphi h})\| = \sup_{\widehat{w}\in\mathcal{K}} \frac{\|\widehat{G}(e^{j\varphi h})\widehat{w}\|_{\mathcal{K}}}{\|\widehat{w}\|_{\mathcal{K}}} \tag{5}$$

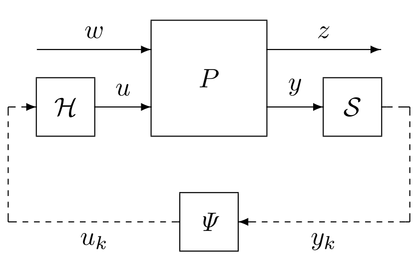

Fig. 1. Sampled-data system Σ.

$$\|\widehat{G}(\zeta)\|_\infty = \max_{\varphi\in\mathcal{I}_0} \|\widehat{G}(e^{j\varphi h})\| \tag{6}$$

where $\|\cdot\|_{\mathcal{K}}$ denotes the norm on $\mathcal{K}$.

The operator $\mathcal{D}$ in (4) can be represented as $\mathcal{D} = \mathbf{D}_{110} + D_{11}(=: \mathbf{D}_{11})$, where the first term on the right-hand side is the Hilbert-Schmidt operator given by

$$\mathbf{D}_{110} : \mathcal{K}_l \ni w \mapsto z \in \mathcal{K}_p,$$
$$z(\theta) = \int_0^\theta C_1 \exp\{A(\theta-\sigma)\} B_1 w(\sigma) d\sigma \tag{7}$$

and the second term is the operator of multiplication by the matrix D_{11}; in this paper, we use the same symbol for the underlying matrix and the associated operator of multiplication for notational simplicity, but they can be easily distinguished from the context. The definitions of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are omitted due to limited space; they are not required explicitly in the following, but we just mention that $\mathcal{A}$ involves the matrices

$$A_d := \exp(Ah),\ B_{2d} := \int_0^h \exp(A\sigma) B_2 d\sigma,\ C_{2d} := C_2 \tag{8}$$

III. Modified Fast-Sample/Fast-Hold Approximation

In this section, we give an extension of the modified FSFH approximation method [7] with nonzero D_{11} taken into account, and show that the frequency response gain and the H_∞-norm can still be evaluated to any degree of accuracy with a discretized generalized plant that is derived in a γ-independent fashion.

A. Application of the Fast-Lifting Technique and Quasi-Finite-Rank Approximation

We first introduce the fast-lifting operator $\mathcal{L}_N$ [5],[7], which plays a key role in modified FSFH approximation. For positive integers N and μ, let us define $h' := h/N$ and $(L_2[0,h'))^\mu =: \mathcal{K}'_\mu$ (which we sometimes denote $\mathcal{K}'$ for simplicity). For $x \in \mathcal{K}$, we define $x^{(i)} \in \mathcal{K}'$ $(i = 1, \cdots, N)$ by

$$x^{(i)}(\theta') := x((i-1)h' + \theta') \quad (0 \le \theta' < h') \tag{9}$$

Then, we define $\check{x} := [(x^{(1)})^T\ \cdots\ (x^{(N)})^T]^T$, and refer to the mapping from $x \in \mathcal{K}$ to $\check{x} \in (\mathcal{K}')^N$ as fast lifting. We denote it by

$$\check{x} = \mathcal{L}_N x \tag{10}$$

It obviously follows from the definition of $\mathcal{L}_N$ that

$$\|\mathcal{L}_N \widehat{G}(e^{j\varphi h})\mathcal{L}_N^{-1}\| = \|\widehat{G}(e^{j\varphi h})\| \tag{11}$$

where the left-hand side of (11) is defined as the induced norm on $\mathcal{K}'$ in a parallel fashion to (5). We call $\mathcal{L}_N \widehat{G}(e^{j\varphi h})\mathcal{L}_N^{-1}$ the fast-lifted frequency operator, and we study how to compute its norm, as suggested by (11). To that end, we first recall that an explicit representation of the fast-lifted frequency operator has been shown in [7] for the case of $D_{11}=0$, which we briefly review as follows.

First, as a result of applying fast-lifting, there arises the operator $\mathbf{D}'_{110}$, which is nothing but $\mathbf{D}_{110}$ given by (7) with the underlying horizon $[0,h)$ replaced by $[0,h')$. Then, to get around the difficulty stemming from its infinite-rank nature and reduce the problem to finite-dimensional computations, this operator was approximated by the finite-rank operator of the form $\mathbf{M}'_1X\mathbf{B}'_1$, where $\mathbf{B}'_1$ and $\mathbf{M}'_1$ are the operators defined by

$$\mathbf{B}'_1 : w \mapsto \int_0^{h'} \exp\{A(h'-\sigma)\}B_1 w(\sigma)d\sigma \tag{12}$$

$$\mathbf{M}'_1 : \begin{bmatrix} x \\ u \end{bmatrix} \mapsto z',$$

$$z'(\theta') = [C_1\ \ D_{12}]\exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}\theta'\right)\begin{bmatrix} x \\ u \end{bmatrix} \tag{13}$$

and X is a matrix introduced for the approximation purpose, which we determine later. We denote the approximation error by

