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Robust Tracking Control for Switched Linear Systems with Time-Varying Delays

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***Abstract*— Robust tracking control for switched linear systems with time-varying delays is investigated in this paper. Sufficient conditions for the solvability of robust tracking control problem are developed. Average dwell time approach and piecewise Lyapunov functional methods are utilized to the stability analysis and controller design, and with free weighting matrix scheme, switching control laws are obtained such that the weighted H_∞ model reference robust tracking performance is satisfied. By using linear matrix inequalities, the controller design problem can be solved efficiently. A simulation example shows the effectiveness of the proposed switching control laws.**

I. Introduction

As an important class of hybrid systems, switched systems involve both a family of subsystems described by continuous or discrete time dynamics, and a rule specifying the switching among them. Due to their significance both in theory development and practical applications, switched systems have been attracting considerable attention during the last decades, see e.g., [5], [9], [13], [15] and [16]. Two key problems in the study of switched systems are the stability analysis and control synthesis. It has been shown that average dwell time approach is an effective tool for choosing certain switching laws, under which asymptotic (or exponential) stability can be obtained ([5], [6], [15]).

On the other hand, time-delays, which are common phenomenon encountered in many engineering process, are known to be great sources of instability and poor performance. Therefore, how to deal with time delays has been a hot topic in the control area, see e.g., [2], [3], [7] and [10]. For switched systems, because of the complicated behavior caused by the interaction between the continuous dynamics and discrete switching, the problem of time delays is more difficult to study. Only a few results have been reported in the literature such as the issues on stability analysis [12], [15]. As for control synthesis, it is much more difficult than stability analysis for switched uncertainty system with time-delays, to the authors' best knowledge, up to now results on such issues are rarely found. However, due to the complexity of system modelling, uncertainty and time-delays are inevitably considered in most cases. We are interested in the issue of tracking control for switched linear systems with time-varying delays, which has been well addressed for non-switched systems without delay [11]. The importance of the study of robust tracking control for switched systems with time-varying delays also arises from the extensive applications in robot tracking control, guided missile tracking control, etc.

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This paper investigates the robust tracking control problem for switched linear systems with time-varying delays. Sufficient conditions for the solvability of the robust tracking control problem are developed. Here are three features of our results compared with existing results (see e.g., [4], [12]). First of all, weighted H_∞ model reference robust tracking performance is given for the switched systems with time-varying delays, whereas most existing results are concerned with stability analysis; secondly, we use average dwell time technique to design a class of switching laws with the chatter bound $N_0 > 0$, while the existing works mostly aimed at arbitrary switching or $N_0 = 0$; thirdly, free weighting matrix scheme is used to design Lyapunov functional candidate, and the number of constraint conditions is reduced.

II. Problem formulation and preliminaries

In this paper, we use $P > 0$ $(\geq, <, \leq 0)$ to denote a positive definite (semi-definite, negative definite, semi-negative definite) matrix P, and $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of P. The superscript "T" stands for matrix transpose; and the symmetric terms in a matrix are denoted by $*$, $\mathbb{R}^n$ denotes the n dimensional Euclidean space; $L_2[0,\infty)$ is the space of square integrable function on $[0,\infty)$. For given $\tau > 0$, let $\mathbb{R}_+ = [0,+\infty]$ and $C_n = C([-\tau,0],\mathbb{R}^n)$ be the Banach Space of continuous mapping from $([-\tau,0],\mathbb{R}^n)$ to $\mathbb{R}^n$ with topology of uniform convergence. Let $x_t \in C_n$ be defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau,0]$. $\|\cdot\|$ denotes the usual 2-norm and $\|x_t\|_{cl} = \sup_{-\tau\leq\theta\leq 0}\{\|x(t+\theta)\|, \|\dot{x}(t+\theta)\|\}$.

