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Transfer equivalence and reduction of nonlinear delta differential equations on homogeneous time scale

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Abstract— **The problem of equivalence is considered for nonlinear single-input single-output systems defined on homogeneous time scales and described by n-th order input-output delta-differential equations. First the concepts of reduction and of irreducibility of an input/output equation are explained. Subsequently, based on these notions, a definition of equivalence is introduced, which generalizes the notion of transfer equivalence. A practical criterion for evaluating irreducibility is given in terms of subspaces of one-forms, classified according to their relative degrees.**

Index Terms— **Time-scale, realization, irreducibility, equivalence.**

I. Introduction

A time scale is a model of time. Both continuous- and discrete-time cases are merged in time scale formalism into a general framework which represents not only a unification of continuous- and discrete-time systems but also an extension. For instance, the notion of the so-called *delta-derivative*, as well as the related definition of a *delta-differential equation*, is not only a generalization of both the standard time-derivative and of the difference operator but accommodates also much more possibilities. For this reason, the time scale approach has become recently very popular in the study of dynamic systems (see, for instance, [1] and [2]) but there are still only a few papers concerning control systems (see, among others, [3], [4], [5]).

The topics studied in this paper are the equivalence and the reduction of nonlinear systems on time scales, i.e. the problem considered is the following: *given an arbitrary i/o delta-differential equation, is it possible to find an accessible (irreducible) lower order representation, which is equivalent to the original system?* In particular, the definitions of transfer equivalence given in [6] and [7] for continuous- and discrete-time nonlinear systems, respectively, are extended to the case of nonlinear control systems described on homogeneous time scale. As in the previous results, the definition is based upon the notions of the autonomous variable and of the irreducible i/o equation of the system. Note that the extended equivalence notion is referred to as the *transfer equivalence*, as in the linear case the definition coincides with the classical definition of transfer equivalence and system reduction corresponds to pole/zero cancellation.

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The notion of transfer equivalence for nonlinear i/o delta-differential equations plays a crucial role in the realization problem [3]. The main result is that an accessible state space realization can be obtained if and only if starting from an irreducible i/o equation. Obviously, an arbitrary i/o equation is not necessarily in the irreducible form and herein a procedure for the reduction of an i/o equation into an irreducible form is proposed. Such a procedure, however, assumes that it is known how to find the integrating factors and to integrate the one-forms. The reduction problem of nonlinear i/o equation on homogeneous time scale was studied earlier in [4] where the necessary and sufficient condition for irreducibility was formulated in terms of the common left factor of two polynomials, describing the behaviour of the tangent linearized system. The purpose of this paper is to provide an alternative criterion for irreducibility and a reduction procedure in terms of certain subspaces of differential one-forms, defined by the system equation. The final condition is formulated in terms of the same sequence of subspaces appearing in the realizability condition pointed out in [3], the only difference being that now one has to compute more elements in this sequence. However, combining the results presented herein with the results in [3], a unified solution to the minimal realization problem is found.

The paper is organized as follows. In Section II the time scale calculus is presented and the concepts necessary for the following analysis are recalled. Section III describes the algebraic machinery of differential one-forms that is used to obtain the main results of the paper. Sections IV and V describe the problem of reducibility and equivalence of an i/o delta-differential equation. Finally, in Section VI conclusions are drawn.

II. Time-scale calculus

The calculus on time scales was initiated by Aulbach and Hilger [1] in order to create a theory that can unify and extend discrete and continuous analysis. For a general introduction, see [2]. The contents presented in the following two sections are not new and have been previously published (see [3]); nevertheless, as the subject is not commonly known, the first part of the paper has been dedicated to give the reader the necessary theoretical basis needed to understand what follows.

In general, a time scale $\mathbb{T}$ is a non-empty closed subset of the set of real numbers $\mathbb{R}$. This definition includes both the discrete time case, $\mathbb{T} = \mathbb{N}$ and the continuous time case, $\mathbb{T} = \mathbb{R}$. The *forward jump* operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

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$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the *backward jump* operator $\rho(t) : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. In addition, if there exists a finite $\max \mathbb{T}$, $\sigma(\max \mathbb{T}) \triangleq \max \mathbb{T}$ and if there exists a finite $\min \mathbb{T}$, $\rho(\min \mathbb{T}) \triangleq \min \mathbb{T}$. As $\mathbb{T}$ is a closed subset of $\mathbb{R}$, both $\sigma(t) \in \mathbb{T}$ and $\rho(t) \in \mathbb{T}$ when $t \in \mathbb{T}$. Finally, for $t \in \mathbb{T}$, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. A time scale $\mathbb{T}$ is *homogeneous* if μ is constant. In the paper only homogeneous time scales are considered, leaving for future research the extension of the results to a more general $\mathbb{T}$.