$$\mathbf{E}' = \mathbf{D}'_{110} - \mathbf{M}'_1X\mathbf{B}'_1 \tag{14}$$

Then, the fast-lifted frequency response operator was shown to be represented by

$$\mathcal{L}_N\widehat{G}(e^{j\varphi h})\mathcal{L}_N^{-1} = \overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{\mathbf{E}'} \tag{15}$$

where $\overline{\mathbf{E}'}$ is defined as

$$\overline{\mathbf{E}'} = \mathrm{diag}[\mathbf{E}',\cdots,\mathbf{E}'] \tag{16}$$

consisting of N copies of $\mathbf{E}'$ and the operators $\overline{\mathbf{B}'_1}$ and $\overline{\mathbf{M}'_1}$ are also defined in a parallel way; the notation $\overline{(\cdot)}$ will be used in the same meaning throughout the paper, not only for operators but also for matrices. The matrix $Z_N(\zeta)$ in (15), on the other hand, is given by

$$Z_N(\zeta) := \begin{bmatrix} I \\ \vdots \\ (A'_{2d})^{N-1} \end{bmatrix} Z(\zeta)[\ (A'_d)^{N-1}\ \cdots\ I\ ] + \begin{bmatrix} X & 0 & \cdots & \cdots & 0 \\ J & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ (A'_{2d})^{N-2}J & \cdots & \cdots & J & X \end{bmatrix} \tag{17}$$

with X in (14), $J := [I, 0]^T$ and A'_d, A'_{2d}, $Z(\zeta)$ defined by

$$A'_d := \exp(Ah'),\ A'_{2d} := \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} h'\right) \tag{18}$$

$$Z(\zeta) := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix}(e^{j\varphi h}I - \mathcal{A})^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \tag{19}$$

Now, let us return to the case with $D_{11}\neq 0$. In this case, we still apply the same approximation regarding $\mathbf{D}'_{110}$ in $\mathbf{D}'_{11} = \mathbf{D}'_{110} + D_{11}$. This leads to approximating the infinite-rank operator $\mathbf{D}'_{11}$ by $\mathbf{M}'_1X\mathbf{B}_1 + D_{11}$, which we call quasi-finite-rank approximation of $\mathbf{D}'_{11}$, since D_{11} is generally of infinite rank. Then, (15) only changes by $\overline{D_{11}}$ so that

$$\mathcal{L}_N\widehat{G}(e^{j\varphi h})\mathcal{L}_N^{-1} = \overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{B'_1} + \overline{D_{11}} + \overline{\mathbf{E}'} \tag{20}$$

Applying the triangle inequality to (20), it follows that

$$\begin{aligned}\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\| - \gamma_N &\le \|\widehat{G}(e^{j\varphi h})\| \\ &\le \|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\| + \gamma_N\end{aligned} \tag{21}$$

with γ_N given by

$$\gamma_N := \|\overline{\mathbf{E}'}\| = \|\mathbf{E}'\| \tag{22}$$

Since $\gamma_N = \|\mathbf{E}'\| \le \|\mathbf{E}'\|_{\mathrm{HS}}$, we also have similar inequalities with γ_N replaced by $\|\mathbf{E}'\|_{\mathrm{HS}}$, where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. There exist methods for finding the matrix X in (14) minimizing $\|\mathbf{E}'\|$ or $\|\mathbf{E}'\|_{\mathrm{HS}}$ [6],[10],[7], together with the resulting norm of $\mathbf{E}'$. In the minimization, dealing with $\|\mathbf{E}'\|_{\mathrm{HS}}$ is much simpler and seems numerically more reliable, and this is why we also consider $\|\mathbf{E}'\|_{\mathrm{HS}}$. In any case, it is shown in [6] and [7] that $\|\mathbf{E}'\|\to 0$ and $\|\mathbf{E}'\|_{\mathrm{HS}}\to 0$ as $N\to\infty$ $(h'\to 0)$ under optimal approximation. This implies that $\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\|$ gives a value that is close enough to the frequency response gain if N is large enough.

B. Computation of $\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\|$

We show that $\|\overline{\mathbf{M}'_1}Z_N(e^{j\varphi h})\overline{\mathbf{B}'_1} + \overline{D_{11}}\|$ can be computed exactly if we introduce an appropriate discretized system. To this end, we begin with a preliminary result on operator compositions, which plays a crucial role in this paper.

Lemma 1 Let $F_{ll}\in\mathbf{R}^{l\times l}$, $F_{lp}\in\mathbf{R}^{l\times p}$ and $F_{pp}\in\mathbf{R}^{p\times p}$ be arbitrary matrices, and let us consider the matrices $\mathbf{B}'_1F_{ll}\left(\mathbf{B}'_1\right)^*$, $\mathbf{B}'_1F_{lp}\mathbf{M}'_1$ and $\left(\mathbf{M}'_1\right)^*F_{pp}\mathbf{M}'_1$ defined as the operator compositions with the operators $\mathbf{B}'_1$ and $\mathbf{M}'_1$ together with the operators of multiplication by the matrices F_{ll}, F_{lp} and F_{pp}. Then, these matrices can be equivalently represented as matrix products in such a way that the underlying matrices F_{ll}, F_{lp} and F_{pp} are left explicitly. More specifically, we have

$$\begin{aligned}\mathbf{B}'_1F_{ll}\left(\mathbf{B}'_1\right)^* &= W'\left(F_{ll}\otimes I_s\right)(W')^T \\ \mathbf{B}'_1F_{lp}\mathbf{M}'_1 &= W'\left(F_{lp}\otimes I_s\right)V' \\ \left(\mathbf{M}'_1\right)^*F_{pp}\mathbf{M}'_1 &= (V')^T\left(F_{pp}\otimes I_s\right)V'\end{aligned} \tag{23}$$

where I_s denotes the $s\times s$ identity matrix with s given in (29), the matrices W' and V' are given by (30), and $\otimes$ denotes the Kronecker product.