Consider the switched linear uncertain system with time-varying delays

$$\begin{cases} \dot{x}(t) = (A_{\sigma(t)} + \triangle A_{\sigma(t)})x(t) \\ \qquad\quad +(D_{\sigma(t)} + \triangle D_{\sigma(t)})x(t-d_\sigma(t)) \\ \qquad\quad +(B_{\sigma(t)} + \triangle B_{\sigma(t)})u(t) + \Pi_{\sigma(t)}\omega(t), \\ x(t) = \phi(t), t \in [-\tau,0],\ x(0) = \phi(0) = 0, \\ y(t) = C_{\sigma(t)}x(t), \quad t \in [0,\infty), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $\omega(t) \in \mathbb{R}^n$ is the exogenous disturbance which belong to $L_2[0,\infty)$, $y(t) \in \mathbb{R}^q$ is the output, $\phi(t)$ is the continuous

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vector valued function specifying the initial state of the system, $d_i(t)$ denote the continuous time-varying delays satisfying the assumption below.

Assumption 1. $0 < d_i(t) \le \tau,\ i \in \underline{N}$.

The right continuous function $\sigma(t) : [0,\infty) \to \underline{N} \triangleq \{1,2,\cdots,N\}$ is the switching signal, corresponding to it, the switching sequence $\Sigma = \{x_0;(i_0,t_0),(i_1,t_1),\cdots,(i_j,t_j),\cdots|i_j \in \underline{N}\}$ means that the i_jth subsystem is active when $t \in [t_j,t_{j+1})$. For simplicity, we denote $\sigma := \sigma(t)$. A_i, D_i, B_i, C_i and Π_i are constant matrices of appropriate dimensions, $i \in \underline{N}$. The uncertainties $\triangle A_i, \triangle D_i$ and $\triangle B_i,\ i \in \underline{N}$ are assumed to satisfy the following assumption.

Assumption 2. $[\triangle A_i, \triangle D_i, \triangle B_i] = E_i\Gamma_i(t)[F_i, L_i, H_i]$, $i \in \underline{N}$, where E_i, F_i, L_i and H_i are constant matrices with appropriate dimensions, and $\Gamma_i(t),\ i \in \underline{N}$ are unknown, real and possibly time-varying matrices satisfying

$$\Gamma_i^T(t)\Gamma_i(t) \le I,\ t \ge 0.$$

Given the reference model and performance index as

$$\dot{x}_r(t) = A_r x_r(t) + Mr(t), \quad x_r(0) = 0, \tag{2}$$

$$\int_0^\infty e^{-\alpha t} e_r^T(t) Q e_r(t) dt \le \gamma^2 \int_0^\infty \varpi^T(t)\varpi(t) dt, \tag{3}$$

where $x_r(t) \in \mathbb{R}^n$ is reference state, A_r is a Hurwitz matrix and M is a constant matrix with appropriate dimmensins, $r(t)$ is reference input which belong to $L_2[0,\infty)$; $e_r(t) = x(t) - x_r(t)$ denotes the error between the real state of the switched system (1) and the reference state of (2); Q is a positive definite weighting matrix; $\varpi(t) = (\omega^T(t)\Pi_\sigma^T, r^T(t)M^T)^T$, $\gamma > 0$ is disturbance attenuation level.

Combining (1) with (2), we get the augmented system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} (A_\sigma + \triangle A_\sigma)x(t) + (D_\sigma + \triangle D_\sigma)x(t - d_\sigma(t)) \\ +(B_\sigma + \triangle B_\sigma)u(t) \\ A_r x_r(t) \end{bmatrix} + \begin{bmatrix} \Pi_\sigma \omega(t) \\ Mr(t) \end{bmatrix}. \tag{4}$$

Definition 1. The system (1) is said to be robust exponentially stabilizable under control law $u = u(t)$ and switching signal $\sigma = \sigma(t)$, if the solution $x(t)$ of switched system (1) through $(t_0, \phi) \in \mathbb{R}_+ \times C_n$ satisfies

$$\|x(t)\| \le \kappa \|x_{t_0}\|_{cl} e^{-\lambda(t-t_0)},\ \forall t \ge t_0$$

for some constants $\kappa \ge 0$ and $\lambda > 0$.

Definition 2. For system (4), if there exist control input $u = u(t)$ and switching signal $\sigma = \sigma(t)$ such that (4) is robust exponentially stabilizable when $\varpi \equiv 0$ and (3) is satisfied when $\varpi \ne 0$ under the initial conditions stated in (1) and (2), then the switched system (1) is said to have weighted H_∞ model reference robust tracking performance.

