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Time Scaling of a Multi-Output Observer Form

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Abstract— **This paper considers the time scaling of a multi-output observer form (TOF) for uncontrolled nonlinear continuous-time systems. It is the multi-output version of an existing single-output result. Time scaling broadens the class of systems which admits an Exact Error Linearization observer design by including Time Scaling Functions (TSFs). Two types of TSFs are considered and the corresponding existence conditions of the time scaling transformation and the change of state coordinates to a TOF are provided. The necessary and sufficient conditions on TSFs to preserve the global exponential stability of the error dynamics are presented.**

I. INTRODUCTION

We consider observer design for uncontrolled multi-output systems in the state space form

$$\begin{aligned}\dot{\zeta} &= f(\zeta),\\ y &= h(\zeta),\end{aligned} \tag{1}$$

where $\zeta \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ vector field, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a C^∞ output function. The well-established Exact Error Linearization (EEL) nonlinear observer design method is based on an Observer Form (OF) and obtains stable LTI state estimate error dynamics in OF coordinates [14], [3]. A multi-output extension of this work is in [15], [26]. Some other extensions to OF-based work include [18], [10], [12], [17], [1], [11], [24], [25], [22], [2], [19]. Recent work [23], [7] considers a generalization of EEL by incorporating an output dependent time scaling transformation for a single-output nonlinear system. This result is generalized to the output dependent observability linear normal form in [27]. Time scaling transformation introduces a to-be-determined time scale function (TSF) into the original system dynamics f, and thus yields an extra degree-of-freedom when transforming the system. The system in the new time scale admits an OF.

This paper considers a multi-output version of work in [23], [6], [27]. In Section II we present two motivational examples, introduce the time scaled multi-output observer form (TOF), and state the problem to be solved. In Section III we discuss the single and multiple time scaling transformation cases, and propose the existence conditions of a TOF. TOF-based observer design ensures global exponential error convergence in the transformed time scales. The error dynamics stability in the original time scale is discussed and the necessary and sufficient conditions on TSFs to preserve the globally exponential stability are given in Section IV.

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II. PROBLEM STATEMENT

The multi-output observer form (OF) existence conditions have been established in [26], [15]. Although an OF allows EEL observer design, the class of systems admitting it is limited. To generalize the class of single-output systems admitting EEL observer design, the following output dependent time scaling transformation was employed in [23]:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = s(y(t)) > 0, \quad \tau(t_0) = \tau_0, \tag{2}$$

where $s(y(t))$ is a non-vanishing positive smooth function, called a time scale function (TSF). The extension to the multi-output system case will be considered in this note.

A. Motivational Examples

Example 1 We consider a two-output system in observable form with observability indices $(2, 2)$ corresponding to the output $y = (y_1, y_2)^T$.

$$\dot{x} = \begin{pmatrix} x_{1,2} \\ x_{1,2}^2 + x_{1,2}x_{2,2} \\ x_{2,2} \\ x_{2,2}^2 + x_{1,2}x_{2,2} \end{pmatrix} \triangleq f(x), y = \begin{pmatrix} x_{1,1} \\ x_{2,1} \end{pmatrix} \triangleq h(x). \tag{3}$$

One can verify system (3) is not transformable to an OF. However, if we introduce the time scaling transformation

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = s(y) = e^{y_1+y_2} > 0, \quad \tau(0) = \tau_0, \tag{4}$$

then system (3) in τ time is given by

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{1}{s} f \triangleq \bar{f}, \quad y = h(x).$$

Following the OF result, one can solve the starting vector fields $\bar{g}_1 = s\partial/\partial x_{1,2}$, $\bar{g}_2 = s\partial/\partial x_{2,2}$, and verify the Lie bracket conditions are satisfied

$$\left[\mathrm{ad}^k_{-\bar{f}}\bar{g}_r, \mathrm{ad}^l_{-\bar{f}}\bar{g}_q\right] = 0, \quad 0 \le k, l \le 1; 1 \le r, q \le 2.$$

System (3) in τ time is transformed into an OF by the transformation $\Phi(x) = (x_{1,1}, x_{1,2}/s, x_{2,1}, x_{2,2}/s)^T$.