Proof: We first define $b_{1\alpha}$ $(\alpha = 1,\cdots,l)$ and $c_{1\beta}$, $d_{12\beta}$ $(\beta = 1,\cdots,p)$ by

$$B_1 =: [\ b_{11}\ \cdots\ b_{1l}\ ],\quad [\ C_1\ \ D_{12}\ ] =: \begin{bmatrix} c_{11} & d_{121} \\ \vdots & \vdots \\ c_{1p} & d_{12p} \end{bmatrix} \tag{24}$$

and then

$$\underline{b_1} := \begin{bmatrix} b_{11} \\ \vdots \\ b_{1l} \end{bmatrix},\ \underline{m_1} := [\,c_{11}\ d_{121}\ \cdots\ c_{1p}\ d_{12p}\,] \tag{25}$$

We also define $\underline{A} := I_l \otimes A$ and $\underline{A_2} := I_p \otimes \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}$ and then introduce the three matrices

$$K' := \int_0^{h'} \exp\{\underline{A}(h'-\sigma)\}\, \underline{b_1}\, \underline{b_1}^T \exp\{\underline{A}^T(h'-\sigma)\} d\sigma \quad (26)$$

$$L' := \int_0^{h'} \exp\left(\underline{A_2}^T \sigma\right) \underline{m_1}^T \underline{m_1} \exp\left(\underline{A_2}\sigma\right) d\sigma \quad (27)$$

$$J' := \int_0^{h'} \exp\{\underline{A}(h'-\sigma)\}\, \underline{b_1}\, \underline{m_1} \exp\left(\underline{A_2}\sigma\right) d\sigma \quad (28)$$

Then, it is easy to see that $\begin{bmatrix} K' & J' \\ (J')^T & L' \end{bmatrix}$ can be represented as an integral of a nonnegative definite matrix function, so that it can be factored into the form

$$\begin{bmatrix} K' & J' \\ (J')^T & L' \end{bmatrix} =: \begin{bmatrix} W'_1 \\ \vdots \\ W'_l \\ (V'_1)^T \\ \vdots \\ (V'_p)^T \end{bmatrix} \begin{bmatrix} W'_1 \\ \vdots \\ W'_l \\ (V'_1)^T \\ \vdots \\ (V'_p)^T \end{bmatrix}^T$$

$$W'_\alpha \in \mathbf{R}^{n\times s} \ (\alpha = 1, \cdots, l),$$
$$V'_\beta \in \mathbf{R}^{s\times(n+m)} \ (\beta = 1, \cdots, p) \quad (29)$$

with an appropriate positive integer s. From the matrices on the right-hand side, we define the matrices W' and V' by

$$W' := [\, W'_1 \ \ \cdots \ \ W'_l \,], \ V' := \begin{bmatrix} V'_1 \\ \vdots \\ V'_p \end{bmatrix} \quad (30)$$

Now, we only prove the first equation in (23); the other two equations can be proved similarly. Let us denote the (α, β) entry of F_{ll} by $f_{\alpha\beta}^{(ll)}$ where $\alpha, \beta = 1, \cdots, l$. Then it follows from (12) and (24) that

$$\begin{aligned} &\mathbf{B}'_1 F_{ll} (\mathbf{B}'_1)^* \\ &= \int_0^{h'} \exp\{A(h'-\sigma)\} \left(\sum_{\alpha=1}^{l} \sum_{\beta=1}^{l} f_{\alpha\beta}^{(ll)} b_{1\alpha} b_{1\beta}^T \right) \\ &\qquad\qquad \cdot \exp\left\{A^T(h'-\sigma)\right\} d\sigma \\ &= \sum_{\alpha=1}^{l} \sum_{\beta=1}^{l} f_{\alpha\beta}^{(ll)} K'_{\alpha\beta} \end{aligned} \quad (31)$$

with $K'_{\alpha\beta}$ defined by

$$K'_{\alpha\beta} = \int_0^{h'} \exp\{A(h'-\sigma)\}\, b_{1\alpha} b_{1\beta}^T \exp\left\{A^T(h'-\sigma)\right\} d\sigma \quad (32)$$

Since $K'_{\alpha\beta}$ is the submatrix at the α-th block column and β-th block row of K' given in (26), it follows from (29) that $K'_{\alpha\beta} = W'_\alpha (W'_\beta)^T$ and hence (31) leads to

$$\begin{aligned} &\mathbf{B}'_1 F_{11} (\mathbf{B}'_1)^* \\ &= [\, W'_1 \ \cdots \ W'_l \,] \begin{bmatrix} f_{11}^{(ll)} I_s & \cdots & f_{1l}^{(ll)} I_s \\ \vdots & \ddots & \vdots \\ f_{l1}^{(ll)} I_s & \cdots & f_{ll}^{(ll)} I_s \end{bmatrix} \begin{bmatrix} (W'_1)^T \\ \vdots \\ (W'_l)^T \end{bmatrix} \\ &= W' \left(F_{ll} \otimes I_s\right) (W')^T \end{aligned} \quad (33)$$

This is nothing but the first equation in (23). ■

By applying the loop-shifting technique and using Lemma 1, we can obtain the following result about the computation of $\|\overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{D_{11}}\|$. It is somewhat related to the results in [4] but is much more general and entirely different in that a general disturbance w and a general controlled output z are considered and thus Lemma 1 plays a crucial role, apart from the fast-lifting context here.