Definition 1: Let $f : \mathbb{T} \to \mathbb{R}$ be a function and[1] $t \in \mathbb{T}^\kappa$. The *delta-derivative* of f at t is defined as the number $f^\Delta(t)$ (provided it exists) such that for each $\epsilon > 0$ there exists a neighborhood $\mathcal{U}(\epsilon)$ of t, $\mathcal{U}(\epsilon) \subset \mathbb{T}$ such that for all $s \in \mathcal{U}(\epsilon)$

$$|f[\sigma(t)] - f(s) - f^\Delta(t)[\sigma(t) - s]| \leqslant \epsilon |\sigma(t) - s| \,. \qquad (1)$$

Remark 2: In Definition 1, a maximal left-scattered point is omitted, since for $t \in \mathbb{T} \backslash \mathbb{T}^\kappa$ $f^\Delta(t)$ is not uniquely defined. For such a point t, small neighborhoods $\mathcal{U}$ of t consist only of t and besides we have $\sigma(t) = t$. Therefore (1) holds for an arbitrary number $f^\Delta(t)$.

Proposition 3: For two delta-differentiable functions $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ one has[2]

(i) $f^\sigma = f + \mu f^\Delta$

(ii) $(\alpha f + \beta g)^\Delta = \alpha f^\Delta + \beta g^\Delta$, $\forall\, \alpha, \beta \in \mathbb{R}$

(iii) $(fg)^\Delta = f^\sigma g^\Delta + f^\Delta g$

(iv) if $g g^\sigma \neq 0$, then $\left(\dfrac{f}{g}\right)^\Delta = \dfrac{f^\Delta g - f g^\Delta}{g g^\sigma}$.

Theorem 4: (Chain Rule). Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $g : \mathbb{T} \to \mathbb{R}$ is delta-differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta-differentiable and

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'[g(t) + h\mu(t) g^\Delta(t)] \mathrm{d}h \right\} g^\Delta(t) \,.$$

For a function $f : \mathbb{T} \to \mathbb{R}$ the delta-derivative of its delta-derivative, namely the second-order delta-derivative $\left(f^\Delta\right)^\Delta$, can be defined provided that f^Δ is delta-differentiable on $\mathbb{T}^{\kappa^2} \triangleq (\mathbb{T}^\kappa)^\kappa$. For the sake of simplicity, in the following the notation $f^{[2]}$ is used instead of $\left(f^\Delta\right)^\Delta$ and, in general, the delta-derivative of i-th order is denoted by $f^{[i]}$. Moreover, for $n \geqslant 1$ we define $\mathbf{f}^{[n]}$ as the vector $\mathbf{f}^{[n]} \triangleq (f, f^\Delta, f^{[2]}, \dots, f^{[n]})$.

III. Algebraic framework

We recall now the algebraic formalism for nonlinear control systems defined on homogeneous time scales, see [3], [4], [5].

Let $y : \mathbb{T} \to \mathbb{R}$ and $u : \mathbb{T} \to \mathbb{R}$ be two functions such that y is delta-differentiable up to the order n and there exists the delta-derivative of any order of u. Consider a single-input single-output dynamic system Σ described by a higher order input-output delta-differential equation on a homogeneous time scale $\mathbb{T}$

$$y^{[n]} = \Phi\left(y, \dots, y^{[n-1]}, u, \dots, u^{[s]}\right). \qquad (2)$$

where $u \in \mathbb{R}$ is the input and $y \in \mathcal{Y} \subset \mathbb{R}$ is the output. Assume s and n to be nonnegative integers, $s < n$ and Φ to be a real analytic function defined on $\mathcal{Y} \times \mathbb{R}^{n+s}$.

Define the real analytic function $\varphi : \mathcal{Y} \times \mathbb{R}^{n+s+1} \to \mathbb{R}$ as

$$\varphi\left(y, y^{[1]}, \dots, y^{[n]}, u, u^{[1]}, \dots, u^{[s]}\right) \triangleq$$
$$\triangleq y^{[n]} - \Phi\left(y, \dots, y^{[n-1]}, u, \dots, u^{[s]}\right).$$

Then Equation (2) can be rewritten as

$$\varphi\left(y, y^{[1]}, \dots, y^{[n]}, u, u^{[1]}, \dots, u^{[s]}\right) = 0. \qquad (3)$$

Associate to system Σ the *extended* state-space model Σ_e with input $v = u^{[s+1]}$ and state $\mathbf{z} = (z_1, z_2, \dots, z_{n+s+1})^\top = \left(\mathbf{y}^{[n-1]}, \mathbf{u}^{[s]}\right)^\top$, whose dynamics is defined by

$$\begin{aligned} \mathbf{z}^\Delta &= [z_2, \dots, z_n, \Phi(\mathbf{z}), z_{n+2}, \dots, z_{n+s+1}, v]^\top \\ &\triangleq \mathbf{f}_\mathrm{e}(\mathbf{z}, v). \end{aligned} \qquad (4)$$

Note that (4) is not claimed to be a realization of (2).

Now, consider the infinite set of independent real indeterminates

$$\mathcal{C} = \left\{ z_i, i = 1, \dots, n+s+1, \quad v^{[k]}, \quad k \geqslant 0 \right\}$$

and denote by $\mathcal{K}$ the (commutative) field of meromorphic functions in the system variables $\mathbf{z}$, v and a finite number of the delta-derivatives of v.