Example 2 We modify the dynamics of system (3) by taking

$$\begin{aligned}\dot{x}_{1,2} &= x_{1,2}^2 + x_{1,1}x_{2,1}\\ \dot{x}_{2,2} &= x_{2,2}^2 + x_{1,1}.\end{aligned}$$

Using the results provided below, one can show that no TSF of the form (4) will transform the system to OF in τ time. However, if we introduce a different time scaling transformation for each subsystem

$$\begin{aligned}\frac{\mathrm{d}\tau_1}{\mathrm{d}t} &= s_1(y) = e^{y_1} > 0, \quad \tau_1(0) = \tau_{10}\\ \frac{\mathrm{d}\tau_2}{\mathrm{d}t} &= s_2(y) = e^{y_2} > 0, \quad \tau_2(0) = \tau_{20},\end{aligned}$$

and rewrite the system in new time scales

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \begin{pmatrix} s_1\mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & s_2\mathbf{I}_2 \end{pmatrix}^{-1} f \triangleq \bar{f},$$

where $\mathrm{d}x/\mathrm{d}\tau = (\mathrm{d}x_1/\mathrm{d}\tau_1, \mathrm{d}x_1/\mathrm{d}\tau_2)^T, x_1 = (x_{1,1}, x_{1,2})^T, x_2 = (x_{2,1}, x_{2,2})^T, \mathbf{I}_k$ is the $k \times k$ dimensional identity matrix, then the system in new time can be put into an OF using $\Phi(x) = (x_{11}, x_{12}/s_1, x_{21}, x_{22}/s_2)^T$.

B. TOF Problem

Given time scaling transformations for each subsystem

$$\frac{\mathrm{d}\tau_i}{\mathrm{d}t} = s_i(y(t)) > 0, \quad \tau_i(t_0) = \tau_{i0}, \tag{5}$$

we define the TOF as an OF in the new times

$$\frac{\mathrm{d}z_i}{\mathrm{d}\tau_i} = A_i z_i + \gamma_i(y), \quad y_i = C_i z_i, \quad 1 \le i \le p, \tag{6}$$

where $z_i = (z_{i,1}, \cdots, z_{i,\lambda_i})^T$, $\gamma_i(y) = (\gamma_{i,1}, \cdots, \gamma_{i,\lambda_i})^T$, λ_i denote the system's observability indices [21], and $A_i = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{\lambda_i - 1} \\ 0 & \mathbf{0} \end{pmatrix}, C_i = (1, \mathbf{0})^T$. The ith subsystem in TOF and t time is given by

$$\dot{z}_i = s_i(A_i z_i + \gamma_i(y)), \quad y_i = C_i z_i, \quad 1 \le i \le p,$$

where s_i abbreviates $s_i(y(t))$. The TOF for the entire system in t time is

$$\dot{z} = S(Az + \gamma(y)), \quad y = Cz, \tag{7}$$

where $z = ((z_1)^T, \cdots, (z_p)^T)^T$, $\gamma(y) = ((\gamma_1)^T, \cdots, (\gamma_p)^T)^T$, $S = \text{Blockdiag}\{s_1\mathbf{I}_{\lambda_1}, \cdots, s_p\mathbf{I}_{\lambda_p}\}$, $A = \text{Blockdiag}\{A_1, \cdots, A_p\}$, $C = \text{Blockdiag}\{C_1, \cdots, C_p\}$. We remark that the difference between multi-output and single-output TOF is in the matrix S. This difference leads to a different approach to derive the TOF existence conditions. Given TSF (5) and TOF (6), we have the following definition.

Definition 2.1: The nonlinear system (1) locally (globally) observable w.r.t. the observability indices [21] is said to be locally (globally) transformable to a TOF (6) if there exists a local (global) diffeomorphism $z = \Phi(x)$, $\Phi(0) = 0$ and time scaling transformations (5) such that the transformed system in the τ times is

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = S^{-1}(y)\frac{\partial \Phi(x)}{\partial x} S(y)\bar{f} = Az + \gamma(y), \tag{8}$$

where $\mathrm{d}z/\mathrm{d}\tau = (\mathrm{d}z_1/\mathrm{d}\tau_1, \cdots, \mathrm{d}z_p/\mathrm{d}\tau_p)^T, \bar{f} = S^{-1}(y)f$.

Remark 2.2: System (1) is transformable to a TOF (6) if and only if there exists a change of coordinates $z = \Phi(x)$, $\Phi(0) = 0$ and time scaling transformations (5) such that the system in the t time is changed to

$$\dot{z} = \frac{\partial \Phi(x)}{\partial x} f(x) = S(y)(Az + \gamma(y)). \tag{9}$$

We can show the sufficiency of this result by assuming system (1) in t time is transformed to (9) by $z = \Phi(x)$, $\Phi(0) = 0$ and time scaling transformations (5). Since TSFs are nonvanishing, we can multiply S^{-1} to both sides of (9) and obtain

$$S^{-1}(y)\dot{z} = S^{-1}(y)\frac{\partial \Phi(x)}{\partial x} f = Az + \gamma(y).$$

This implies (8) holds since it is straightforward to check $S^{-1}(y)\dot{z} = (\mathrm{d}z_1/\mathrm{d}\tau_1, \cdots, \mathrm{d}z_p/\mathrm{d}\tau_p)$.