Definition 3[5]. For any $T_2 > T_1 \ge 0$, let $N_\sigma(T_1, T_2)$ denote the number of switching of $\sigma(t)$ over (T_1, T_2). If $N_\sigma(T_1,T_2) \le N_0 + \frac{T_2 - T_1}{T_\alpha}$ holds for $T_\alpha > 0, N_0 \ge 0$, then T_α is called average dwell time.

Our purpose is to design robust tracking controller $u(t) = K_{\sigma(t)} e_r(t)$ and a switching law such that system (1) has the weighted H_∞ model reference robust tracking performance.

To conclude this section, we recall the following lemmas which will be used in the proof of our main results.

Lemma 1[1]. Let M, N be real matrices of appropriate dimensions. For any matrix $Q > 0$ of appropriate dimension and any scalar $\gamma > 0$, the following inequality holds

$$MN + N^T M^T \le \gamma^{-1} M Q^{-1} M^T + \gamma N^T Q N.$$

Lemma 2[14]. Given matrices $Q = Q^T, H, E$ and $R = R^T > 0$ of appropriate dimensions, $Q + HFE + E^T F^T H^T < 0$ holds for all F satisfying $F^T F \le R$, if and only if there exists scalar $\beta > 0$ such that

$$Q + \beta H H^T + \beta^{-1} E^T R E < 0.$$

III. Performance analysis and controller design

In this section, we will show how to design state feedback gain K_i and switching law $\sigma(t) = i, (i \in \underline{N})$ for switched time-varying delays system (1), such that the weighted H_∞ model reference robust tracking performance is satisfied. We first consider the nominal system of switched system (1),

$$\begin{cases} \dot{x}(t) = A_\sigma x(t) + D_\sigma x(t - d_\sigma(t)) + B_\sigma u(t) + \Pi_\sigma \omega(t), \\ x(t) = \phi(t),\ t \in [-\tau, 0],\ x(0) = \phi(0) = 0, \\ y(t) = C_\sigma x(t), \quad t \in [0,\infty). \end{cases} \tag{5}$$

In this case, the augmented system (4) can be reduced to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} A_\sigma x(t) + D_\sigma x(t - d_\sigma(t)) + B_\sigma u(t) \\ A_r x_r(t) \end{bmatrix} + \begin{bmatrix} \Pi_\sigma \omega(t) \\ Mr(t) \end{bmatrix}. \tag{6}$$

Consider the ith subsystem with the state feedback controller $u(t) = K_i e_r(t)$. The augmented system (6) can be rewritten as

$$\dot{\bar{x}}(t) = \bar{A}_i \bar{x}(t) + \bar{D}_i \bar{x}(t - d_i(t)) + \varpi(t), \tag{7}$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}, \qquad \bar{A}_i = \begin{bmatrix} A_i + B_i K_i & -B_i K_i \\ 0 & A_r \end{bmatrix},$$
$$\bar{D}_i = \begin{bmatrix} D_i & 0 \\ 0 & 0 \end{bmatrix}, \varpi(t) = \begin{bmatrix} \Pi_\sigma \omega(t) \\ Mr(t) \end{bmatrix}, \bar{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}. \tag{8}$$

Consider the following closed-loop switched linear system with time-varying delays,

$$\dot{\bar{x}}(t) = \bar{A}_\sigma \bar{x}(t) + \bar{D}_\sigma \bar{x}(t - d_\sigma(t)) + \varpi(t). \tag{9}$$

We have the following result.

Theorem 1. Suppose that the augmented system (9) satisfies Assumption 1. For given positive constants α, γ, if there exist positive definite matrices P_i, S_i, matrices K_i, and any matrices Y_i, T_i with appropriate dimensions, such that

$$\Theta_i := \begin{bmatrix} \varphi_{11}^i + \bar{Q} & \varphi_{12}^i & -Y_i & \bar{A}_i^T S_i & P_i \\ * & \varphi_{22}^i & -T_i & \bar{D}_i^T S_i & 0 \\ * & * & -\tau^{-1} e^{-\alpha\tau} S_i & 0 & 0 \\ * & * & * & -\tau^{-1} S_i & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0,$$
$$i \in \underline{N} \tag{10}$$