Assume that the map $\mathbf{z} \mapsto \tilde{\mathbf{f}}(\mathbf{z}, v) = \mathbf{z} + \mu \mathbf{f}_\mathrm{e}(\mathbf{z}, v)$ generically defines a *submersion*, namely (see [8]) assume that the following condition holds:

$$\mathrm{rank}_\mathcal{K} \frac{\partial \tilde{\mathbf{f}}(\mathbf{z}, v)}{\partial\left(z_1, \dots, z_{n+s+1}, v\right)} = n + s + 1 \,. \qquad (5)$$

Remark 5: One can show that for (5) to hold either

$$1 + \sum_{i=1}^{n} (-1)^{i+1} \mu^i \frac{\partial \Phi}{\partial y^{[n-i]}} \not\equiv 0$$

or

$$\sum_{j=0}^{s} (-1)^j \mu^{j+2} \frac{\partial \Phi}{\partial u^{[s-j]}} \not\equiv 0$$

has to be satisfied.

The operators $\Delta : \mathcal{K} \to \mathcal{K}$ and $\sigma : \mathcal{K} \to \mathcal{K}$ are defined as[3]

$$\Delta\left[F\left(\mathbf{z}, v, \dots, v^{[k]}\right)\right] \triangleq$$
$$\triangleq \int_0^1 \Big\{ \mathrm{grad}\, F\left(\mathbf{z} + h\mu \mathbf{f}_\mathrm{e}(\mathbf{z}, v), v + h\mu v^\Delta, \dots\right.$$
$$\left. \dots, v^{[k]} + h\mu v^{[k+1]}\right) \cdot \begin{bmatrix} \mathbf{f}_\mathrm{e}(\mathbf{z}, v) \\ v^\Delta \\ \vdots \\ v^{[k+1]} \end{bmatrix} \Big\} dh$$

[1] The notation $\mathbb{T}^\kappa$ is used for the set consisting of $\mathbb{T}$ except for a possible left-scattered (i.e. a point t such that $\rho(t) < t$) maximal point.

[2] f^σ is a shortened notation for $f \circ \sigma$.

[3] The notations $\Delta(F)$ and F^Δ equivalently denote the delta-derivative of a meromorphic function. Analogously both $\sigma(F)$ and F^σ denote the operator σ acting on F.

and

$$\sigma\left[F(\mathbf{z},v,\ldots,v^{[k]})\right]\triangleq F\left[\mathbf{z}^{\sigma},v^{\sigma},\ldots,(v^{[k]})^{\sigma}\right],$$

respectively, where $\mathbf{z}^{\sigma}=\mathbf{z}+\mu\mathbf{f}_{\mathrm{e}}(\mathbf{z},v)$ and $\left(v^{[i]}\right)^{\sigma}=v^{[i]}+\mu v^{[i+1]}$, $i=0,\ldots,k$.

The map σ is an endomorphism. If the extended state-space system (4) satisfies (5), the kernel of σ is $\{\mathbf{0}\}$ and the endomorphism σ is well-defined on the field $\mathcal{K}$. For homogeneous time scales, if $F\in\mathcal{K}$ then both $F^{\sigma}\in\mathcal{K}$ and $F^{\Delta}\in\mathcal{K}$.

The operator Δ satisfies:

(i) $(F_1+F_2)^{\Delta}=F_1^{\Delta}+F_2^{\Delta}$, for all $F_1,F_2\in\mathcal{K}$
(ii) $(F_1F_2)^{\Delta}=F_1^{\Delta}F_2+F_1^{\sigma}F_2^{\Delta}$, for all $F_1,F_2\in\mathcal{K}$.

According to (ii), operator Δ satisfies a suitable generalization of the Leibniz rule. An operator satisfying the rule (ii) is called a *σ-derivation* (see, for instance, [9]) while a commutative field endowed with the σ-derivation is called a *σ-differential field*.

The field $\mathcal{K}$ associated to the control system (4) and endowed with the σ-derivation Δ is a σ-differential field. For $\mu=0$, $\sigma=\sigma^{-1}=\mathrm{id}$ and $\mathcal{K}$ is inversive, i.e. every element of $\mathcal{K}$ has a pre-image. However, $\mathcal{K}$ is not inversive in general. Nevertheless, it is always possible to embed $\mathcal{K}$ into an inversive differential overfield $\mathcal{K}^*$, called the *inversive closure* (see [9]) of $\mathcal{K}$. This inversive closure is unique up to an isomorphism. Since σ is an injective endomorphism, it can be extended to $\mathcal{K}^*$ in such a way that $\sigma:\mathcal{K}^*\to\mathcal{K}^*$ is an automorphism.

Hereinafter the inversive closure of differential field $\mathcal{K}$ is assumed to be given and the symbol $\mathcal{K}$ is used to denote both the differential field and its inversive closure.