III. EXISTENCE CONDITIONS

We first introduce some notation, then present the existence conditions for a TOF. Next, the necessary and sufficient conditions for a TOF where the same time scaling transformation is used for all subsystems are given; these conditions can be specified in a concise form and are similar to the established result for an OF. Following [26] we define two co-distributions Q_i, Q:

$$\begin{aligned} Q_i = \text{span}\{&\mathrm{d}L_f^k h_r, 0 \le k \le \lambda_i - 1, 1 \le r \le p, \\ &\text{where } \mathrm{d}L_f^{\lambda_i - 1} h_i \text{ is omitted}\}, \quad 1 \le i \le p, \\ Q = \text{span}\{&\mathrm{d}L_f^k h_r, 0 \le k \le \lambda_r - 1, 1 \le r \le p\}. \end{aligned}$$

For system (1), it has been shown in [26] that $Q_i = Q_i \cap Q$ guarantees the existence of the starting vector g_i satisfying

$$L_{g_i} L_f^k h_r = \delta_{k,\lambda_i - 1}\delta_{i,r}, \quad 0 \le k \le \lambda_i - 1, \ 1 \le r \le p. \tag{10}$$

A. Multiple Time Scaling Transformation Case

Theorem 3.1: The nonlinear system (1) is locally transformable to a TOF (8) if and only if, locally at x_0,

1) The TSF of the ith subsystem (5), denoted by s_i, satisfies the PDEs

$$\begin{aligned} \mathrm{d}L_{g_i} L_f^{\lambda_i} h_i = \frac{1}{s_i}\Bigg(& l_{\lambda_i}\frac{\partial s_i}{\partial z_{i,1}} \mathrm{d}L_f h_i \\ & + (l_{\lambda_i} - 1)\sum_{j=1, j\neq i}^{p} \frac{\partial s_i}{\partial z_{j,1}} \mathrm{d}L_f h_j \Bigg) \ \mathrm{mod}\ \{\mathrm{d}y\}, \end{aligned} \tag{11}$$

where $l_k = \frac{k(k-1)}{2} + 1, \ 1 \le k \le \lambda_i$, and g_i is the starting vector field in the original time and defined by (10).

2) Q_i=$Q_i \cap Q$.

3) The Lie brackets conditions are satisfied

$$[\eta_{i,r}, \eta_{l,s}] = 0, \quad 1 \le r, s \le \lambda_i; 1 \le i, l \le p, \tag{12}$$

where for $1 \le i \le p$,

$$\eta_{i,1} = \bar{g}_i, \quad \eta_{i,j} = \frac{1}{s_i}\mathrm{ad}_{-f}\eta_{i,j-1}, \quad 2 \le j \le \lambda_i, \tag{13}$$

and $\bar{g}_i$ are the starting vector fields and defined by

$$L_{\bar{g}_i} L_f^k h_l = s_i^{\lambda_i - 1}\delta_{k,\lambda_i - 1}\delta_{l,i}, \quad \begin{cases} 0 \le k \le \lambda_i - 1, \\ 1 \le l \le p. \end{cases} \tag{14}$$

Remark 3.2: The transformation $z = \Phi(x)$ is the solution of the n^2 PDEs

$$\frac{\partial \Phi(x)}{\partial x}\left[\eta_{1,\lambda_1}, \cdots, \eta_{1,1}, \cdots, \eta_{p,\lambda_p}, \cdots, \eta_{p,1}\right] = \mathbf{I}_n. \tag{15}$$

Remark 3.3: The TOF coordinates are globally defined if the system is globally observable and the vector fields $\eta_{i,j}, 1 \le j \le \lambda_i, 1 \le i \le p$ are complete.

Remark 3.4: Given the matrix TSF S one can verify

$$\begin{aligned}\mathrm{d}h_i \frac{\partial S}{\partial z_{i,1}}(Az+\gamma) &= \mathrm{d}h_i \frac{\partial S}{\partial z_{i,1}} S^{-1} \underbrace{S(Az+\gamma)}_{f} \\ &= [\mathbf{0}, \cdots, \underbrace{\frac{\partial s_i}{\partial z_{i,1}}\frac{1}{s_i}, 0, \cdots, 0}_{\text{ith block}}, \cdots, \mathbf{0}]f \\ &= \frac{\partial s_i}{\partial z_{i,1}}\frac{1}{s_i} L_f h_i.\end{aligned}$$

Remark 3.5: From Definition (14), we know $\bar{g}_i = s_i^{\lambda_i - 1} g_i$ and its existence is guaranteed by Condition 2 [26].