Proposition 1 Let us define

$$\Phi_N(\zeta) := \overline{V'} Z_N(\zeta) \overline{W'} + \overline{D_{11} \otimes I_s} \quad (34)$$

Then, we have

$$\|\overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{D_{11}}\| = \max\left(\|D_{11}\|, \|\Phi_N(e^{j\varphi h})\|\right) \quad (35)$$

Proof: We first show that for any γ such that $\gamma > \|D_{11}\|$, the condition

$$\|\overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{D_{11}}\| < \gamma \quad (36)$$

is equivalent to the condition $\|\Phi_N(e^{j\varphi h})\| < \gamma$. Once this claim is established, the proposition follows readily from the well-known fact [15] that $\|\overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{D_{11}}\| \geq \|D_{11}\|, \ \forall \varphi \in \mathcal{I}_0$.

To establish the above claim, we first note that (36) is equivalent to the condition

$$\begin{aligned} &\gamma^2 I - (\overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{D_{11}})^* \\ &\qquad\qquad \cdot (\overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} + \overline{D_{11}}) > 0 \end{aligned} \quad (37)$$

Here, we define the Hermitian matrix $E\ (> 0)$ as follows.

$$E := \gamma^2 \left(\gamma^2 I - D_{11}^T D_{11}\right)^{-1} \quad (38)$$

Following the well-known technique of the loop-shifting, we multiply $\overline{E^{1/2}}$ from left and right of (37), which leads to the equivalent condition

$$\begin{aligned} &\gamma^2 I - \overline{E^{1/2}} \Big\{ \left(\overline{D_{11}}\right)^* \overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} \\ &\quad + \left(\overline{\mathbf{B}'_1}\right)^* Z_N(e^{j\varphi h})^* \left(\overline{\mathbf{M}'_1}\right)^* \overline{D_{11}} \\ &+ \left(\overline{\mathbf{B}'_1}\right)^* Z_N(e^{j\varphi h})^* \left(\overline{\mathbf{M}'_1}\right)^* \overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1} \Big\} \overline{E^{1/2}} > 0 \end{aligned} \quad (39)$$

or equivalently,

$$\gamma^2 I - \mathbf{Y}_1 \mathbf{Y}_2 > 0 \quad (40)$$

with

$$\mathbf{Y}_1 := \left[\overline{E^{1/2} D_{11}^*}\, \overline{\mathbf{M}'_1} \quad \overline{E^{1/2}} \left(\overline{\mathbf{B}'_1}\right)^* \quad \overline{E^{1/2}} \left(\overline{\mathbf{B}'_1}\right)^* \right]$$

$$\mathbf{Y}_2 := \begin{bmatrix} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1}\, \overline{E^{1/2}} \\ Z_N(e^{j\varphi h})^* \left(\overline{\mathbf{M}'_1}\right)^* \overline{D_{11} E^{1/2}} \\ Z_N(e^{j\varphi h})^* \left(\overline{\mathbf{M}'_1}\right)^* \overline{\mathbf{M}'_1} Z_N(e^{j\varphi h}) \overline{\mathbf{B}'_1}\, \overline{E^{1/2}} \end{bmatrix} \quad (41)$$

Since $\mathbf{Y}_1 \mathbf{Y}_2$ is obviously a compact operator, the condition (40) is equivalent to the condition that the eigenvalues of $\gamma^2 I - \mathbf{Y}_1 \mathbf{Y}_2$ are all positive (e.g., [9]). They are all positive if and only if the eigenvalues of $\gamma^2 I - \mathbf{Y}_2\mathbf{Y}_1$ are, and thus we consider $\mathbf{Y}_2\mathbf{Y}_1$ instead; $\mathbf{Y}_2\mathbf{Y}_1$ is actually a matrix and can be computed by applying Lemma 1. In fact, since we have

$$\overline{\mathbf{B}_1'}\,\overline{E}\,\left(\overline{\mathbf{B}_1'}\right)^* = \left(\overline{W'}\,\overline{E^{1/2}\otimes I_s}\right)\left(\overline{E^{1/2}\otimes I_s}\,\left(\overline{W'}\right)^T\right) \quad (42)$$

$$\overline{\mathbf{B}_1'}\,\overline{ED_{11}^*}\,\overline{\mathbf{M}_1'} = \left(\overline{W'}\,\overline{E^{1/2}\otimes I_s}\right) \cdot \left(\overline{E^{1/2}D_{11}^T\otimes I_s}\,\overline{V'}\right) \quad (43)$$

$$\left(\overline{\mathbf{M}_1'}\right)^*\,\overline{D_{11}ED_{11}^*}\,\overline{\mathbf{M}_1'} = \left(\left(\overline{V'}\right)^T\,\overline{D_{11}E^{1/2}\otimes I_s}\right) \cdot \left(\overline{E^{1/2}D_{11}^T\otimes I_s}\,\overline{V'}\right) \quad (44)$$