A. The subspaces of one-forms

Consider the infinite set of symbols

$$\mathrm{d}\mathcal{C}=\left\{\mathrm{d}z_i,\ i=1,\ldots,n+s+1,\quad \mathrm{d}v^{[k]},\quad k\geqslant 0\right\}$$

and denote by $\mathcal{E}$ the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}\mathcal{C}$, namely $\mathcal{E}=\mathrm{span}_{\mathcal{K}}\mathrm{d}\mathcal{C}$. Any element of $\mathcal{E}$ is a vector of the form

$$\omega=\sum_{i=1}^{n+s+1}\alpha_i\mathrm{d}z_i+\sum_{k\geqslant 0}\beta_k\mathrm{d}v^{[k]},$$

where only a finite number of coefficients β_k are nonzero elements of $\mathcal{K}$. A differential operator $\mathrm{d}:\mathcal{K}\to\mathcal{E}$ is defined in the standard manner:

$$\mathrm{d}F\left(\mathbf{z},\mathbf{v}^{[k]}\right)\triangleq\sum_{i=1}^{n+s+1}\frac{\partial F}{\partial z_i}\mathrm{d}z_i+\sum_{j=0}^{k}\frac{\partial F}{\partial v^{[j]}}\mathrm{d}v^{[j]}\,.$$

The elements of $\mathcal{E}$ will be called *one-forms*; $\omega\in\mathcal{E}$ is an *exact one-form* if $\omega=\mathrm{d}F$ for some $F\in\mathcal{K}$. Finally, $\mathrm{d}F$ is referred to as the *total differential* (or simply the *differential*) of F.

The operators $\Delta:\mathcal{K}\to\mathcal{K}$ and $\sigma:\mathcal{K}\to\mathcal{K}$ induce the operators $\Delta:\mathcal{E}\to\mathcal{E}$ and $\sigma:\mathcal{E}\to\mathcal{E}$ by[4]

$$\begin{aligned}\Delta\left(\sum_{i=1}^{n+s+1}\alpha_i\mathrm{d}z_i+\sum_{k\geqslant 0}\beta_k\mathrm{d}v^{[k]}\right)&\triangleq\\ \triangleq\sum_{i=1}^{n+s+1}&\left[\alpha_i^{\Delta}\mathrm{d}z_i+\alpha_i^{\sigma}\mathrm{d}(z_i^{\Delta})\right]+\\ &+\sum_{k\geqslant 0}\left[\beta_k^{\Delta}\mathrm{d}v^{[k]}+\beta_k^{\sigma}\mathrm{d}v^{[k+1]}\right],\end{aligned}\quad(6)$$

$$\begin{aligned}\sigma\left(\sum_{i=1}^{n+s+1}\alpha_i\mathrm{d}z_i+\sum_{k\geqslant 0}\beta_k\mathrm{d}v^{[k]}\right)&\triangleq\\ \triangleq\sum_{i=1}^{n+s+1}&\left[\alpha_i^{\sigma}\mathrm{d}z_i^{\sigma}+\beta_k^{\sigma}\mathrm{d}\left(v^{[k]}\right)^{\sigma}\right]\end{aligned}\quad(7)$$

for $\alpha_i,\ \beta_k\in\mathcal{K}$. For homogeneous time scales the total differential commutes with operators Δ and σ defined by (6) and (7), i.e. $(\mathrm{d}F)^{\Delta}=\mathrm{d}\left(F^{\Delta}\right)$ and $(\mathrm{d}F)^{\sigma}=\mathrm{d}\left(F^{\sigma}\right)$.

Definition 6: The *relative degree* r of a one-form $\omega\in\mathcal{E}$ (with respect to v) is defined to be the least integer such that $\Delta^r(\omega)\notin\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathbf{z}\}$. If such an integer does not exist, define $r=\infty$. The relative degree of a meromorphic function $\varphi(\mathbf{z},v)$ is defined as the relative degree of $\mathrm{d}\varphi(\mathbf{z},v)$.

Denote $\mathrm{d}\mathbf{g}^{[k]}\triangleq\left(\mathrm{d}g,\ldots,\mathrm{d}g^{[k]}\right)$. Introduce the sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ defined by

$$\begin{aligned}\mathcal{H}_0&=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathbf{z},\mathrm{d}v\}=\mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}\mathbf{y}^{[n-1]},\mathrm{d}\mathbf{u}^{[s+1]}\right\}\\ \mathcal{H}_k&=\mathrm{span}_{\mathcal{K}}\left\{\omega\in\mathcal{H}_{k-1}\mid\ \omega^{\Delta}\in\mathcal{H}_{k-1}\right\},\ k\geqslant 1.\end{aligned}\quad(8)$$

It is clear that at the first step, the above induction yields $\mathcal{H}_1=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathbf{z}\}$ and that

$$\mathcal{E}\supset\mathcal{H}_0\supset\ldots\supset\mathcal{H}_{k^*}\supset\mathcal{H}_{k^*+1}=\cdots\triangleq\mathcal{H}_{\infty}\,.\quad(9)$$

The existence of the integer $k^*>0$ comes from the fact that each $\mathcal{H}_k$ is a finite dimensional $\mathcal{K}$-vector space so that, at each step either its dimension decreases or $\mathcal{H}_{k+1}=\mathcal{H}_k$. Moreover [5]$k^*\leqslant n+s+1=\dim_{\mathcal{K}}\mathcal{H}_1$.

Remark 7: From (8) it is obvious that $\mathcal{H}_k$ contains the one-forms whose relative degree is greater than or equal to k. Additionally, $\mathcal{H}_{\infty}$ is the largest subspace of $\mathcal{H}_1$, invariant under σ-differentiation Δ.