Proof: $\Leftarrow$: Taking $\eta_{i,1} = \partial/\partial z_{i,\lambda_i}, 1 \le i \le p$ and following definition (13) of $\eta_{i,k}, 2 \le k \le \lambda_i$, we have $\eta_{i,k} = \partial/\partial z_{i,\lambda_i - k+1}$. Clearly, $\eta_{i,k}, 1 \le k \le \lambda_i, 1 \le i \le p$ are unit vectors and commute, i.e., condition (12) is necessary. Next we derive the definition of the starting vector $\bar{g}_i$ (14). Since $\bar{g}_i = \eta_{i,1}, 1 \le i \le p$, we have $\partial h_l/\partial z_{i,\lambda_i} = 0 \Rightarrow L_{\bar{g}_i} h_l = 0$ for $1 \le l \le p$. Further computation gives

$$\begin{aligned}0 = \frac{\partial h_l}{\partial z_{i,\lambda_i - 1}} &= \langle \mathrm{d}h_l, \eta_{i,2}\rangle = \left\langle \mathrm{d}h_l, \frac{1}{s_i}[-f, \bar{g}_i]\right\rangle \\ &= \frac{1}{s_i}\langle \mathrm{d}L_f h_l, \bar{g}_i\rangle - \frac{1}{s_i} L_f \langle \mathrm{d}h_l, \bar{g}_i\rangle \\ &= \frac{1}{s_i} L_{\bar{g}_i} L_f h_l,\end{aligned}$$

for $1 \le l \le p$. By induction, one can show

$$L_{\bar{g}_i} L_f^k h_l = \begin{cases} s_i^k \frac{\partial h_l}{\partial z_{i,\lambda_i - k}} = 0, & 0 \le k \le \lambda_i - 2; 1 \le l \le p, \\ s_i^{\lambda_i - 1} \frac{\partial h_l}{\partial z_{i,1}} = s_i^{\lambda_i - 1}, & k = \lambda_i - 1; 1 \le l \le p.\end{cases}$$

Hence, the starting vector $\bar{g}_i$ satisfies (14).

To derive the condition on the TSF, we first state the equations ensuring the existence of state transformation $\Phi(x)$, for $1 \le i \le p$

$$s_i \frac{\partial W}{\partial z_{i,j}} = \mathrm{ad}_{-f} \frac{\partial W}{\partial z_{i,j+1}}, \quad 1 \le j \le \lambda_i - 1, \tag{16a}$$

$$\frac{\partial W}{\partial z}\frac{\partial}{\partial z_{i,1}}(S(Az+\gamma)) = \mathrm{ad}_{-f}\frac{\partial W}{\partial z_{i,1}}, \tag{16b}$$

$$\mathrm{d}h_r \frac{\partial W}{\partial z_{i,k}} = \delta_{k,1}\delta_{r,i}, \quad 0 \le k \le \lambda_i - 1; 1 \le r \le p, \tag{16c}$$

where $W = \Phi^{-1}(z)$, and $\partial W/\partial z_{i,\lambda_i}$ is the starting vector $\bar{g}_i$. One can see from (16a) that $\partial W/\partial z_{i,j} = \eta_{i,\lambda_i - j+1}, 1 \le j \le \lambda_i, 1 \le i \le p$.

The left hand side of (16b) is

$$\frac{\partial W}{\partial z}\frac{\partial}{\partial z_{i,1}}(S(Az+\gamma)) = \frac{\partial W}{\partial z}\left(\frac{\partial S}{\partial z_{i,1}}(Az+\gamma) + S\frac{\partial \gamma}{\partial z_{i,1}}\right).$$

Given the right hand side of (16b) in Remark 3.6, (16b) multiplied by $\mathrm{d}h_i(\partial W/\partial z)^{-1}$ is

$$\begin{aligned}\mathrm{d}h_i \frac{\partial S}{\partial z_{i,1}}(Az+\gamma) + \mathrm{d}h_i S \frac{\partial \gamma}{\partial z_{i,1}} &= \mathrm{d}h_i \frac{1}{s_i^{\lambda_i - 1}} \mathrm{ad}_{-f}^{\lambda_i} \bar{g}_i \\ &+ \mathrm{d}h_i \frac{\sum_{j=1}^{\lambda_i - 1} j}{s_i^{\lambda_i}} L_f(s_i) \mathrm{ad}_{-f}^{\lambda_i - 1}\bar{g}_i.\end{aligned} \tag{17}$$

