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Robust control of uncertain multi-inventory systems via Linear Matrix Inequality

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***Abstract*— We consider a continuous time linear multi–inventory system with unknown demands bounded within ellipsoids and controls bounded within ellipsoids. We address the problem of ϵ-stabilizing the inventory since this implies some reduction of the inventory costs. As main result, we provide conditions under which ϵ-stabilizability is possible through a saturated linear state feedback control. All the results are based on a Linear Matrix Inequalities (LMIs) approach.**

I. Introduction

We consider a continuous time linear multi–inventory system with unknown demands bounded within ellipsoids and controls bounded within ellipsoids. The system is modelled as a first order one integrating the discrepancy between controls and demands at different sites (buffers). Thus, the state represents the buffer levels. We wish to study conditions under which the state can be driven within an a-priori chosen target set through a saturated linear state feedback control. Let ϵ be a maximal dimension of the target set, the above problem corresponds to ϵ-stabilizing the state.

Motivations for ϵ-stabilizing the state derive from the benefits associated to keeping the state and consequently also the inventory costs bounded. This work is in line with some recent literature on robust optimization [1], [6] and control [2] of inventory systems. Here as well as in [2] we focus on saturated linear state feedback controls since such controls arise naturally in any system with bounded controls.

The main results of this work can be summarized as follows. Initially we introduce the necessary and sufficient conditions for the ϵ-stabilizability in the form of an inclusion between convex sets. In the case where both demands and controls are bounded within polytopes, it is well known that verifying such conditions is NP-hard [9]. Here, we prove that verification becomes easy when both demands and controls are bounded within ellipsoids. This is possible by rewriting the inclusion between ellipsoids in terms of unconstrained quadratic maximization.

We first characterize invariant sets through a fourth degree condition. As verifying such a condition is difficult, we then propose the best quadratic approximation of the same condition. We proceed by describing the region of linearity of the control and conclude by providing LMI conditions on the target set under which the saturated control ϵ-stabilizes the system. The case where demands are bounded within ellipsoids and controls are bounded within polytopes is an open problem and sufficient LMI conditions to solve it are presented in [3].

All the results are based on a Linear Matrix Inequalities (LMIs) approach in line with the recent work [7] on inventory/manufacturing systems.

This paper is arranged as follows. In Section II, we formulate the problem. In Section III, we introduce necessary and sufficient conditions for the admissibility of the problem. In Section IV we study the problem with ellipsoidal constraints. In Section V, we provide numerical illustrations. Finally, in Section VI, we draw some conclusions.

II. Problem Formulation

Consider the continuous time linear multi–inventory system

$$\dot{x}(t) = Bu(t) - w(t), \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is a vector whose components are the buffer levels, $u(t) \in \mathbb{R}^m$ is the controlled flow vector, $B \in \mathbb{Q}^{n\times m}$, with $m \geq n$ and $rank(B) = n$ is the controlled process matrix and $w(t) \in \mathbb{R}^n$ is the unknown demand. To model backlog $x(t)$ may be less than zero. Demands and controls are bounded within ellipsoids, i.e.,

$$w(t) \in \mathcal{W} = \{w \in \mathbb{R}^n : w^T R_w w \leq 1\} \qquad (2)$$
$$u(t) \in \mathcal{U} = \{u \in \mathbb{R}^m : u^T R_u u \leq 1\}. \qquad (3)$$

For any positive definite matrix $P \in \mathbb{R}^{n\times n}$, define the function $V(x) = x^T P x$ and the ellipsoidal target set $\Pi = \{x \in \mathbb{R}^n : V(x) \leq 1\}$. In addition, for any matrix $K \in \mathbb{R}^{n\times n}$, define as saturated linear state feedback control any policy

$$u = sat\{-Kx\} = \begin{cases} -Kx & \text{if } Kx \in \mathcal{U} \\ u(x) \in \partial\mathcal{U} & \text{otherwise} \end{cases} \qquad (4)$$

where hereafter ∂F indicates the frontier of a given set F.

Problem 1: (ϵ-stabilizing) Given system (1), find conditions on the positive definite matrix $P \in \mathbb{R}^{n\times n}$, under which there exists a saturated linear state feedback control $u = sat\{-Kx\}$ such that it is possible to drive the state $x(t)$ within the target set Π.

Solving the above problem corresponds to ϵ-stabilizing the state x where the relation between ϵ and Π is

$$\epsilon := \max_x \{\|x\|_\infty : x \in \Pi\}. \qquad (5)$$

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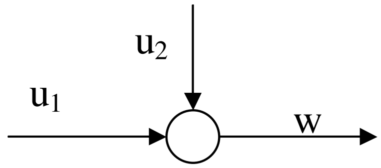

Fig. 1. Graph with one node and two arcs.