The quantities defined so far are now used to prove the following fact.

Lemma 8: The sequence $\{\mathcal{H}_k\}$ is invariant under any diffeomorphism on the state variables.

Proof. The lemma follows directly from Remark 7 and from the fact that the relative degree is invariant under the state diffeomorphism. ■

The following algorithm allows to explicitly construct the bases vectors for the subspaces $\mathcal{H}_k\neq\mathcal{H}_{\infty}$.

[4]Like in the case of functions, the notations $\Delta(\omega)$ and ω^{Δ}, as well as $\sigma(\omega)$ and ω^{σ}, are equivalent.

[5]$\dim_{\mathcal{K}}\mathcal{H}_k$ is the dimension of the space $\mathcal{H}_k$ over the field $\mathcal{K}$.

Step 1. Take $\{\mathrm{d}z_1, \ldots, \mathrm{d}z_{n+s+1}, \mathrm{d}v\}$ and $\{\mathrm{d}z_1, \ldots, \mathrm{d}z_{n+s+1}\}$ as bases of $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

Step k+1. Suppose that $\{\eta_1, \ldots, \eta_{n+s-k+2}, \vartheta\}$ and $\{\eta_1, \ldots, \eta_{n+s-k+2}\}$ are the bases of $\mathcal{H}_{k-1}$ and $\mathcal{H}_k$, respectively, and construct a basis for $\mathcal{H}_{k+1}$ as follows. The elements of $\mathcal{H}_{k+1}$ are the one-forms $\omega \in \mathcal{H}_k$ such that $\omega^\Delta \in \mathcal{H}_k$. In order to compute $\mathcal{H}_k$ explicitly, let

$$\omega = \sum_{j=1}^{n+s-k+2} \lambda_j \eta_j \in \mathcal{H}_k,$$

where $\lambda_j \in \mathcal{K}$. Then by (6)

$$\omega^\Delta = \sum_{j=1}^{n+s-k+2} \left(\lambda_j^\Delta \eta_j + \lambda_j^\sigma \eta_j^\Delta\right) .$$

It is clear that $\omega^\Delta \in \mathcal{H}_k$ if and only if

$$\sum_{j=1}^{n+s-k+2} \lambda_j^\sigma \eta_j^\Delta \in \mathcal{H}_k.$$

Now, since $\eta_j \in \mathcal{H}_k$, η_j^Δ must be in $\mathcal{H}_{k-1}$, so $\sum_j \lambda_j^\sigma \eta_j^\Delta$ may be written in the following form:

$$\sum_{j=1}^{n+s-k+2} \lambda_j^\sigma \eta_j^\Delta = \sum_{j=1}^{n+s-k+2} \lambda_j^\sigma \left(\sum_l \alpha_{lj} \eta_l + \beta_j \vartheta \right).$$

Thus, $\omega^\Delta \in \mathcal{H}_k$ if and only if the coefficients λ_j satisfy the following linear equation

$$\sum_{j=1}^{n+s-k+2} \lambda_j^\sigma \beta_j = 0 \tag{10}$$

This equation has $n+s-k+1$ linearly independent solutions $\boldsymbol{\lambda}_i^\sigma = (\lambda_{i,1}^\sigma, \ldots \lambda_{i,n+s-k+1}^\sigma)^\top$, for $i=1, \ldots, n+s-k+1$. To find $\boldsymbol{\lambda}_i$ we have to apply σ^{-1} which is uniquely determined as σ is an automorphism. Hence, a basis of $\mathcal{H}_{k+1}$ can be computed as

$$\bar{\omega}_i = \sum_{j=1}^{n+s-k+2} \lambda_{i,j} \eta_j, \qquad i = 1, \ldots, n+s-k+1 .$$

where $\lambda_{i,j}$ is the j-th component of the i-th solution $\boldsymbol{\lambda}_i$.

Lemma 9: Let $\{\omega_1, \ldots, \omega_{r_\infty}\}$ be a basis for $\mathcal{H}_\infty$. Then there exists (locally) a basis for $\mathcal{H}_\infty$ composed of exact one-forms, i.e. $\mathcal{H}_\infty$ is integrable.

Proof. For $\mu = 0$ the proof of Lemma 9 is given in [10]. If $\mu \neq 0$, from (i) in Proposition 3 one obtains

$$\sigma(\omega) = \omega + \mu\Delta(\omega). \tag{11}$$

Hence, $\mathcal{H}_k$ in (8) can be alternatively defined as $\mathcal{H}_k = \mathrm{span}_\mathcal{K}\{\omega \in \mathcal{H}_{k-1} \mid \sigma(\omega) \in \mathcal{H}_{k-1}\}$. As a matter of fact, by (11), $\omega \in \mathcal{H}_k$ if and only if $\omega \in \mathcal{H}_{k-1}$ and $\sigma(\omega) \in \mathcal{H}_{k-1}$. Therefore the subspace $\mathcal{H}_\infty$ is invariant both under delta-differentiation and under shift operator σ. Hence in the case $\mu \neq 0$ the integrability of $\mathcal{H}_\infty$ can be deducted from the results in [8]. ■

IV. Irreducibility

Definition 10: A function $\varphi_\mathrm{r} \in \mathcal{K}$ is an *autonomous variable* (see [7], [6]) for (3) (or for (4)) if there exist an integer $\nu \geqslant 1$ and a non-zero meromorphic function G so that $G\left(\varphi_\mathrm{r}, \varphi_\mathrm{r}^\Delta, \ldots, \varphi_\mathrm{r}^{[\nu]}\right) = 0$.