hold, then under the feedback controller $u(t) = K_\sigma e_r(t)$ for system (6), the weighted H_∞ model reference robust tracking performance in (1) is guaranteed for any switching signal with average dwell time satisfying

$$T_\alpha > T_\alpha^* = \frac{\ln\mu}{\alpha}, \tag{11}$$

where $\mu \geq 1$ satisfies

$$\begin{aligned}
&P_i \leq \mu P_j,\ S_i \leq \mu S_j,\ \forall i,j \in \underline{N}, \\
&\varphi_{11}^i = \bar{A}_i^T P_i + P_i \bar{A}_i + \alpha P_i + Y_i^T + Y_i, \\
&\varphi_{12}^i = P_i \bar{D}_i + T_i^T - Y_i, \\
&\varphi_{22}^i = -T_i^T - T_i.
\end{aligned} \tag{12}$$

Proof. By Schur complement lemma, the conditions (10) are equivalent to the following inequalities when $i \in \underline{N}$

$$\begin{bmatrix} \Omega_{11}^i + \gamma^{-2} P_i P_i + \bar{Q} & \Omega_{12}^i & -Y_i \\ * & \Omega_{22}^i & -T_i \\ * & * & -\tau^{-1} e^{-\alpha\tau} S_i \end{bmatrix} < 0. \tag{13}$$

where

$$\begin{aligned}
\Omega_{11}^i &= \varphi_{11}^i + \tau \bar{A}_i^T S_i \bar{A}_i, \\
\Omega_{12}^i &= \varphi_{12}^i + \tau \bar{A}_i^T S_i \bar{D}_i, \\
\Omega_{22}^i &= \varphi_{22}^i + \tau \bar{D}_i^T S_i \bar{D}_i.
\end{aligned}$$

Multiplying both sides of (13) by symmetric matrix $diag\{I, I, d_i(t)I\}$, and noticing $0 < d_i(t) \leq \tau$, we have

$$\begin{bmatrix} \Omega_{11}^i + \gamma^{-2} P_i P_i + \bar{Q} & \Omega_{12}^i & -d_i(t) Y_i \\ * & \Omega_{22}^i & -d_i(t) T_i \\ * & * & -d_i(t) e^{-\alpha\tau} S_i \end{bmatrix} < 0. \tag{14}$$

Define the piecewise Lyapunov functional candidate

$$\begin{aligned}
V(\bar{x}(t)) =& V_{\sigma(t)}(\bar{x}_t) = \bar{x}^T(t) P_{\sigma(t)} \bar{x}(t) \\
&+ \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(s) e^{-\alpha(t-s)} S_{\sigma(t)} \dot{\bar{x}}(s) ds d\theta,
\end{aligned} \tag{15}$$

which is positive definite since P_i and S_i ($i \in \underline{N}$) are positive definite matrices.

First, we will prove that system (9) is exponentially stable while $\varpi \equiv 0$.

When $t \in [t_k, t_{k+1})$, for the simplicity of notations, suppose that the ith subsystem is active, i.e., $\sigma(t) = i$. Differentiating (15) along the trajectory of (9) and noticing $d_i(t) \leq \tau$, we obtain

$$\begin{aligned}
\dot{V}_i(\bar{x}_t) \leq& 2\bar{x}^T(t) P_i (\bar{A}_i \bar{x}(t) + \bar{D}_i \bar{x}(t - d_i(t))) \\
&+ \tau \dot{\bar{x}}^T(t) S_i \dot{\bar{x}}(t) - \int_{t-d_i(t)}^{t} \dot{\bar{x}}^T(s) e^{-\alpha\tau} S_i \dot{\bar{x}}(s) ds \\
&- \alpha \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(s) e^{-\alpha(t-s)} S_i \dot{\bar{x}}(s) ds d\theta.
\end{aligned} \tag{16}$$

Note that

$$\begin{aligned}
\tau \dot{\bar{x}}^T(t) S_i \dot{\bar{x}}(t) =& \bar{x}^T(t) \tau \bar{A}_i^T S_i \bar{A}_i \bar{x}(t) \\
&+ 2\bar{x}^T(t) \tau \bar{A}_i^T S_i \bar{D}_i \bar{x}(t - d_i(t)) \\
&+ \bar{x}^T(t - d_i(t)) \tau \bar{D}_i^T S_i \bar{D}_i \bar{x}(t - d_i(t)).
\end{aligned} \tag{17}$$