Example 1: Throughout this paper we consider, as illustrative example, the graph with one node and two arcs depicted in Fig. 1. The incidence matrix is $B = [1 \quad 1]$. The continuous time dynamics is

$$\dot{x}(t) = \underbrace{[1 \quad 1]}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{u} - w = u_1(t) + u_2(t) - w(t),$$

with demand bounded in the ellipsoid

$$w^2 \leq 1$$

and with the following either ellipsoidal constraints on the control u

$$(u_1 + u_2)^2 \leq 1, \tag{6}$$

Finally, the target set is the sphere of unitary radius $\Pi = \{x \in \mathbb{R} : x^2 \leq 1\}$.

III. Stability necessary and sufficient conditions

System (1) is ϵ-stabilizable if and only if for all $w \in \mathcal{W}$, there exists $u \in int\{\mathcal{U}\}$ such that $Bu = w$ (see, e.g., [4]). For the short of notation, the previous condition is usually expressed as

$$B\mathcal{U} \supset \mathcal{W}. \tag{7}$$

Deciding whether (7) holds is NP-hard, when $\mathcal{U}$ and $\mathcal{W}$ are polytopes. Here, we prove that verifying (7) becomes easy when both $\mathcal{U}$ and $\mathcal{W}$ are ellipsoids. Observe that we can rewrite $Bu = w$ as $u_{\mathcal{B}} = \mathcal{B}^{-1}w - \mathcal{B}^{-1}Nu_N$, where $B = [\mathcal{B}|N]$ being $\mathcal{B}$ a basis of B and N the remaining columns of B, correspondingly $u_{\mathcal{B}}$ are the n components of u associated to the basis $\mathcal{B}$ and u_N are the $m - n$ components of u associated to the columns in N.

As we observe that (7) is equivalent to

$$\max_{w \in \mathcal{W}} \quad \min_{u \in \mathbb{R}^m : Bu = w} uR_u u < 1,$$

Condition (7) holds if and only if

$$\begin{aligned} &\max_{w \in \mathcal{W}} \min_{u_N \in \mathbb{R}^{m-n}} f(u_{\mathcal{B}}(w, u_N), u_N) = \\ &= \left[w^T \mathcal{B}^{-T} - u_N^T N^T \mathcal{B}^{-T} | u_N^T\right] R_u \\ &\begin{bmatrix} \mathcal{B}^{-1}w - \mathcal{B}^{-1}Nu_N \\ u_N \end{bmatrix} < 1 \end{aligned} \tag{8}$$

When we consider the illustrative example in Section 1, we have $\mathcal{B} = [1]$, $N = [1]$ then problem (9) becomes

$$\max_{-1 \leq w \leq 1} \min_{u_2 \in \mathbb{R}} f(u_{\mathcal{B}}(w, u_2), u_2) =$$

$$= [w - u_2 | u_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w - u_2 \\ u_2 \end{bmatrix} = (w - u_2)^2 + u_2^2 < 1$$

Now consider, function $f(u_{\mathcal{B}}(w, u_N), u_N)$. It is a differentiable convex function in u_N. Then, for any $w \in \mathcal{W}$ we can analytically determine the best response $u_N^*(d) = arg \min_{u_N \in \mathbb{R}^{m-n}} f(u_{\mathcal{B}}(w, u_N), u_N)$, by imposing

$$\nabla_{u_N} f(u_{\mathcal{B}}(w, u_N), u_N) =$$
$$2\left[-N^T \mathcal{B}^{-T} | I\right] R_u \begin{bmatrix} \mathcal{B}^{-1}w - \mathcal{B}^{-1}Nu_N \\ u_N \end{bmatrix} = 0,$$

where I is the $(m-n) \times (m-n)$ identical matrix. We obtain

$$u_N^*(w) = -Mw,$$

where 0 is the $(m - n) \times n$ null matrix and

$$M = \left(\left[-N^T \mathcal{B}^{-T} | I\right] R_u \begin{bmatrix} -\mathcal{B}^{-1}N \\ I \end{bmatrix}\right)^{-1}$$
$$\left[\ -N^T \mathcal{B}^{-T} | I\ \right] R_u \begin{bmatrix} \mathcal{B}^{-1} \\ 0 \end{bmatrix}.$$

In the example under consideration, we have

$$u_2^*(w) = -\left([-1|1] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)^{-1} [-1|1]$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} w = \tfrac{w}{2}.$$

For any $w \in \mathcal{W}$ the minimal value of $f(u_{\mathcal{B}}(w, u_N), u_N)$ is

$$f(u_{\mathcal{B}}(w, u_N^*(w)), u_N^*(w)) = w^{*T} \Phi w^*,$$

where

$$\Phi = \underbrace{\left[\mathcal{B}^{-T} + M^T N^T \mathcal{B}^{-T} | - M^T\right]}_{H^T} R_u \underbrace{\begin{bmatrix} \mathcal{B}^{-1} + \mathcal{B}^{-1}NM \\ -M \end{bmatrix}}_{H} = H^T R_u H \tag{9}$$

is a positive definite $n \times n$ matrix, as M is full rank. So far, we have shown that we can find the optimal value of problem (9) by solving problem

$$\max_{w \in \mathcal{W}} w^T \Phi w, \tag{10}$$

and checking that the optimal value is less than one.