Proposition 11: If function $\varphi_\mathrm{r} \in \mathcal{K}$ is an autonomous variable for (3) (or for (4)), then φ_r has infinite relative degree.

Proof. If a non-constant function φ_r has a finite relative degree, then it is eventually influenced by the input v and by its delta derivatives, and therefore, since $\mathrm{d}v$, $\mathrm{d}v^{[1]}$, $\mathrm{d}v^{[2]}$, …are independent vectors, we have

$$\dim \mathrm{span}_\mathcal{K}\{\mathrm{d}\varphi_\mathrm{r}, \ldots, \mathrm{d}\varphi_\mathrm{r}^{[k-1]}\} = k,$$

for any $k \geqslant 1$. This contradicts Definition 10. ■

Definition 12: If there does not exist any non-zero autonomous variable in $\mathcal{K}$ for (3) and (4), then system (2) is said to be *irreducible* (see [7]) and system (4) is said to be *accessible*. Otherwise system (2) is called *reducible*.

Since the mathematical tools we employ require that instead of working with the equations themselves we work with their differentials, the systems $\varphi(\cdot) = 0$ and $\varphi(\cdot) + c = 0$ are not distinguished for an arbitrary constant c. In order to avoid such situations we fix the constant c and assume it to be defined by the equilibrium point of the system.

If system (3) is reducible (or system (4) is not accessible), then by Definition 12 there exists a non zero autonomous variable $\varphi_\mathrm{r}(\mathbf{z})$. Then, since φ_r has infinite relative degree, $\mathrm{d}\varphi_\mathrm{r} \in \mathcal{H}_\infty$. Consider, now, the system of equations

$$\begin{array}{rcl} \varphi_\mathrm{r}(\mathbf{z}) &=& 0\,, \\ \varphi_\mathrm{r}^\Delta(\mathbf{z}) &=& 0\,, \\ & \vdots & \\ \varphi_\mathrm{r}^{[k]}(\mathbf{z}) &=& 0\,, \end{array} \tag{12}$$

where $k \geqslant \nu$ and ν is some integer greater or equal to 1. The Jacobian matrix over $\mathcal{K}$ of the left hand side of (12), namely

$$\begin{bmatrix} \mathrm{d}\varphi_\mathrm{r}(\mathbf{z}) \\ \mathrm{d}\varphi_\mathrm{r}^\Delta(\mathbf{z}) \\ \vdots \\ \mathrm{d}\varphi_\mathrm{r}^{[k]}(\mathbf{z}) \end{bmatrix},$$

has a limiting rank for some ν. This implies locally that if in the neighborhood of the equilibrium point $\mathbf{z}$ of the system there is no point at which all entries of the Jacobian matrix are zero, then there exist a function G such that

$$G\left(\varphi_\mathrm{r}, \varphi_\mathrm{r}^\Delta, \ldots, \varphi_\mathrm{r}^{[\nu]}\right) = 0.$$

Since also $\varphi(\cdot) = 0$, one gets the following corollary.

Corollary 13: If the system (3) is reducible, then its behaviour can be expressed as

$$\varphi = kG(\varphi_\mathrm{r}, \varphi_\mathrm{r}^\Delta, \ldots, \varphi_\mathrm{r}^{[\nu]}) = 0,$$

where $\varphi_\mathrm{r} = \varphi_\mathrm{r}(y, \ldots, y^{[m]}, u, \ldots, u^{[l]})$, with $m < n$, $l < s$ and $k \neq 0$ is an element of $\mathcal{K}$.

Now the main result of the section can be proven.

Theorem 14: A necessary and sufficient condition for system (3) to be irreducible is that $\mathcal{H}_\infty = \{0\}$ for the extended system (4).

Proof. (*Sufficiency*). We prove by contradiction. Suppose that $\mathcal{H}_\infty = \{0\}$ for the extended system (4) and simultaneously (3) is reducible. Then, by Corollary 13 there exist two non-zero meromorphic functions $\varphi_{\mathrm{r}}(\mathbf{y}^{[m]}, \mathbf{u}^{[l]})$ and $G(\varphi_{\mathrm{r}}, \varphi_{\mathrm{r}}^{\Delta}, \dots, \varphi_{\mathrm{r}}^{[\nu]})$ such that

$$\mathrm{d}\varphi_{\mathrm{r}}^{[\nu]} = \sum_{i=0}^{\nu-1} \alpha_i \mathrm{d}\varphi_{\mathrm{r}}^{[i]}, \tag{13}$$

where $\alpha_i = \dfrac{\dfrac{\partial G}{\partial \varphi_{\mathrm{r}}^{[i]}}}{\dfrac{\partial G}{\partial \varphi_{\mathrm{r}}^{[\nu]}}} \in \mathcal{K}$. Note that $r + \nu = n$, $l + \nu = s$ and φ_{r} depends on $\mathbf{y}^{[m]}$ and $\mathbf{u}^{[l]}$. By (13), φ_{r} has infinite relative degree which implies $\mathrm{d}\varphi_{\mathrm{r}} \in \mathcal{H}_\infty$. Therefore $\mathcal{H}_\infty \neq \{0\}$, what gives rise to a contradiction.