From the Leibniz-Newton formula, we obtain

$$\begin{aligned}
&2[\bar{x}^T(t), \bar{x}^T(t - d_i(t))] \begin{bmatrix} Y_i \\ T_i \end{bmatrix} \\
&\times [\bar{x}(t) - \bar{x}(t - d_i(t)) - \int_{t-d_i(t)}^{t} \dot{\bar{x}}(s) ds] = 0.
\end{aligned} \tag{18}$$

Substituting (17) and (18) into (16) yields

$$\begin{aligned}
&\dot{V}_i(\bar{x}_t) + \alpha V_i(\bar{x}_t) \\
&\leq \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t - d_i(t)) \end{bmatrix}^T \begin{bmatrix} \Omega_{11}^i & \Omega_{12}^i \\ * & \Omega_{22}^i \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t - d_i(t)) \end{bmatrix} \\
&- 2\left[\bar{x}^T(t) Y_i + \bar{x}^T(t - d_i(t)) T_i\right] \int_{t-d_i(t)}^{t} \dot{\bar{x}}(s) ds \\
&- \int_{t-d_i(t)}^{t} \dot{\bar{x}}^T(s) e^{-\alpha\tau} S_i \dot{\bar{x}}(s) ds.
\end{aligned}$$

Let $\xi(t,s) = \left[\bar{x}^T(t) \bar{x}^T(t - d_i(t)) \dot{\bar{x}}^T(s)\right]^T$. We have

$$\begin{aligned}
&\dot{V}_i(\bar{x}_t) + \alpha V_i(\bar{x}_t) \leq \frac{1}{d_i(t)} \times \\
&\int_{t-d_i(t)}^{t} \xi^T(t,s) \begin{bmatrix} \Omega_{11}^i & \Omega_{12}^i & -d_i(t) Y_i \\ * & \Omega_{22}^i & -d_i(t) T_i \\ * & * & -d_i(t) e^{-\alpha\tau} S_i \end{bmatrix} \xi(t,s) ds.
\end{aligned}$$

Taking (14) into account, we get

$$\begin{aligned}
&\dot{V}_i(\bar{x}_t) + \alpha V_i(\bar{x}_t) < \frac{1}{d_i(t)} \times \\
&\int_{t-d_i(t)}^{t} \xi^T(t,s) diag\{-\bar{Q} - \gamma^{-2} P_i P_i, 0, 0\} \xi(t,s) ds.
\end{aligned}$$

Noticing that $\bar{Q} \geq 0$ and $\gamma^{-2} P_i P_i > 0$, we can obtain

$$\dot{V}_i(\bar{x}_t) + \alpha V_i(\bar{x}_t) < 0, \quad i \in \underline{N}. \tag{19}$$

When $t \in [t_k, t_{k+1})$, integrating (19) from t_k to t gives

$$V(\bar{x}_t) = V_{\sigma(t)}(\bar{x}_t) \leq e^{-\alpha(t-t_k)} V_{\sigma(t_k)}(\bar{x}_{t_k}). \tag{20}$$

Using (12) and (15), at the switching instant t_i, we have

$$V_{\sigma(t_i)}(\bar{x}_{t_i}) \leq \mu V_{\sigma(t_i^-)}(\bar{x}_{t_i^-}),\ i = 1, 2, \cdots. \tag{21}$$

Therefore, it follows from (20), (21) and the relation $k = N_\sigma(t_0, t) \leq N_0 + \frac{t - t_0}{T_\alpha}$, noticing $N_0 > 0$, that

$$\begin{aligned}
V(\bar{x}_t) \leq& e^{-\alpha(t-t_k)} \mu V_{\sigma(t_k^-)}(\bar{x}_{t_k^-}) \\
\leq& e^{-\alpha(t-t_{k-1})} \mu V_{\sigma(t_{k-1})}(\bar{x}_{t_{k-1}}) \leq \cdots \\
\leq& e^{-\alpha(t-t_0)} \mu^k V_{\sigma(t_0)}(\bar{x}_{t_0}) \\
\leq& \mu^{N_0} \cdot e^{-(\alpha - \frac{\ln\mu}{T_\alpha})(t-t_0)} V_{\sigma(t_0)}(\bar{x}_{t_0}).
\end{aligned} \tag{22}$$