We are ready to observe that problem (10) is easy as it reduces to determining the eigenvectors of an $n \times n$ matrix.

Theorem 3.1: System (1) is ϵ-stabilizable if and only if $w^{*T} \Phi w^* < 1$, for all w^* eigenvectors associated to the maximum eigenvalue of matrix $R_w^{-1}\Phi$ whose weighted quadratic norm $w^{*T} R_w w^*$ is equal to 1.

Proof: As $w^T \Phi w$ is convex, its optimal value w^* lays on the frontier $\partial\mathcal{W}$ of the set $\mathcal{W}$, i.e., for $w^{*T} R_w w^* = 1$. Imposing the Karush Kuhn Tucker first order optimality condition, we obtain $2(\Phi - \lambda R_w)w^* = 0$. Then the optimal values of w^* are some of the matrix $R_w^{-1}\Phi$ eigenvectors whose weighted quadratic norm $w^{*T} R_w w^*$ is equal to 1. In particular, w^* are the eigenvectors associated to the maximal eigenvalues of $R_w^{-1}\Phi$. $\square$

In the example under consideration $\Phi = \left[\frac{1}{2}\right]$ and $w^* = \pm 1$ then $w^{*T} \Phi w^* = \frac{1}{2} < 1$, hence the associated system is ϵ-stabilizable.

In the following we discuss for which initial state the system is certainly ϵ-stabilizable through a (pure) linear state feedback control; hence we show that if we saturated the previous linear policy the system is ϵ-stabilizable for any initial state.

IV. Ellipsoidal constraints

Let us start by considering only the constraints (2) on w and neglect the ellipsoidal constraints (3) on u. Among the saturated linear state feedback control (4) we prove that we can solve Problem 1 using controls of type $u = sat\{-kHx\}$, with $k \in \mathbb{R}$ and $H \in \mathbb{R}^n$ as defined in (9). Note that matrix H is a right inverse of B, that is $BH = I$. We motivate the choice of $u = sat\{-kHx\}$ with H as defined in (9) as such a control describes the best response of u under the worst w as proved in the previous section. Also, note that the scalar $k \in \mathbb{R}$ must be lower than a certain value, which means that we cannot use a bang-bang control. This is motivated by the following reason. If we use a control $u = sat\{-kHx\}$, then the necessary and sufficient condition (7) becomes

$$B\mathcal{U}_{lin} \supset \mathcal{W} \qquad (11)$$

where

$$\mathcal{U}_{lin} = \{u \in \mathbb{R}^m : u = -kHx,\ k^2 x^T H^T R_u H x \leq 1\}.$$

Following the derivation of (10) in the previous Section, we have that (11) holds if and only if

$$k^2 w^{*T} \Phi w^* < 1.$$

For $k = 1$ the above condition holds true as it reduces to (10). Obviously, the value $\hat{k} = \sqrt{\frac{1}{w^{*T}\Phi w^*}}$ is an upper bound for k, namely, we must choose k such that $k < \hat{k}$ if we wish the necessary and sufficient condition (11) be satisfied.

With the above considerations in mind, we can conclude that the dimensions of the target Π where it is possible to drive the state are lower bounded.

Denote by $\lambda_{max}(Z)$ the maximum eigenvalue of a given matrix Z. In the following theorem we prove that $\dot{V}(x) < 0$ within a given set (*invariant* set). This result will allow exploiting $V(x)$ as a Lyapunov function to prove the convergence to the target set Π.

Theorem 4.1: Consider system (1) subject to the only ellipsoidal constraints (2) on w, and controlled via linear state feedback $u = -kHx$, with H such that $BH = I$. Then condition $\dot{V} < 0$ holds if and only if

$$k^2(x^T P x)^2 - x^T P R_w^{-1} P x > 0. \qquad (12)$$

Proof: For H such that $BH = I$, condition $\dot{V} < 0$ is equivalent to

$$2kx^T P x + 2w^T P x > 0. \qquad (13)$$

We aim at proving that $\dot{V} < 0$ holds for any x external to an appropriate smooth closed surface. To do this, we look for an $x \in \mathbb{R}^n$ inducing a solution strictly greater than zero for the following problem

$$\min_{w \in \mathcal{W}} \zeta(x, w) = 2kx^T P x + 2w^T P x. \qquad (14)$$

As $\zeta(x, w)$ is linear in w, the optimal w^* must lay on the boundary of set $\mathcal{W}$. The Karush Kuhn Tucker conditions impose that $Px = -\lambda R_w w^*$ for some $\lambda \geq 0$, that is $w^* = -\frac{1}{\lambda} R_w^{-1} P x$. Note that being P full rank, it necessarily holts that$\lambda \neq 0$ for all $x \neq 0$. Then, $\zeta(x, w^*) = 2kx^T P x - \frac{2}{\lambda} x^T P R_w^{-1} P x > 0$. As w^* lays on the boundary of $\mathcal{W}$, we have $w^{*T} R_w w^* = \frac{x^T P R_w^{-1} P x}{\lambda^2} = 1$ from which $\lambda = \sqrt{x^T P R_w^{-1} P x}$. Hence, $\zeta(x, w^*) > 0$, and therefore also (13) holds, if and only if (12) holds. □

We now exploit $V(x) = x^T P x$ as a Lyapunov function to prove the convergence to the target set Π. We determine under which conditions on P and k we have that $\dot{V} < 0$ or, equivalently, inequality (12) hold for any $x \notin \Pi$.