According to Remark 3.4, (17) is modified into

$$\begin{aligned}\frac{\partial s_i}{\partial z_{i,1}}\frac{1}{s_i} L_f h_i + \rho(y) &= \frac{1}{s_i^{\lambda_i - 1}} L_{\mathrm{ad}_{-f}^{\lambda_i}\bar{g}_i} h_i \\ &+ \frac{\sum_{j=1}^{\lambda_i - 1} j}{s_i^{\lambda_i}} L_f(s_i) L_{\mathrm{ad}_{-f}^{\lambda_i - 1}\bar{g}_i} h_i,\end{aligned} \tag{18}$$

where $\rho(y) = \mathrm{d}h_i S \partial\gamma/\partial z_{i,1}$ is some function of y. From [9, Lem. 4.1.2], [20, Thm. A.3.1]

$$\begin{aligned}L_{\mathrm{ad}_{-f}^{\lambda_i - 1}\bar{g}_i} h_i &= L_{\bar{g}_i} L_f^{\lambda_i - 1} h_i = s_i^{\lambda_i - 1}, \\ L_{\mathrm{ad}_{-f}^{\lambda_i}\bar{g}_i} h_i &= L_{\bar{g}_i} L_f^{\lambda_i} h_i,\end{aligned}$$

we can rearrange (18) as

$$\frac{\partial s_i}{\partial z_{i,1}}\frac{1}{s_i} L_f h_i + \rho(y) = \frac{1}{s_i^{\lambda_i - 1}} L_{\bar{g}_i} L_f^{\lambda_i} h_i - \frac{l_{\lambda_i} - 1}{s_i} L_f(s_i). \tag{19}$$

Collecting the terms of (19) and taking the differential, we have

$$\begin{aligned}\mathrm{d}L_{\bar{g}_i} L_f^{\lambda_i} h_i &= l_{\lambda_i} s_i^{\lambda_i - 2}\frac{\partial s_i}{\partial y_i}\mathrm{d}L_f h_i \\ &+ (l_{\lambda_i} - 1)s_i^{\lambda_i - 2}\sum_{j=1, j\ne i}^{p}\frac{\partial s_i}{\partial y_j}\mathrm{d}L_f h_j \mod \{\mathrm{d}y\},\end{aligned} \tag{20}$$

where $y_i = z_{i,1}, y_j = z_{j,1}$. Since $\bar{g}_i = s_i^{\lambda_i - 1} g_i$, we have $\mathrm{d}L_{\bar{g}_i} L_f^{\lambda_i} h_i = s^{\lambda_i - 1}\mathrm{d}L_{g_i} L_f^{\lambda_i} h_i \mod \{\mathrm{d}y\}$. Hence, we have condition (11) by plugging the above equation into (20).

$\Rightarrow$: Given the TSFs of each subsystem s_i solved from (11), it is readily shown Conditions 2)–3) are sufficient to guarantee the existence of state coordinate $z = \Phi(x)$ which puts system (1) into a TOF (7) by following the standard proof in [14], [26], [20]. ■

Remark 3.6: Given (16a), one can compute $\partial W/\partial z_{i,j}, 1 \le j \le \lambda_i - 1$ iteratively and have

$$\begin{aligned}\frac{\partial W}{\partial z_{i,\lambda_i - k}} &= \frac{1}{s_i^k}\mathrm{ad}_{-f}^k \bar{g}_i + \frac{\sum_{j=1}^{k-1} j}{s_i^{k+1}} L_f(s_i)\mathrm{ad}_{-f}^{k-1}\bar{g}_i \\ &\mod \{\mathrm{ad}_{-f}^j \bar{g}_i, 0 \le j \le k-2\}, 1 \le k \le \lambda_i - 1.\end{aligned}$$

Further calculation yields the right hand side of (16b)

$$\begin{aligned}\mathrm{ad}_{-f}\frac{\partial W}{\partial z_{i,1}} = &\frac{1}{s_i^{\lambda_i - 1}}\mathrm{ad}_{-f}^{\lambda_i}\bar{g}_i + \frac{\sum_{j=1}^{\lambda_i - 1} j}{s_i^{\lambda_i}} L_f(s_i)\mathrm{ad}_{-f}^{\lambda_i - 1}\bar{g}_i \\ &\mod \{\mathrm{ad}_{-f}^j \bar{g}_i, 0 \le j \le \lambda_i - 2\}.\end{aligned}$$

Remark 3.7: The multiple time scaling transformation case has a different TSF for each subsystem. This can be generalized by employing a TSF for each state, i.e., $S = \mathrm{Blockdiag}\{s_1, \cdots, s_n\}$, which leads to the multi-output extension of the output dependent observability linear normal form in [27]. A similar procedure can be followed to obtain the existence conditions of the corresponding TOF.