In view of (15) again, it holds that

$$a\|\bar{x}(t)\|^2 \leq V(\bar{x}_t),\ V_{\sigma(t_0)}(\bar{x}_{t_0}) \leq b\|\bar{x}_{t_0}\|_{cl}^2, \tag{23}$$

where $a = \min_{i \in \underline{N}} \lambda_{\min}(P_i)$, $b = \max_{i \in \underline{N}} \lambda_{\max}(P_i) + \tau^2 \max_{i \in \underline{N}} \lambda_{\max}(S_i)$.

Let $\lambda = \frac{1}{2}(\alpha - \frac{\ln\mu}{T_\alpha})$. Combining (22) and (23) gives rise to

$$\|\bar{x}(t)\|^2 \leq \frac{1}{a} V(\bar{x}_t) \leq \frac{b}{a} \mu^{N_0} \cdot e^{-(\alpha - \frac{\ln\mu}{T_\alpha})(t-t_0)} \|\bar{x}_{t_0}\|_{cl}^2.$$

Therefore, $\|\bar{x}(t)\| \leq \sqrt{\frac{b}{a}}\mu^{\frac{N_0}{2}} \cdot e^{-\lambda(t-t_0)}\|\bar{x}_{t_0}\|_{cl}$, which means that system (9) is exponentially stable with $\varpi \equiv 0$.

Next, we will show under the zero initial condition with $\varpi \neq 0$ that $\int_0^{\infty} e^{-\alpha t} e_r^T(t) Q e_r(t) dt \leq \gamma^2 \int_0^{\infty} \varpi^T(t)\varpi(t)dt$.

Suppose that $t \in [t_k, t_{k+1})$, and the ith subsystem is active. Differentiating the Lyapunov functional candidate along the trajectory $\bar{x}(t)$ of system (9), we can easily get

$$\begin{aligned}&\dot{V}_i(\bar{x}_t) + \alpha V_i(\bar{x}_t)\\ \leq& \frac{1}{d_i(t)}\int_{t-d_i(t)}^{t} \xi^T(t,s)\begin{bmatrix}\Omega_{11}^i & \Omega_{12}^i & -d_i(t)Y_i\\ * & \Omega_{22}^i & -d_i(t)T_i\\ * & * & -d_i(t)e^{-\alpha\tau}S_i\end{bmatrix}\xi(t,s)ds\\ &+ 2\bar{x}^T(t)P_i\varpi(t). \end{aligned} \tag{24}$$

Applying Lemma 1 gives

$$2\bar{x}^T(t)P_i\varpi(t) \leq \gamma^{-2}\bar{x}^T(t)P_iP_i\bar{x}(t) + \gamma^2\varpi^T(t)\varpi(t). \tag{25}$$

Substituting (25) into (24) yields

$$\begin{aligned}&\dot{V}_i(\bar{x}_t) + \alpha V_i(\bar{x}_t) \leq \frac{1}{d_i(t)}\int_{t-d_i(t)}^{t}\xi^T(t,s)\\ &\times\begin{bmatrix}\Omega_{11}^i + \gamma^{-2}P_iP_i & \Omega_{12}^i & -d_i(t)Y_i\\ * & \Omega_{22}^i & -d_i(t)T_i\\ * & * & -d_i(t)e^{-\alpha\tau}S_i\end{bmatrix}\xi(t,s)ds\\ &+\gamma^2\varpi^T(t)\varpi(t). \end{aligned} \tag{26}$$

Taking (14) into account again, and noticing the structure of $\bar{Q}$, we obtain

$$\begin{aligned}&\dot{V}_i(\bar{x}_t) + \alpha V_i(\bar{x}_t)\\ <& \frac{1}{d_i(t)}\int_{t-d_i(t)}^{t}\xi^T(t,s)\begin{bmatrix}-\bar{Q} & 0 & 0\\ * & 0 & 0\\ * & * & 0\end{bmatrix}\xi(t,s)ds + \gamma^2\varpi^T(t)\varpi(t)\\ =& -e_r^T(t)Qe_r(t) + \gamma^2\varpi^T(t)\varpi(t). \end{aligned} \tag{27}$$