When $P = \nu R_w$, (12) becomes $k^2 x^T P x > \nu$. Then, in this case, we can use $V(x)$ to prove the convergence of the system to Π for $k^2 \geq \nu$.

In the following, we consider the general case when $P \neq \nu R_w$.

Lemma 4.2: Consider system (1) subject to the only ellipsoidal constraints (2) on w, and controlled via linear state feedback $u = -kHx$, with H such that $BH = I$. Then, $k^2(x^T P x)^2 - x^T P R_w^{-1} P x > 0$ holds for any $x \notin \Pi$ if and only if $k^2 - x^T P R_w^{-1} P x \geq 0$ holds for any $x \in \partial\Pi$.

Proof: (*Necessity*). Assume that there exists $\hat{x} \in \partial\Pi$ such that $k^2 - x^T P R_w^{-1} P x < 0$. Then, there also exists a ball $Ball(\hat{x}, r)$ centered in $\hat{x}$ with a sufficiently small radius $r > 0$ such that for all $x \in Ball(\hat{x}, r)$ we have $k^2 - x^T P R_w^{-1} P x < 0$. This implies that there exist $x \notin \Pi$ for which condition (12) does not hold.

(*Sufficiency*). Assume that $k^2 - x^T P R_w^{-1} P x \geq 0$ holds for any $x \in \partial\Pi$. By contradiction, consider $\hat{x} \notin \Pi$, i.e., $\hat{x}^T P \hat{x} = \rho > 1$, such that $k^2(\hat{x}^T P \hat{x})^2 - \hat{x}^T P R_w^{-1} P \hat{x} < 0$, that is $k^2\rho^2 - \hat{x}^T P R_w^{-1} P \hat{x} < 0$. Then, there exists $\tilde{x} = \frac{\hat{x}}{\sqrt{\rho}} \in \partial\Pi$ such that $k^2\rho^2 - \rho\tilde{x}^T P R_w^{-1} P \tilde{x} < 0$, that is $k^2\rho - \tilde{x}^T P R_w^{-1} P \tilde{x} < 0$. This latter result is contradictory as we cannot have $k^2\rho < \tilde{x}^T P R_w^{-1} P \tilde{x} \leq k^2$, for $\rho > 1$. □

Lemma 4.3: Consider system (1) subject to the only ellipsoidal constraints (2) on w, and controlled via linear state feedback $u = -kHx$, with H such that $BH = I$. We can use $V(x)$ to prove the convergence of the system to Π for $k^2 \geq \lambda_{max}(R_w^{-1} P)$.

Proof: Condition $k^2 - x^T P R_w^{-1} P x \geq 0$ holds for any $x \in \partial\Pi$ if and only if $\min_{x \in \partial\Pi}\{k^2 - x^T P R_w^{-1} P x\} \geq 0$. Imposing the Karush Kuhn Tucker first order optimality condition, we obtain $2(P R_w^{-1} P - \lambda P)x^* = 0$. Then the optimal values of x^* are some of the matrix $R_w^{-1} P$ eigenvectors whose weighted quadratic norm $x^{*T} P x^*$ is equal to 1. In particular, x^* are the eigenvectors associated to the maximal eigenvalues of $R_w^{-1} P$. For vectors x^*, condition $k^2 - x^{*T} P R_w^{-1} P x^* \geq 0$ becomes $k^2 - \lambda_{max}(R_w^{-1} P) x^{*T} P x^* \geq 0$, that is $k^2 - \lambda_{max}(R_w^{-1} P) \geq 0$. □

Observe that the system converges to the target set $\Pi_R = \{x : k^2 x^T R_w x \leq 1\}$ as any feasible target set $\Pi = \{x : x^T P x \leq 1\}$, with $k^2 \geq \lambda_{max}(R_w^{-1} P)$ includes Π_R. Indeed, $\Pi \supseteq \Pi_R$ if $x^T P x - k^2 x^T R_w x = x^T(P - k^2 R_w)x \leq 0$ or equivalently if $P - k^2 R_w \preceq 0$. In turn, the latter condition is equivalent to $R_w^{-1} P - k^2 I \preceq 0$ that certainly holds as $k^2 \geq \lambda_{max}(R_w^{-1} P)$

In the next theorem we introduce the constraints on controls (3). To this end, we need to define the family of ellipsoids

$$\Sigma_0(\xi) = \{x \in \mathbb{R}^n : x^T P x \leq x(0)^T P x(0) := \xi\} \qquad (15)$$

parametrized in $\xi \geq 1$.