$$\left(\overline{\mathbf{M}_1'}\right)^*\,\overline{\mathbf{M}_1'} = \left(\overline{V'}\right)^T\,\overline{V'} \quad (45)$$

by Lemma 1, we see that

$$\mathbf{Y}_2\mathbf{Y}_1 = Y_2 Y_1 \quad (46)$$

with the matrices

$$Y_1 := \Big[\ \overline{E^{1/2}D_{11}^T\otimes I_s}\,\overline{V'} \quad \overline{E^{1/2}\otimes I_s}\,\left(\overline{W'}\right)^T \quad \overline{E^{1/2}\otimes I_s}\,\left(\overline{W'}\right)^T\ \Big]$$

$$Y_2 := \begin{bmatrix} Z_N(e^{j\varphi h})\overline{W'}\,\overline{E^{1/2}\otimes I_s} \\ Z_N(e^{j\varphi h})^*\left(\overline{V'}\right)^T\,\overline{D_{11}E^{1/2}\otimes I_s} \\ Z_N(e^{j\varphi h})^*\left(\overline{V'}\right)^T\,\overline{V'}Z_N(e^{j\varphi h})\overline{W'}\,\overline{E^{1/2}\otimes I_s} \end{bmatrix} \quad (47)$$

Note that Y_1 and Y_2 are nothing but $\mathbf{Y}_1$ and $\mathbf{Y}_2$ with the operators $\mathbf{B}_1'$ and $\mathbf{M}_1'$ replaced by the matrices W' and V' respectively and the operators $E^{1/2}$ and $D_{11}E^{1/2}$ replaced by the matrices $E^{1/2}\otimes I_s$ and $D_{11}E^{1/2}\otimes I_s$ respectively. Since the eigenvalues of $\gamma^2 I - \mathbf{Y}_2\mathbf{Y}_1 = \gamma^2 I - Y_2 Y_1$ are all positive if and only if those of $\gamma^2 I - Y_1 Y_2$ are and since $Y_1 Y_2$ is a Hermitian matrix, we readily have the equivalent condition $\gamma^2 I - Y_1 Y_2 > 0$. If we write down $Y_1 Y_2$ explicitly, it is easy to see that this condition is nothing but (39) with the same replacement as above. Hence, it is easy to see that multiplying $\overline{E^{-1/2}\otimes I_s}$ from left and right leads to the equivalent condition

$$\gamma^2 I - \left(\overline{V'}Z_N(e^{j\varphi h})\overline{W'} + \overline{D_{11}\otimes I_s}\right)^* \cdot \left(\overline{V'}Z_N(e^{j\varphi h})\overline{W'} + \overline{D_{11}\otimes I_s}\right) > 0 \quad (48)$$

which naturally has a form of (37) with the same replacement of $\mathbf{B}_1'$ and $\mathbf{M}_1'$ as above, together with the replacement of the operator D_{11} with the matrix $\overline{D_{11}\otimes I_s}$. Hence, by the definition of $\Phi_N(\zeta)$, the claim has been established. ∎

C. γ-Independent H_∞-Discretization

We are now ready to give a γ-independent H_∞-discretization method via modified FSFH approximation; the following arguments are mostly the same as those in the case of $D_{11} = 0$ [7], but are given explicitly to make the discussions clearer.

It follows by (17) and the definitions of $\overline{W'}$ and $\overline{V'}$ that $\Phi_N(\zeta)$ in (34) can be rewritten as

$$\Phi_N(\zeta) = [\,V_{1N} \quad V_{2N}\,]\,Z(\zeta)W_N + \Delta_{ND} \quad (49)$$

with W_N, V_{1N}, V_{2N} and Δ_{ND} given by

$$W_N := [\,(A_d')^{N-1}W' \quad \cdots \quad W'\,] \quad (50)$$

$$[\,V_{1N} \quad V_{2N}\,] := \begin{bmatrix} V' \\ \vdots \\ V'\,(A_{2d}')^{N-1} \end{bmatrix} \quad (51)$$

$$\Delta_{ND} := \Delta_N + \overline{D_{11}\otimes I_s} \quad (52)$$

respectively; in the above, Δ_N is given by

$$\Delta_N := \begin{bmatrix} V'XW' & 0 & \cdots & \cdots & 0 \\ V_A'W' & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ V_A'\,(A_d')^{N-2}\,W' & \cdots & \cdots & V_A'W' & V'XW' \end{bmatrix} \quad (53)$$

with V_A' defined by partitioning V' into $V' = [V_A', V_B']$ according to the partitioning of A_{2d}' in (18).

Now, let us consider the discrete-time system shown in Fig. 2 with the discrete-time generalized plant Π_N given by

$$\begin{aligned} x_{k+1} &= A_d x_k + W_N\rho_k + B_{2d}u_k \\ \upsilon_k &= V_{1N}x_k + \Delta_{ND}\rho_k + V_{2N}u_k \\ y_k &= C_{2d}x_k \end{aligned} \quad (54)$$

Then, it can be seen that the discrete-time transfer matrix from ρ to υ is equal to $\Phi_N(\zeta)$ defined in (34). Thus, $\|\overline{\mathbf{M}_1'}Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'} + \overline{D_{11}}\|$ can be evaluated exactly with the discrete-time frequency response gain $\|\Phi_N(e^{j\varphi h})\|$ by Proposition 1. Taking account of the inequality (21), together with the fact that $\|D_{11}\| \le \|\widehat{G}(e^{j\varphi h})\|, \forall\varphi \in \mathcal{I}_0$ [15], we readily obtain the following main result that gives the γ-independent H_∞-discretization method for the case of $D_{11} \neq 0$ with modified FSFH approximation.