Let $\Gamma(t) = e_r^T(t)Qe_r(t) - \gamma^2\varpi^T(t)\varpi(t)$. According to the theory of the first order linear nonhomogeneous differential inequality, for any $t \in [t_k, t_{k+1})$, we have

$$V(\bar{x}_t) \leq e^{-\alpha(t-t_k)}V_{\sigma(t_k)}(\bar{x}_{t_k}) - \int_{t_k}^{t} e^{-\alpha(t-s)}\Gamma(s)ds, \tag{28}$$

It follows from (12) that

$$V(\bar{x}_{t_i}) \leq \mu V_{\sigma(t_i^-)}(\bar{x}_{t_i^-}), \quad i = 1, 2, \cdots. \tag{29}$$

Let $t_0 = 0$, combining (28) and (29) gives rise to

$$\begin{aligned}V(\bar{x}_t) \leq& \mu V_{\sigma(t_k^-)}(\bar{x}_{t_k^-})e^{-\alpha(t-t_k)} - \int_{t_k}^{t} e^{-\alpha(t-s)}\Gamma(s)ds\\ \leq& \mu^k V_{\sigma(t_0)}(\bar{x}_0)e^{-\alpha t} - \mu^k\int_0^{t_1} e^{-\alpha(t-s)}\Gamma(s)ds\\ &- \mu^{k-1}\int_{t_1}^{t_2} e^{-\alpha(t-s)}\Gamma(s)ds\\ &- \cdots - \int_{t_k}^{t} e^{-\alpha(t-s)}\Gamma(s)ds\\ =& e^{-\alpha t + N_\sigma(0,t)\ln\mu}V(\bar{x}_0)\\ &- \int_0^t e^{-\alpha(t-s)+N_\sigma(s,t)\ln\mu}\Gamma(s)ds. \end{aligned} \tag{30}$$

Under the zero initial condition, (30) becomes

$$0 \leq -\int_0^t e^{-\alpha(t-s)+N_\sigma(s,t)\ln\mu}\Gamma(s)ds. \tag{31}$$

Multiplying both sides of (31) by $e^{-N_\sigma(0,t)\ln\mu}$ yields

$$\begin{aligned}&\int_0^t e^{-\alpha(t-s)-N_\sigma(0,s)\ln\mu}e_r^T(s)Qe_r(s)ds\\ \leq& \int_0^t e^{-\alpha(t-s)-N_\sigma(0,s)\ln\mu}\gamma^2\varpi^T(s)\varpi(s)ds. \end{aligned} \tag{32}$$

Note that $N_\sigma(0,s) \leq N_0 + \frac{s}{T_\alpha}$, $N_0 > 0$ and $T_\alpha > \frac{\ln\mu}{\alpha}$, we have

$$N_\sigma(0,s)\ln\mu \leq N_0\ln\mu + \alpha s. \tag{33}$$

Therefore, it follows from (32) and (33) that

$$\int_0^t e^{-\alpha t}e_r^T(s)Qe_r(s)ds < \int_0^t e^{-\alpha(t-s)}\gamma^2\varpi^T(s)\varpi(s)ds. \tag{34}$$

Integrating both sides of (34) from 0 to ∞ results in

$$\int_0^{\infty} e^{-\alpha s}e_r^T(s)Qe_r(s)ds \leq \int_0^{\infty}\gamma^2\varpi^T(s)\varpi(s)ds.$$

This completes the proof.

Remark 1. When $\mu = 1$, we have $T_\alpha^* = 0$, which implies that the switching signal can be arbitrary. Note that $N_0 = 0$ corresponds to the case of no switching on any interval of length smaller than T_α, this case degenerates into dwell time case, that is, if we discard the first N_0 switches, then the average time between consecutive switches is at least T_α (cf. [5], [8]). Therefore, our adopting $N_0 > 0$ in this paper is more general and natural.

Now, we will design robust tracking controllers.