Remark 3.8: The type of freedom time scaling transformation introduces is apparent by comparing the characteristic equations with and without TSFs. More specifically we derive a necessary condition on the expression of $\varphi_i = L_f^{\lambda_i} h_i$ in observable coordinates for a system (1) admitting a TOF. This condition is less restrictive than the polynomial condition of a system admitting OF [15]. Assuming (1) is in observable form with observability indices $\lambda_i, 1 \le i \le p$, the ith subsystem is

$$\dot{x}_i = \begin{pmatrix} x_{i,2} \\ \vdots \\ \varphi_i(x) \end{pmatrix}, \quad y_i = x_{i,1}.$$

When $\lambda_i \ge 2$ and taking the typical starting vector $g_i = \partial/\partial x_{i,\lambda_i}$, condition (11) is formulated as

$$\mathrm{d}\frac{\partial \varphi_i(x)}{\partial x_{i,\lambda_i}} = \frac{1}{s_i(y)} \left[l_{\lambda_i} \frac{\partial s_i(y)}{\partial y_i} \mathrm{d}x_{i,2} + (l_{\lambda_i} - 1) \sum_{j=1, j\neq i}^{p} \mathrm{d}x_{j,2} \right] \mod \{\mathrm{d}y\}.$$

Performing coefficient matching on the above equation, one can solve s_i only if the coefficients of x_{i,λ_i} in φ_i are of the form $\alpha_1(y)x_{j,2}, \alpha_2(y)$. We note a system having $x_{j,k}x_{i,\lambda_i}, k \ge 3$ in φ_i is not transformable to a TOF.

As a special case, if $\lambda_i = 1$, condition (11) is

$$\mathrm{d}\frac{\partial \varphi_i(x)}{\partial x_{i,1}} = \frac{1}{s_i(y)} \frac{\partial s_i(y)}{\partial y_i} \mathrm{d}\varphi_i(x) \mod \{\mathrm{d}y\}.$$

If $\varphi_i(x)$ depends on $x_{j,k}, k \ge 2$ and y, no TSF can be solved. On the other hand, if $\varphi_i(x)$ depends on y only, no s_i is required for the ith subsystem is already in OF. Hence, we conclude, time scaling transformation is not helpful in transforming 1-dimensional subsystems into an OF. Following a similar procedure, one can verify unlike the single-output case, time scaling transformation is not equivalent to output transformation for a p-output ($p > 2$) system with observability indices $\lambda_k = 2, 1 \le k \le p$.

Example 2 (Continued) One can apply Theorem 3.1 to solve the candidate matrix TSF and compute the state transformation. Condition (12) is reduced to

$$2\mathrm{d}x_{1,2} = s_1^{-1}(2\frac{\partial s_1}{\partial y_1}\mathrm{d}x_{1,2} + \frac{\partial s_1}{\partial y_2}\mathrm{d}x_{2,2}) \mod \{\mathrm{d}y\}$$
$$2\mathrm{d}x_{2,2} = s_2^{-1}(2\frac{\partial s_2}{\partial y_2}\mathrm{d}x_{2,2} + \frac{\partial s_2}{\partial y_1}\mathrm{d}x_{1,2}) \mod \{\mathrm{d}y\},$$

which yields the PDEs

$$\frac{\partial s_1}{\partial y_1} = s_1, \quad \frac{\partial s_1}{\partial y_2} = 0, \quad \frac{\partial s_2}{\partial y_1} = 0, \quad \frac{\partial s_2}{\partial y_2} = s_2.$$

Hence, we solve the TSF $s_1 = e^{y_1}, s_2 = e^{y_2}$. Lie bracket condition (12) is satisfied for $1 \le r, s, i, l \le 2$ and the system is transformable to a TOF.

B. Single Time Scaling Transformation Case

The existence conditions are given in the following theorem without proof.