Theorem 1 Consider the discrete-time system shown in Fig. 2, where Π_N is given by (54) with X determined appropriately, and let γ_N be defined by (14) and (22). Then, with the closed-loop transfer matrix $\Phi_N(\zeta)$ from ρ to υ, we have the following inequalities for the frequency response gain and H_∞ norm of the sampled-data system Σ in Fig. 1.

$$\max\left(\|D_{11}\|, \|\Phi_N(e^{j\varphi h})\| - \gamma_N\right) \le \|\widehat{G}(e^{j\varphi h})\| \le \max\left(\|D_{11}\| + \gamma_N, \|\Phi_N(e^{j\varphi h})\| + \gamma_N\right),\ \forall\varphi \in \mathcal{I}_0 \quad (55)$$

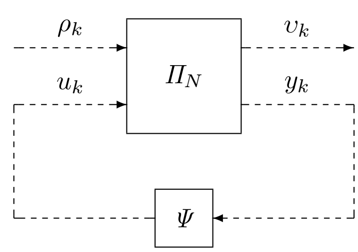

Fig. 2. Discrete-time system Σ_d.

$$\max\left(\|D_{11}\|, \|\Phi_N(\zeta)\|_\infty - \gamma_N\right) \le \|\widehat{G}(\zeta)\|_\infty$$
$$\le \max\left(\|D_{11}\| + \gamma_N, \|\Phi_N(\zeta)\|_\infty + \gamma_N\right) \qquad (56)$$

Remark 1 The matrix X is usually chosen to minimize either $\|\mathbf{E}'\|$ or $\|\mathbf{E}'\|_{\mathrm{HS}}$. See the last paragraph of Section III-A (also recall that $\gamma_N \le \|\mathbf{E}'\|_{\mathrm{HS}}$); the above inequalities ensure that the method is asymptotically exact in the sense that the H_∞-norm and the frequency response gain can be computed to any degree of accuracy by choosing N that is large enough. In the context of designing the H_∞ controller Ψ for the sampled-data system Σ, on the other hand, the above theorem still implies that we can simply deal with the H_∞ controller design problem for the discrete-time system Σ_d in Fig. 2. This is because the H_∞ norm of the sampled-data system Σ cannot be less than $\|D_{11}\|$ whatever Ψ we may take, so that we always assume that $\gamma > \|D_{11}\|$ in the H_∞ design $\|\widehat{G}(\zeta)\|_\infty < \gamma$. Hence, it follows from the proof of Proposition 1 that the H_∞ controller design minimizing the H_∞-norm of Σ_d is equivalent to minimizing $\max_{\varphi\in\mathcal{I}_0} \|\overline{\mathbf{M}_1'} Z_N(e^{j\varphi h})\overline{\mathbf{B}_1'} + \overline{D_{11}}\|$, which in turn is equivalent to minimizing the upper bound of $\|\widehat{G}(\zeta)\|_\infty$ that follows readily from (21). Since γ_N is independent of the controller Ψ, the minimization of $\|\widehat{G}(\zeta)\|_\infty$ can be carried out within the error by γ_N with the discrete-time system Σ_d, where the only point is that the necessary condition $\gamma > \|D_{11}\|$ must be imposed explicitly in the γ-iteration process with Σ_d.

Remark 2 The discretized generalized plant (54) is similar to that given in [7] for the case of $D_{11} = 0$, and at a glance, the appearance of the second term on the right-hand side of (52) might look the only difference. This, however, is not the case; when $D_{11} = 0$, the matrices W' and V' are given by the Cholesky factors of the matrices $\mathbf{B}_1'(\mathbf{B}_1')^*$ and $(\mathbf{M}_1')^*\mathbf{M}_1'$, respectively, so that we do not have to consider the coupling between the operators $\mathbf{B}_1'$ and $\mathbf{M}_1'$ and thus Lemma 1 is irrelevant. The existence of $D_{11} = 0$, on the other hand, leads to such coupling as well as other more involved operator compositions, for which Lemma 1 plays a crucial role. The resulting W' and V' are thus different from those in the case $D_{11} = 0$. The operator $D_{11} \ne 0$ is noncompact and whatever sort of finite-rank approximation one may apply to D_{11} alone, the approximation error cannot be less than $\|D_{11}\|$. Hence, such an approach always fails to give an asymptotically exact result. In this sense, no simple interpretation will be possible even as to the reason why the second term appears on the right-hand side of (52); at least, this term is not a result of some independent finite-rank approximation of the operator D_{11} alone. As such, the treatment of $D_{11} = 0$ in this paper is a nontrivial extension of the previous result for the case $D_{11} = 0$ [7]. The idea of employing such relations as in Lemma 1 is actually closely related to a technique employed in the recent studies on analysis and design of sampled-data systems [8] via noncausal linear periodically time-varying scaling [5].