(*Necessity*). Let $\mathcal{A}$ be the set of all autonomous variables for (3); then, for $\varphi_{\mathrm{r}} \in \mathcal{A}$, an one-form $\mathrm{d}\varphi_{\mathrm{r}}$ has infinite relative degree. Therefore, by (8) and since $\mathcal{H}_\infty$ contains one-forms whose relative degree is infinite, $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi_{\mathrm{r}} | \ \varphi_{\mathrm{r}} \in \mathcal{A}\} = \mathcal{H}_\infty$. Hence, if the system is irreducible, then, from Definition 12, $\mathcal{A} = \emptyset$, which implies $\mathcal{H}_\infty = \{0\}$. ∎

Corollary 15: From Definition 12 the irreducibility of system (2) is equivalent to the accessibility of system (4). Therefore the extended system (4) is accessible if and only if $\mathcal{H}_\infty = \{0\}$.

This section is concluded by presenting some examples useful to understand the concepts explained so far. The calculations carried out to find the quantities which are of interest in the examples are simple and hence omitted. In particular, they can be performed with any software for symbolic calculations.

Example 16: Consider the system described by the i/o delta-differential equation

$$\varphi_0 = y^{\Delta} - uy = 0. \tag{14}$$

The extended state-space system associated to (14), with $\mathbf{z} = (y, u)$, has the following form

$$\begin{array}{rcl} z_1^{\Delta} &=& z_1 z_2\,, \\ z_2^{\Delta} &=& v. \end{array} \tag{15}$$

One can compute

$$\begin{array}{rcl} \mathcal{H}_1 &=& \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}u\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_1, \mathrm{d}z_2\}\,, \\ \mathcal{H}_2 &=& \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_1\}\,, \\ \mathcal{H}_k &=& \{0\}\,,\ k \geqslant 3. \end{array}$$

Hence, according to Theorem 14, system (14) is irreducible and (15) is accessible.

Example 17: Consider the system described by the i/o delta-differential equation

$$\varphi_0^{\Delta} + y\varphi_0 = y^{[2]} - y^{\Delta}u - yu^{\Delta} - \mu y^{\Delta}u^{\Delta} + \\ + yy^{\Delta} - uy^2 = 0\,. \tag{16}$$

The extended state-space system associated to (16), with $\mathbf{z} = (y, y^{\Delta}, u, u^{\Delta})$, has the following form

$$\begin{array}{rcl} z_1^{\Delta} &=& z_2\,, \\ z_2^{\Delta} &=& z_2(z_3 - z_1 + \mu z_4) + z_1(z_4 + z_3 z_1)\,, \\ z_3^{\Delta} &=& z_4\,, \\ z_4^{\Delta} &=& v\,. \end{array} \tag{17}$$

One can compute

$$\begin{array}{rcl} \mathcal{H}_1 &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta}, \mathrm{d}u, \mathrm{d}u^{\Delta}\right\} \\ &=& \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3, \mathrm{d}z_4\}\,, \\ \mathcal{H}_2 &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta}, \mathrm{d}u\right\} \\ &=& \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3\}\,, \\ \mathcal{H}_3 &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta} - y\mathrm{d}u\right\} \\ &=& \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_1, \mathrm{d}z_2 - z_1\mathrm{d}z_3\}\,, \\ \mathcal{H}_4 &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y^{\Delta} - y\mathrm{d}u - u\mathrm{d}y\right\} \\ &=& \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z_2 - z_1\mathrm{d}z_3 - z_3\mathrm{d}z_1\}\,, \\ \mathcal{H}_k &=& \{0\}\,,\ \text{for}\ \ k \geqslant 5\,. \end{array}$$

Since $\mathcal{H}_\infty = \{0\}$, system (16) is irreducible and system (17) is accessible.

Example 18: Consider the system described by the i/o delta-differential equation

$$\varphi_0^{\Delta} + \varphi_0 = y^{[2]} - y^{\Delta}u - yu^{\Delta} - \mu y^{\Delta}u^{\Delta} + \\ + y^{\Delta} - uy = 0. \tag{18}$$

One can compute

$$\begin{array}{rcl} \mathcal{H}_1 &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta}, \mathrm{d}u, \mathrm{d}u^{\Delta}\right\}\,, \\ \mathcal{H}_2 &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta}, \mathrm{d}u\right\}\,, \\ \mathcal{H}_3 &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta} - y\mathrm{d}u\right\}\,, \\ \mathcal{H}_k &=& \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}(y^{\Delta} - yu)\right\}\,,\ k \geqslant 4\,. \end{array}$$

Since $\mathcal{H}_\infty = \{\mathrm{d}(y^{\Delta} - yu)\}$, system (18) is reducible.