Theorem 4.4: Given system (1), we can drive the state $x(t)$ from any initial value $x(0) \in \Sigma_0(\xi)$ to the target set Π via linear state feedback $u = -kHx$ if the following conditions hold

$$k^2 \geq \lambda_{max}(R_w^{-1} P) \qquad (16)$$

$$k^2 \xi \lambda_{max}(P^{-1}\Phi) \leq 1. \qquad (17)$$

Proof: By Lemma 4.3, under condition (16) it holds $\dot{V}(t) < 0$ for all $x(t) \notin \Pi$ and then $V(x)$ can be considered as a Lyapunov function for the convergence of the state to the set Π when the linear control $u = -kHx$ is implemented. Condition $\dot{V}(t) < 0$ also implies that $\Sigma_0(\xi)$ is invariant with respect to the same linear feedback as $\xi \geq 1$ which means $\Sigma_0(\xi) \supseteq \Pi$. Then

$$\max_{t \geq 0} u^T(t) R_u u(t) \leq \max_{x \in \Sigma_0(\xi)} k^2 x^T H^T R_u H x =$$
$$= \max_{x \in \Sigma_0(\xi)} k^2 x^T \Phi x = k^2 \xi \lambda_{max}(P^{-1}\Phi).$$

Therefore the constraint $u = -kHx(t) \in \mathcal{U}$ for all $t \geq 0$ is enforced if (17) holds true. □

The following theorem provides a solution to Problem 1. Let us denote by X the set of states x where we can define a linear control $u(x) = -kHx$, i.e., $X = \{x : -kHx \in \mathcal{U}\}$. Consider the saturated linear state feedback control of type

$$u(x) = \begin{cases} -kHx & \text{if } x \in X \\ -\frac{Hx}{\sqrt{x^T H^T R_u H x}} & \text{if } x \notin X \end{cases} . \qquad (18)$$

Theorem 4.5: Given system (1), for any positive definite matrix $P \in \mathbb{R}^{n \times n}$ satisfying condition (16), the saturated linear state feedback control (18) drives the state $x(t)$ within the target set Π for any initial state $x(0)$.

Proof: By construction, $u(x)$ is a continuous function with $\mathcal{U}$ as codmain. When we use such a control, we know that $\dot{V}(x) < 0$ also holds for any $x \notin \Pi$, if $\Pi \subset X$ and $k^2 \geq \lambda_{max}(R^{-1}P)$ (see Lemma 4.3).

First observe that, for all $x \in \partial X$, we have $x^T P x > k^2 x^T H^T R_u H x = 1$, where the latter inequality holds as $\Pi \subset X$. Then, for any $x \notin X$, that is for $k^2 x^T H^T R_u H x > 1$, we have $\frac{x^T P x}{x^T H^T R_u H x} > k^2 \geq \lambda_{max}(R^{-1}P)$ since both $x^T P x$ and $x^T H^T R_u H x$ are positive definite quadratic forms.

α (10^{-2})	1	2	3	4	5	6	7	8	9
ξ	31	15	10	7.7	6.2	5.1	4.4	3.8	3.4
α (10^{-2})	10	15	20	25	30	35	40	45	50
ξ	3	2	1.5	1.2	1	0.8	0.7	0.6	0.6

TABLE I

DEPENDENCE OF ξ ON α IN THE CASE WHERE $R_u := \alpha I$ AND $k = 1$: THE HIGHER α THE BIGGER THE REGION $\Sigma_0(\xi)$ AS IN (15) AND ALSO THE REGION OF LINEARITY $X = \{x : -kHx \in \mathcal{U}\}$.

In Lemma 4.3, we have proved that $\dot{V}(x) < 0$ for $x \in X \setminus \Pi$. Now, we consider $x \notin X$. We have $\dot{V}(x) < 0$ if and only if $-x^T PBu(x) + x^T P w > 0$, for all $w \in \mathcal{W}$, that is

$$\min_{w \in \mathcal{W}} \left\{ \frac{x^T P x}{\sqrt{x^T H^T R_u H x}} + x^T P w \right\} > 0 \qquad (19)$$

must hold. Applying the Karush-Kuhn-Tuker conditions, we transform (19) in $\frac{x^T P x}{\sqrt{x^T H^T R_u H x}} - \sqrt{x^T P^T R_w^{-1} P x} > 0$. In turn, the latter inequality holds if $\frac{x^T P x}{x^T H^T R_u H x} - \lambda_{max}(R^{-1}P) > 0$, as $x^T P^T R_w^{-1} P x \leq \lambda_{max}(R^{-1}P) x^T P x$. We then conclude that $\dot{V}(x) < 0$ since $\frac{x^T P x}{x^T H^T R_u H x} > k^2 \geq \lambda_{max}(R^{-1}P)$. □

Observe that the saturated linear state feedback control (18) is not decentralized in the sense that the generic ith control u_i in general depends on the demand at different nodes and on the other controls u_j, $j \neq i$. This is due to either the structure of matrix H or the ellipsoidal constraints (3).