Remark 3 When X is determined to minimize $\|\mathbf{E}'\|_{\mathrm{HS}}$, we have

$$\|\Phi_N(e^{j\varphi h})\| \le \|\widehat{G}(e^{j\varphi h})\|$$
$$\le \left(\|\Phi_N(e^{j\varphi h})\|^2 + \|\mathbf{E}'\|_{\mathrm{HS}}^2\right)^{1/2}, \quad \forall\varphi\in\mathcal{I}_0 \qquad (57)$$

provided that $D_{11} = 0$ [7], which gives sharper evaluation than (55) with γ_N replaced by $\|\mathbf{E}'\|_{\mathrm{HS}}$. A parallel result, however, seems hard to derive when $D_{11} \ne 0$ since the existence of nonzero D_{11} prevents us from developing an orthogonality argument, which plays a crucial role in the derivation of (57) under $D_{11} = 0$.

IV. Numerical Example

In this section, we give a numerical example of H_∞ analysis with modified FSFH approximation, and demonstrate its effectiveness in comparison with the conventional FSFH approximation [16]. For sampled-data systems with $D_{11} = 0$, however, the new method introduced in this paper is essentially equivalent to the one proposed in [7], and the effectiveness has already been verified there. Thus, we consider a slightly modified numerical example of [7] so that D_{11} becomes nonzero. More precisely, let us consider the continuous-time system shown in Fig. 3 [1] so that we have $z = [w-u,\ y]^T$ and thus $D_{11} = [1, 0]^T$, where the plant $G(s)$ and the controller $C_r(s)$ are given respectively by

$$G(s) = \frac{1}{s^2}\cdot\frac{(s/a+1)\prod_{i=0}^{1}\{(s/\omega_i)^2 + 2\zeta_i\,(s/\omega_i) + 1\}}{\prod_{i=2}^{4}\{(s/\omega_i)^2 + 2\zeta_i\,(s/\omega_i) + 1\}}$$

$$a = 4.84,\ \zeta_0 = 0.02,\ \ \zeta_1 = -0.4,\ \zeta_2 = \zeta_3 = \zeta_4 = 0.02,$$

$$\omega_0 = 1,\ \omega_1 = 5.65,\ \omega_2 = 0.765,\ \omega_3 = 1.41,\ \omega_4 = 1.85$$

$$C_r(s) = \frac{0.0513s^3 + 0.00424s^2 + 0.0296s + 0.00157}{s^4 + 0.693s^3 + 0.779s^2 + 0.293s + 0.0739}$$

We then discretize the controller $C_r(s)$ by the Tustin transformation with $h = 8$, and consider the sampled-data system shown in Fig. 4. We analyze its H_∞-norm from w to z. As in [7], we determine the matrix X minimizing the Hilbert-Schmidt norm $\|\mathbf{E}'\|_{\mathrm{HS}}$ with the method of [6] and evaluate the H_∞-norm based on (56) with γ_N replaced by $\|\mathbf{E}'\|_{\mathrm{HS}}$. All computations are executed with MATLAB on a PC with Pentium 4, 3.0 GHz.

Table I shows the results of H_∞ analysis by the conventional and modified FSFH approximation. The exact value of the H_∞-norm obtained by the γ-dependent exact discretization method is 111.9771, so the modified FSFH approximation can be seen to give accurate enough upper and lower bounds of the H_∞-norm at $N = 4$, while the conventional FSFH approximation [16] gives 111.9757, which is not satisfactorily accurate, even at $N = 100$.

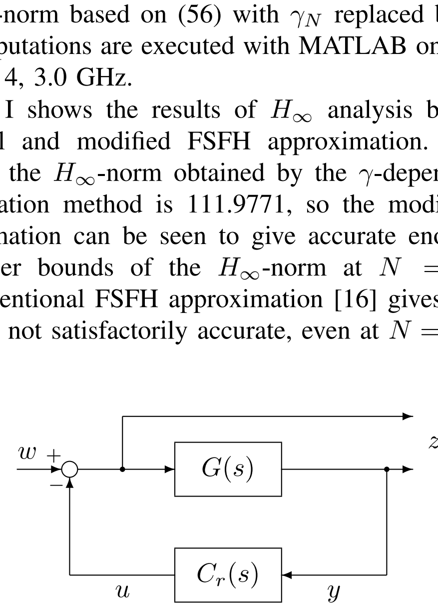

Fig. 3. Continuous-time control system.

TABLE I

H_∞-NORM ANALYSIS.

N	1	2	3	4	5
conv. FSFH	102.8916	111.2174	110.8725	111.1307	111.4949
mod. FSFH (upper)	112.0104	111.9774	111.9771	111.9771	111.9771
mod. FSFH (lower)	111.9437	111.9768	111.9770	111.9771	111.9771
$\|\|\mathbf{E}'\|\|_{\mathrm{HS}}$	0.0334	2.7891×10^{-4}	3.1299×10^{-5}	1.0757×10^{-5}	5.4312×10^{-6}

TABLE II

COMPARISON OF THE COMPUTATION TIME.

N	1	2	3	4	5
conv. FSFH	0.060	0.070	0.070	0.070	0.070
mod. FSFH	0.080	0.090	0.100	0.110	0.110

Table II shows the computation time in seconds required for the computations about Table I. For the same value of N, the modified FSFH approximation method takes much more time because the resulting discrete-time system has larger numbers of input and output than in the conventional FSFH approximation method. However, the conventional method takes about three times as much time (i.e., 0.32 seconds) even at $N = 100$ that is still small for accurate computations.