V. Transfer Equivalence

The notion of irreducibility studied in the previous section is now used to define the notion of transfer equivalence. For, some further definitions are needed.

Definition 19: An exact non-zero one-form $\mathrm{d}\varphi_{\mathrm{r}} \in \mathcal{H}_\infty$ is said to be a *reduced differential form* of system (3).

Definition 20: Consider the input-output system

$$\varphi_{\mathrm{r}}(\cdot) = 0. \tag{19}$$

System (19) is said to be a *reduced form* of system (3) if either $\mathrm{d}\varphi_{\mathrm{r}}$ is a reduced differential form of system (3), or (19) is irreducible.

Definition 21: If system (19) is irreducible, then $\mathrm{d}\varphi_{\mathrm{r}}$ is said to be an *irreducible differential form.*

A. Reduced forms

In this subsection we present an algorithm to possibly reduce a differential form and to find, for a system in the form (3), an irreducible (accessible) realization. This procedure is also helpful, as explained in the final part of the paper, to check the equivalence of two systems.

To begin with, observe that by Corollary 13 the reducibility of system (3) implies that there exists a meromorphic function ψ such that the original equation $\varphi(\cdot) = 0$ can be replaced by the equation $\psi(\cdot) = 0$.

Now, if system (3) is reducible, then, according to Theorem 14, $\mathcal{H}_\infty \neq \{0\}$. Moreover, Lemma 9 guarantees that there exists a basis of exact one-forms, thus one may pick as ψ any non-zero function such that $\mathrm{d}\psi$ belongs to[6] $\mathcal{H}_\infty$. Then $\mathrm{d}\psi$ is a reduced differential form and $\psi(\cdot) = 0$ is a reduced equation of the i/o system (3). Now, the system $\psi(\cdot) = 0$ may be either irreducible or not. We may repeat the reduction procedure for the system $\psi(\cdot) = 0$ provided that it can be solved uniquely (at least locally) for the highest order delta derivative of y, i.e.

$$\begin{aligned} &\psi(y, \ldots, y^{[k]}, u, \ldots, u^{[\ell]}) = \\ &\qquad = y^{[k]} - \Psi(y, \ldots, y^{[k-1]}, u, \ldots, u^{[\ell]}) \end{aligned} \tag{20}$$

and either

$$1 + \sum_{i=1}^{k} (-1)^{i+1} \mu^i \frac{\partial \Psi}{\partial y^{[k-i]}} \not\equiv 0 \tag{21}$$

or

$$\sum_{j=0}^{\ell} (-1)^j \mu^{j+2} \frac{\partial \Psi}{\partial u^{[\ell-j]}} \not\equiv 0 \tag{22}$$

is satisfied. At each step the order of the i/o equation decreases and eventually the reduction procedure converges to an irreducible i/o equation $\varphi_{ir}(\cdot) = 0$ and the form $\mathrm{d}\varphi_{ir}$ is an irreducible differential form of system (3), provided the assumptions (20) and (21) or (22) are satisfied at each intermediate step. If at some step the assumptions are not satisfied we say that i/o equation does not admit an irreducible form.

B. Transfer equivalence

With respect to the subclass of input-output equations that admit an irreducible form, the equivalence relation is defined as follows.

Definition 22: Two systems Σ_1 and Σ_2, which are assumed to admit an irreducible form, are *transfer equivalent* if they have the same irreducible differential form.

Example 23: Consider the i/o delta differential equations (14) and (18). System (14) is irreducible (see Example 16). For system (18), which is reducible (see Example 18), one can define $\mathrm{d}\varphi_r = \mathrm{d}(y^\Delta - yu)$. Hence, according to Definition 22, systems (14) and (18) are equivalent since both have the same irreducible differential form $\mathrm{d}(y^\Delta - yu)$.

Example 24: Compare the two systems described by i/o delta differential equations (14) and (16). They are not equivalent because both are irreducible and their irreducible differential forms are not the same.

VI. Conclusions

The problem of transfer equivalence and reduction of nonlinear delta-differential equations on homogeneous time scale has been addressed. A necessary and sufficient condition for irreducibility is provided in terms of a sequence of subspaces of differential one-forms, associated to control system. The reduction procedure is described in details and the reduced system it provides is accessible and transfer equivalent to the original system. Compared to the condition for irreducibility previously given, our condition matches well with the realizability condition, providing in this way a unified framework to solve the minimal realization problem for i/o delta-differential equation on homogeneous time scale.

Though in this paper we focus on *homogeneous* time scales, which are models of continuous-time systems or uniformly time-sampled (discrete time) systems, one of the future goals is to build a framework that allows to extend the results to the non-homogeneous case. This paper is aimed to be a first step towards this goal. Very recently, the notion of transfer function was generalized for a class of nonlinear systems, that includes also the nonlinear i/o equation on homogeneous time scale (see [11]). Another open problem is to prove that the notion of transfer equivalence introduced in this paper has the same meaning as the equality of transfer functions, like in the linear case. Finally, a further extension of the notion of transfer equivalence and of the reduction procedure should concern the MIMO case.

[6]Finding a basis of $\mathcal{H}_\infty$ may sometimes require finding the integrating factors.

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