Example 2: Consider the graph depicted in Fig. 1, with one node and two arcs and incidence matrix $B = [1 \quad 1]$. Controls are subject to ellipsoidal constraints (6). Then we have, $R_w = 1$, $R_u = I$ and $\Phi = \frac{1}{2}$. We can stabilize the system within $\Pi = \{x \in \mathbb{R} : x^2 \leq 1\}$ for any initial state $x(0) \leq \sqrt{2}$ via a pure linear state feedback $u = -kHx$. To see this take $Q = I$, and observe that the matrix inequality on Q (**??**) is satisfied for any $k \geq 1$. Furthermore, if we assume $k = 1$, then from (17) we must have $k^2 = 1 \leq \frac{2}{\xi^2} = \frac{2}{x(0)^2}$.

V. NUMERICAL ILLUSTRATIONS

Consider the constrained dynamics (1)-(3) for the flow network system with $n = 5$ nodes and $m = 9$ arcs depicted in Fig. 2 and take without loss of generality $R_w = I$ and $R_u = \alpha I$ for different values of $\alpha = 0.01, \ldots, 0.5$. Trivially, the higher the value of the parameter α, the weaker the constraints on the control (3). Also, from condition (17), we have that the weaker the constraints (3), the bigger the region $\Sigma_0(\xi)$ as defined in (15) and also the region of linearity $X = \{x : -kHx \in \mathcal{U}\}$. In Table I, we display the dependence of ξ on increasing values of α when $k = 1$.

Now, for a specific value of $\alpha = 0.5$, apply the control (18)

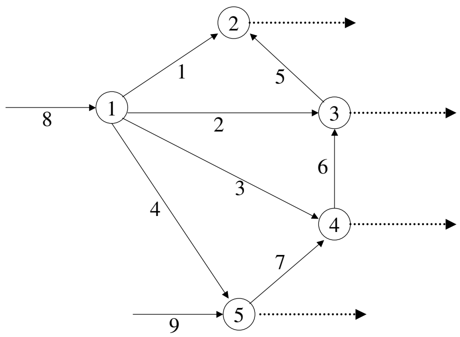

Fig. 2. Example of a system with 5 nodes and 9 arcs.

with $k = \frac{1}{3}, \frac{1}{2}, 1$ and matrix $H \in \mathbb{R}^n$ defined as

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ -0.1 & 0 & 0.5 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0 \\ 0.6 & 1 & 1 & 0 & 0 \\ 0.4 & 0 & 0 & 1 & 1 \end{bmatrix}. \qquad (20)$$

Note that matrix H is a right inverse of B, that is $BH = I$. Basically, the columns of the above matrix establish that i) the demand at node 2 is satisfied by a flow through arc 8 and 1, ii) the demand at node 3 is satisfied by a flow through arc 8, which splits in two equal parts, the first one going through arc 2 and the second one through arc 3 and 6, iii) the demand at node 4 is entirely satisfied by a flow through arc 9 and 7, iv) finally the demand at node 5 is satisfied by a flow through arc 9. Obviously, the first column has no particular meaning since the demand at node 1 is null.

Now, we simulate the system with initial state $x(0) = [0\,4\,4\,4\,4]^T$ and random demand $w(t)$ for (a) $k = \frac{1}{3}$, (b) $k = \frac{1}{2}$ and (c) $k = 1$. Demand $w(t)$ is randomly extracted from the set $\{w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}\}$ with uniform probability where

$$w^{(1)} = [0 \quad \pm 1 \quad 0 \quad 0 \quad 0]^T \quad w^{(2)} = [0 \quad 0 \quad \pm 1 \quad 0 \quad 0]^T$$
$$w^{(3)} = [0 \quad 0 \quad 0 \quad \pm 1 \quad 0]^T \quad w^{(4)} = [0 \quad 0 \quad 0 \quad 0 \quad \pm 1]^T.$$

Actually, imposing a maximal (in this case the maximal demand componentwise is 1) non null demand only at one node at each time translates into larger oscillations of the buffers (variable x). For this reason the above demand can be reviewed as a sort of "worst case" demand.

Fig. 3 displays the time plot of the state variable $x(t)$ and observe that in all of the three cases, from about $t > 10$ on, the state $x(t)$ never exceeds the interval $[-k, k]$ componentwise. With the above choices of $k = \frac{1}{3}, \frac{1}{2}, 1$, and $R_w = I$, the possible values for P satisfying condition (16) are $P = k^2 I$. Fig. 4 plots the evolution of function $V(x(t)) - 1$ with $V(x(t)) = k^2 x^T x$ for $k = \frac{1}{3}, \frac{1}{2}, 1$. The latter function decreases and from a certain time on (about $t > 10$) we always have $V(x(t)) \leq 1$. This means that in all the three cases, we can drive the state within the target sets $\Pi = \{x \in \mathbb{R}^n : k^2 x^T x \leq 1\}$.