From the above arguments, we can see that the modified FSFH approximation method is a more effective method for H_∞ analysis of sampled-data systems in the sense of accuracy and efficient computation.

V. Conclusion

In this paper, we gave a method for H_∞-discretization through the fast-lifting technique, which we call modified FSFH approximation. This method can lead to a γ-independent discretized generalized plant even for the case with nonzero D_{11}, while the previous study in [7] only dealt with the case of $D_{11} = 0$; the method developed in this paper is a nontrivial generalization of the previous result as discussed in Remark 2, and special factorizations of matrices defined as operator compositions (Lemma 1) played a crucial role in the derivation, together with other techniques such as quasi-finite-rank approximation of an infinite-rank operator and the loop-shifting technique.

The discretization procedure becomes slightly more involved than the case with $D_{11} = 0$, but the result still possesses similarity to the conventional FSFH approximation method [16] in the structure of the resulting discretized generalized plant and in the respect that the discretization is ensured to be asymptotically exact as the approximation parameter N is made larger. A distinctive advantage of the modified FSFH approximation method over the conventional FSFH method, however, is that the former can give both the upper and lower bounds of the approximation error in terms of N. Since these bounds are independent of the discrete-time controller, the modified FSFH method is more suitable for control system design with guaranteed performance. In this respect, some relationship of the arguments of this paper to the recent study of control system design via noncausal linear periodically time-varying scaling [5],[8] was also suggested.

We also verified the effectiveness of the new method with a numerical example of H_∞ analysis in comparison with the conventional FSFH approximation [16]. The result shows that the modified FSFH approximation method is more effective than the conventional FSFH method in the sense of accuracy and computational efficiency.

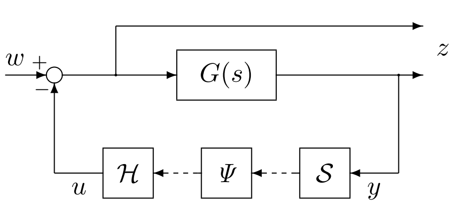

Fig. 4. Sampled-data system with controller discretization.

References

[1] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice-Hall (1990).

[2] M. Araki, Y. Ito and T. Hagiwara, Frequency response of sampled-data systems, *Automatica*, Vol. 32, No. 4, pp. 483–497 (1996).

[3] B. A. Bamieh and J. B. Pearson, A general framework for linear periodic systems with applications to H_∞ sampled-data control, *IEEE Trans. Automat. Contr.*, Vol. 37, No. 4, pp. 418–435 (1992).

[4] J. H. Braslavsky, R. H. Middleton and J. S. Freudenberg, L_2-induced norms and frequency-gains of sampled-data sensitivity operators, *IEEE Trans. Automat. Contr.*, Vol. 43, No. 2, pp. 252–258 (1998).

[5] T. Hagiwara, Causal/noncausal linear periodically time-varying scaling for robust stability analysis and their properties, Proc. 17th International Symposium on Mathematical Theory of Networks and Systems, pp. 742–752 (2006).

[6] T. Hagiwara, M. Suyama and M. Araki, Upper and lower bounds of the frequency response gain of sampled-data systems, *Automatica*, Vol. 37, No.9, pp.1363–1370 (2001).

[7] T. Hagiwara and H. Umeda, Modified fast-sample/fast-hold approximation for sampled-data system analysis, Proc. European Control Conference, pp. 5262–5269 (2007).

[8] T. Hagiwara and H. Umeda, Robust stability analysis of sampled-data systems with noncausal periodically time-varying scaling: optimization of scaling via approximate discretization and error bound analysis, Proc. Conf. Decision and Control, pp. 450–457 (2007).

[9] Y. Ito, T. Hagiwara, H. Maeda and M. Araki, Bisection algorithm for computing the frequency response gain of sampled-data systems—Infinite-dimensional congruent transformation approach, *IEEE Trans. Automat. Contr.*, Vol. 46, No. 3, pp. 369–381 (2001).

[10] L. Mirkin and and Z. J. Palmor, Computation of the frequency response gain of sampled-data systems via projection in the lifted domain, *IEEE Trans. Automat. Contr.*, Vol. 47, No. 9, (2002).

[11] E. N. Rosenvasser and B. P. Lampe, *Computer Controlled Systems: Analysis and Design with Process-Oriented Models*, Springer (2000).

[12] G. Tadmor, H_∞ optimal sampled-data control in continuous time systems, *Int. J. Control*, Vol. 56, No. 1, pp. 99-141 (1992).

[13] H. T. Toivonen, Sampled-data control of continuous-time systems with an H_∞ optimality criterion, *Automatica*, Vol. 28, No. 1, pp. 45–54 (1992).

[14] Y. Yamamoto, A function space approach to sampled-data control systems and tracking problems, *IEEE Trans. Automat. Contr.*, Vol. 39, pp. 703–712 (1994).

[15] Y. Yamamoto and P. P. Khargonekar, Frequency response of sampled-data systems, *IEEE Trans. Automat. Contr.,* Vol. 41, No. 2, pp. 166–176 (1996).

[16] Y. Yamamoto, A. G. Madievski and B. D. O. Anderson, Approximation of frequency response for sampled-data control systems, *Automatica*, Vol. 35, pp. 729–734 (1999).