Theorem 3.9: The nonlinear system (1) is locally transformable to a TOF (8) if and only if, locally at x_0,

1) Condition 1) in Theorem 3.1 with $s = s_i, 1 \le i \le p$ holds.
2) Q_i=$Q_i \cap Q$.
3) The Lie brackets conditions are satisfied

$$\left[\mathrm{ad}^k_{-\bar{f}}\bar{g}_r, \mathrm{ad}^l_{-\bar{f}}\bar{g}_q\right] = 0, \quad \begin{cases} 0 \le k \le \lambda_r - 1, \\ 0 \le l \le \lambda_q - 1, \\ 1 \le r, q \le p, \end{cases} \tag{21}$$

where $\bar{g}_i$ is the starting vector field in τ time and defined by

$$L_{\bar{g}_i} L^k_{\bar{f}} h_r = \delta_{k,\lambda_i - 1}\delta_{i,r}, \quad 0 \le k \le \lambda_i - 1, \; 1 \le r \le p. \tag{22}$$

Remark 3.10: The transformation $z = \Phi(x)$ is the solution of the n^2 PDEs

$$\frac{\partial \Phi(x)}{\partial x}\left[\mathrm{ad}^{\lambda_1 - 1}_{-f} g_1 \cdots g_1 \cdots \mathrm{ad}^{\lambda_p - 1}_{-f} g_p \cdots \mathrm{ad}_{-f} g_p\right] = \mathbf{I}_n. \tag{23}$$

Remark 3.11: Comparing Theorem 3.9 with [26, Thm. 3.1], one can see the difference is the additional Condition 1) on the TSF. Provided the existence of a TSF, the necessity and sufficiency of Condition 2) – 3) have been shown in [26]. Also, Condition 1) is obvious given Condition 1) in Theorem 3.1.

Remark 3.12: Assuming for system (1), the starting vectors $g_i, 1 \le i \le p$ in the original time can be solved from (10), we have the starting vectors defined by (22) $\bar{g}_i = s^{\lambda_i - 1} g_i, 1 \le i \le p$. This is because by induction, one can derive that for a fixed i and any $r, 1 \le r \le p$,

$$\mathrm{d}L^k_{\bar{f}} h_r = \begin{cases} s^{-1}\mathrm{d}L_f h_r \mod \{\mathrm{d}h_r\}, & k = 0, \\ s^{-k}\mathrm{d}L^k_f h_r \mod \{\mathrm{d}L^j_f h_r, 0 \le j \le k-1\}, \end{cases}$$

for $1 \le k \le \lambda_i - 1$. Therefore, it is straightforward to verify if g_i satisfies (10) then $\bar{g}_i$ satisfies (22).

Remark 3.13: The case of multiple time scaling transformation is a generalization of the single time scaling transformation. This is because replacing the matrix TSF with a scalar TSF, Theorem 3.1 is equivalent to Theorem 3.9. We can verify the $\bar{g}_i$ solved from (14) is the same as $\bar{g}_i$ solved from (22), and $\mathrm{ad}^{k-1}_{-\bar{f}}\bar{g}_i = \eta_{i,k}, 1 \le k \le \lambda_i, 1 \le i \le p$. Thus the Lie bracket conditions are equivalent. When $p = 1$, Theorem 3.9, Theorem 3.1 lead to the same existence conditions as [23, Thm. 1].

Example 1 (Continued) One can apply Theorem 3.9 to solve the scalar TSF, and compute the transformation. Since $L_{g_i}L^2_f h_i, \mathrm{d}L_f h_i, i = 1, 2$ can be readily obtained, condition (11) is reduced to

$$\mathrm{d}(2x_{1,2} + x_{2,2}) = s^{-1}(2\frac{\partial s}{\partial y_1}\mathrm{d}x_{1,2} + \frac{\partial s}{\partial y_2}\mathrm{d}x_{2,2}) \mod \{\mathrm{d}y\} \tag{24a}$$

$$\mathrm{d}(2x_{2,2} + x_{1,2}) = s^{-1}(2\frac{\partial s}{\partial y_2}\mathrm{d}x_{2,2} + \frac{\partial s}{\partial y_1}\mathrm{d}x_{1,2}) \mod \{\mathrm{d}y\}. \tag{24b}$$

(24a) yields the PDEs

$$\frac{\partial s}{\partial y_1} = \frac{\partial s}{\partial y_2} = s.$$

Also, one can obtain the same PDEs for s from (24b). Hence, we solve the scalar TSF (4). Lie bracket conditions (21) are satisfied and the state transformation is computed by solving (23).

IV. ERROR DYNAMICS STABILITY

Assuming the existence of a TOF and considering the standard Luenberger observer in TOF coordinates and the new times

$$\frac{\mathrm{d}\hat{z}}{\mathrm{d}\tau} = A\hat{z} + \gamma(y) + L(y - C\hat{z}), \tag{25}$$

we have the LTI error dynamics

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau} = (A - LC)\tilde{z},$$

which is globally exponentially stable (GES). The error dynamics of the ith subsystem in the original time is a LTV system:

$$\dot{\tilde{z}}_i = s_i(A_i - L_iC_i)\tilde{z}_i. \tag{26}$$

We study the error dynamics stability of (26) by examining the stability of the LTV system

$$\dot{e} = \alpha(t)A_c e, \quad \alpha(t) > 0, \forall t \in [t_0, \infty), \tag{27}$$

where $e = (e_1, \ldots, e_n)^T$, and $A_c \in \mathbb{R}^{n\times n}$ is Hurwitz. Since the observer gain allows for arbitrary eigenvalue assignment, we further assume A_c is diagonalizable. We first give the stability result of (27) when $n = 1$.