From Table I we have that the value of ξ associated to α is 0.62. Such a value identifies the region $\Sigma(\xi) = \{x \in \mathbb{R}^n : x^T x \leq 0.62\}$ used to approximate the region of linearity $X = \{x : -kHx \in \mathcal{U}\}$. Actually, condition (17) guarantees the condition $\Sigma(\xi) \subseteq X$.

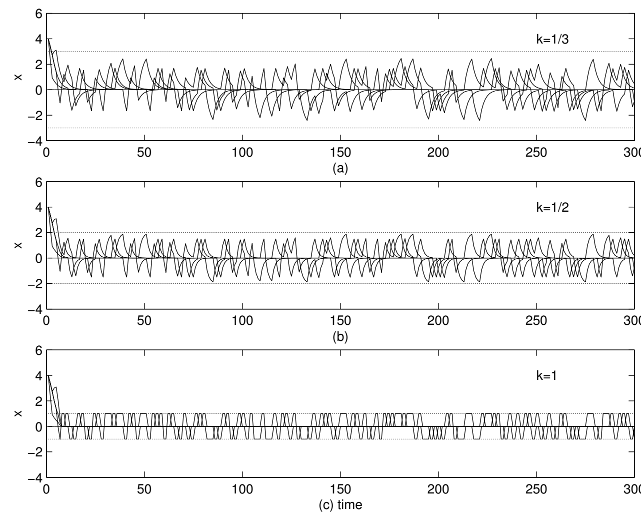

Fig. 3. Time plot of the state variable $x(t)$ when the saturated linear feedback control (18) is applied with H as in (20) and with gain (a) $k = \frac{1}{3}$, (b) $k = \frac{1}{2}$ and (c) $k = 1$. Demand $w(t)$ is randomly generated.

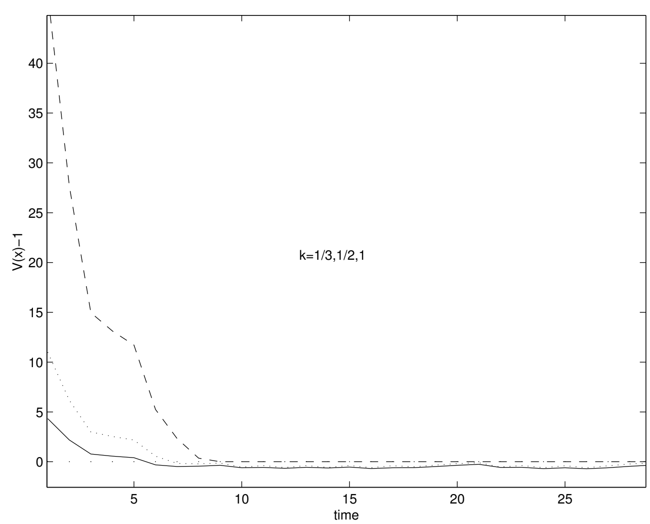

Fig. 4. Time plot of function $V(x(t)) - 1$ when the saturated linear feedback control (18) is applied with H as in (20), and $k = \frac{1}{3}$ (solid line), $k = \frac{1}{2}$ (dotted line), and $k = 1$ (dashed line). Function $V(x(t))$ decreases and for about $t > 8$ it satisfies the condition $V(x(t)) \leq 1$.

In Fig. 5 we show the projection onto the plane x_3-x_4 of the simulated state trajectory for $k = \frac{1}{2}$ and displayed in Fig.

3 (a). Starting at point $[4 \quad 4]^T$, the trajectory (dotted) is soon confined within the target set $\Pi = \{x \in \mathbb{R}^n : k^2 x^T x \leq 1\}$ described by the dashed sphere of radius 3 and centered in the origin.

Finally we choose a different matrix

$$R_w = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

Note that with the new choice of R_w we have bilinear terms in w_3 and w_4 in the constraints 2. Then, a possible value for P satisfying condition (16) is $P = k^2 R_w$. In Fig. 6 we show the projection onto the plane x_3-x_4 of the simulated state trajectory for $k = \frac{1}{2}$ with the new choice of R_w. Again, starting at point $[4 \quad 4]^T$, the trajectory (dotted) is soon confined within the target set $\Pi = \{x \in \mathbb{R}^n : k^2 x^T R_w x \leq 1\}$ described by the dashed ellipsoid centered at zero and with axes $\frac{1}{k\sqrt{\lambda_1}} q_1$ and $\frac{1}{k\sqrt{\lambda_2}} q_2$ where $q_1 = [\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}]^T$, $q_2 = [-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}]^T$, $\lambda_1 = 1/2$ and $\lambda_2 = 1$ are the eigenvectors and eigenvalues of the submatrix $\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$ of R_w. To simulate a worst case scenario in the sense clarified above, demand $w(t)$ is randomly extracted from the set $\{w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}\}$ with uniform probability where

$$w^{(1)} = [0 \ \pm 1 \ 0 \ 0 \ 0]^T \quad w^{(2)} = [0 \ 0 \ \pm[2 \quad 2] \ 0]^T$$
$$w^{(3)} = [0 \ 0 \ \pm[-2\sqrt{2} \ 2\sqrt{2}] \ 0]^T \quad w^{(4)} = [0 \ 0 \ 0 \ 0 \ \pm 1]^T.$$

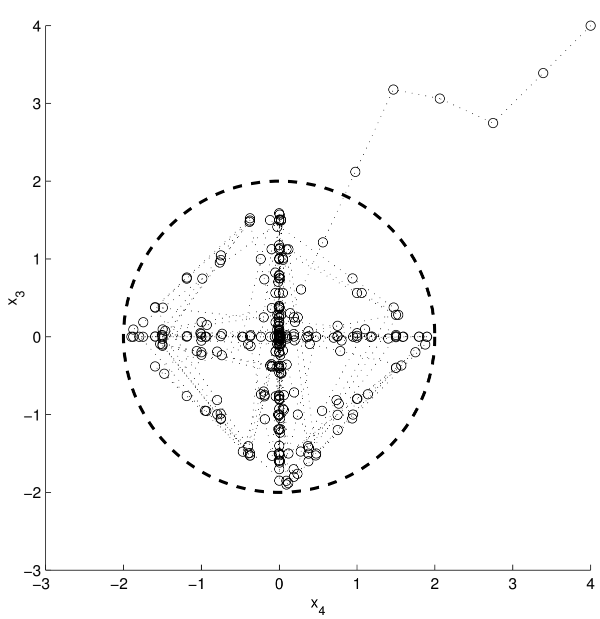

Fig. 5. Projection onto the plane x_3-x_4 of the simulated state trajectory for $k = \frac{1}{2}$, see Fig. 3 (b). Starting at point $[4 \quad 4]^T$, the trajectory (dotted) is soon confined within the sphere of radius 3 and centered in the origin.

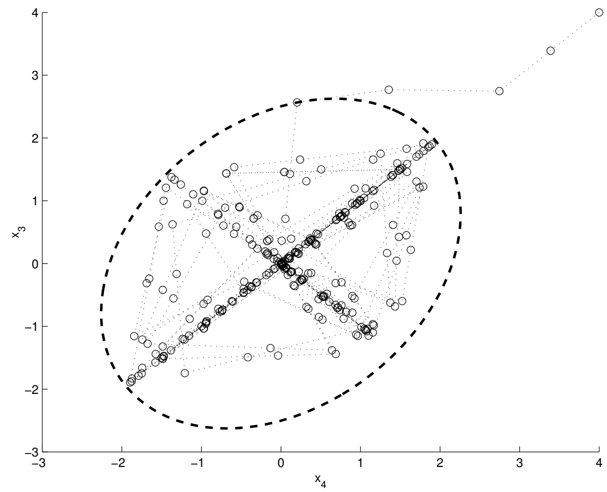

Fig. 6. Projection onto the plane x_3-x_4 of the simulated state trajectory for $k = \frac{1}{2}$, when R_w is as in (21). Starting at point $[4 \quad 4]^T$, the trajectory (dotted) is soon confined within the target set (dashed ellipsoid).

VI. Conclusions and Future Works

This work is a continuation of [2] and is in line with some recent applications of LMI techniques to inventory/manufacturing systems [7]. In a future work, we will study the validity in probability of the LMI conditions derived in this paper. This is in accordance with some recent literature on *chance LMI constraints* developed in the area of robust optimization [5], [8].

References

[1] E. Adida, and G. Perakis. A Robust Optimization Approach to Dynamic Pricing and Inventory Control with no Backorders. *Mathematical Programming*, Ser. B 107:97–129, 2006.

[2] D. Bauso, F. Blanchini, and R. Pesenti. Robust control policies for multi-inventory systems with average flow constraints. *Automatica*, Special Issue on Optimal Control Applications to Management Sciences, 42(8): 1255–1266, 2006.

[3] D. Bauso, L. Giarré and R. Pesenti. Robust control of uncertain multi-inventory systems and consensus problems. *Accepted to the 17th IFAC World Congress*, Seoul, Korea, July 2008.

[4] F. Blanchini, F. Rinaldi and W. Ukovich. A network design problem for a distribution system with uncertain demands. *SIAM Journal on Optimization*, 7: 560–578, 1997.

[5] A. Ben-Tal, A. Nemirovsky. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM Journal on Optimization*, 12: 811–833, 2002.

[6] D. Bertsimas, A. Thiele. A Robust Optimization Approach to Inventory Theory. *Operations Research*, 54(1): 150–168, 2006.

[7] E. K. Boukas. Manufacturing Systems: LMI Approach. *IEEE Transactions on Automatic Control*, 51(6): 1014-1018, 2006.

[8] G. Calafiore, M. C. Campi. Uncertain Convex Programs: Randomized Solutions and Confidence Levels. *Mathematical Programming*, 102: 25–46, 2005.

[9] S.T. McCormick. Submodular containment is hard, even for networks. *Operations Research Letters*, 19: 95–99, 1996.