Proposition 4.1: Given a 1-dimensional system

$$\dot{x} = -\sigma\alpha(t)x, \quad x_0 = x(t_0), \sigma > 0, \alpha(t) > 0, \forall t \in [t_0, \infty), \tag{28}$$

its equilibrium point $x = 0$ is GES if and only if there exist positive constants t_0, T_0, and $\epsilon > 0$ such that

$$\int_t^{t+T_0} \alpha(\xi)\mathrm{d}\xi \geq \epsilon, \quad \forall t \geq t_0. \tag{29}$$

Proof: The solution of the LTV system (28) is

$$x(t) = \exp\left(-\sigma \int_{t_0}^{t} \alpha(\xi)\mathrm{d}\xi\right) x(t_0). \tag{30}$$

$\Rightarrow$: From (29) we have

$$\left(\frac{(t-t_0)}{T_0} - 1\right)\epsilon \leq \int_{t_0}^{t} \alpha(\xi)\mathrm{d}\xi.$$

Substituting the above equation into (30) we know

$$c_1 e^{-m(t-t_0)}|x(t_0)| \leq |x(t)| \leq c_2 e^{-l(t-t_0)}|x(t_0)|, \tag{31}$$

where c_1, m are some positive constants, and $c_2 = \exp(\sigma\epsilon/T_0), l = \sigma\epsilon/T_0$. Hence, the equilibrium point $x = 0$ is GES and the sufficiency of condition (29) is shown.

$\Leftarrow$: Since the origin $x = 0$ is GES, the system trajectory satisfies (31) where c_1, c_2, m, l are some positive constants. Using (30), we have

$$\int_{t_0}^{t} \alpha(\xi)\mathrm{d}\xi \geq c_3(t - t_0) - c_4, \quad \forall t \geq t_0,$$

where c_3, c_4 are some appropriate positive constants. Letting $T_0 > c_4/c_3$ and $\epsilon = c_3T_0 - c_4 > 0$, and computing the integral from t to $t + T_0$, we have

$$\int_t^{t+T_0} \alpha(\xi)\mathrm{d}\xi \geq \epsilon, \quad \forall t \geq t_0.$$

Thus the necessity of condition (29) is proven. ■

Proposition 4.2: The equilibrium point $e = 0$ of the LTV system (27) is GES if and only if there exist positive constants t_0, T_0, and $\epsilon > 0$ such that (29) holds.

Proof: Since A_c is assumed diagonalizable into $A_d = \mathrm{Diag}\{\sigma_1, \cdots, \sigma_n\}$ by a linear transformation $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_n)^T = He$, the system (27) is transformed into n decoupled scalar systems

$$\dot{\bar{e}}_i = -\sigma_i\alpha(t)\bar{e}_i, \quad 1 \leq i \leq n$$

whose equilibrium points $\bar{e}_i = 0$ are GES. According to Proposition 4.1, (29) is necessary and sufficient to ensure $e_i = 0$ is GES. ■

Finally, we state the theorem without proof for the stability of error dynamics (26), $1 \leq i \leq p$.

Theorem 4.3: Assume system (1) is globally transformed to a TOF (6). Given the observer (25) with $A - LC$ Hurwitz, the zero solution of the error dynamics in the original time (26), $1 \leq i \leq p$, is GES if and only if there exist positive constants t_0, T_0, and $\epsilon > 0$ such that (29) holds with $\alpha(\xi) = s_i(y(\xi))$.

Remark 4.4: A non-vanishing positive TSF is required to preserve the error dynamics stability in the sense of Lyapunov. For a linear time varying system

$$\dot{x} = A(t)x, \quad x(t_0) = x_0,$$

$x = 0$ is a GUAS equilibrium point if and only if $x = 0$ is an GES equilibrium point. Hence, GUAS is guaranteed by (29).

V. CONCLUSION

Time scaling of a multi-output observer form (TOF) for uncontrolled nonlinear continuous-time systems is considered in this note. Two cases of TOF are discussed and necessary and sufficient conditions for the existence of these TOF are provided. Since time scaling transformation affects the error dynamics stability, the necessary and sufficient condition on TSFs to preserve the GES of the error dynamics are presented.

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