Consider the ith subsystem with state feedback controller of the form $u(t) = K_ie_r(t)$. Augmenting system (4) we have

$$\dot{\bar{x}}(t) = (\bar{A}_i + \triangle\bar{A}_i)\bar{x}(t) + (\bar{D}_i + \triangle\bar{D}_i)\bar{x}(t - d_i(t)) + \varpi(t),$$

where $\bar{x}(t), \bar{A}_i, \bar{D}_i$ and $\varpi(t)$ are defined in (8), and

$$\triangle\bar{A}_i = \begin{bmatrix}\triangle A_i + \triangle B_iK_i & -\triangle B_iK_i\\ 0 & 0\end{bmatrix}, \triangle\bar{D}_i = \begin{bmatrix}\triangle D_i & 0\\ 0 & 0\end{bmatrix}.$$

We adopt the following notations

$$\bar{E}_i = \begin{bmatrix}E_i & 0\\ 0 & 0\end{bmatrix}, \bar{F}_i = \begin{bmatrix}F_i + H_iK_i & -H_iK_i\\ 0 & 0\end{bmatrix},$$

$$\bar{\Gamma}_i(t) = \begin{bmatrix}\Gamma_i(t) & 0\\ 0 & 0\end{bmatrix}, \bar{L}_i = \begin{bmatrix}L_i & 0\\ 0 & 0\end{bmatrix}, \bar{H}_i = \begin{bmatrix}H_i\\ 0\end{bmatrix}.$$

A simple calculation shows

$$\left[\triangle\bar{A}_i, \triangle\bar{D}_i, \triangle\bar{B}_i\right] = \bar{E}_i\bar{\Gamma}_i(t)\left[\bar{F}_i, \bar{L}_i, \bar{H}_i\right],\ i \in \underline{N}$$

and

$$\bar{\Gamma}_i^T(t)\bar{\Gamma}_i(t) \leq I,$$

which means Assumption 2 is satisfied.

Consider the closed loop switched linear uncertain system with time-varying delays,

$$\dot{\bar{x}}(t) = (\bar{A}_\sigma + \triangle\bar{A}_\sigma)\bar{x}(t) + (\bar{D}_\sigma + \triangle\bar{D}_\sigma)\bar{x}(t - d_\sigma(t)) + \varpi(t). \tag{35}$$

$$\begin{bmatrix} \varphi_{11}^i+\bar{Q} & \varphi_{12}^i+\beta^{-1}\bar{F}_i^T\bar{L}_i & -Y_i & \bar{A}_i^T S_i+\beta P_i\bar{E}_i\bar{E}_i^T & P_i & P_i\bar{E}_i & \bar{F}_i \\ * & \varphi_{22}^i+\beta^{-1}\bar{L}_i^T\bar{L}_i & -T_i & \bar{D}_i^T S_i & 0 & 0 & 0 \\ * & * & -\tau^{-1}e^{-\alpha\tau}S_i & 0 & 0 & 0 & 0 \\ * & * & * & -\tau^{-1}S_i+\beta\bar{E}_i\bar{E}_i^T & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & -\beta^{-1}I & 0 \\ * & * & * & * & * & 0 & -\beta I \end{bmatrix} < 0,\ i\in\underline{N} \tag{38}$$

We have the following result.

Theorem 2. Suppose that the augmented system (35) satisfies Assumption 1-2. For given positive constants α, γ, if there exist scalar $\beta > 0$, positive definite matrices P_i, S_i, matrices K_i, and any matrices Y_i, T_i with appropriate dimensions, such that (38) hold, then under the feedback controller $u(t) = K_\sigma e_r(t)$ for system (4), the weighted H_∞ model reference robust tracking performance in (1) is guaranteed for any switching signal with average dwell time satisfying

$$T_\alpha > T_\alpha^* = \frac{\ln\mu}{\alpha}, \tag{37}$$

where $\mu \geq 1$ satisfies

$$\begin{aligned} P_i &\leq \mu P_j, \quad S_i \leq \mu S_j, \quad \forall i,j\in\underline{N}, \\ \varphi_{11}^i &= \bar{A}_i^T P_i + P_i\bar{A}_i + \alpha P_i + Y_i^T + Y_i, \\ \varphi_{12}^i &= P_i\bar{D}_i + T_i^T - Y_i, \\ \varphi_{22}^i &= -T_i^T - T_i. \end{aligned} \tag{38}$$

Proof. Define the piecewise Lyapunov functional candidate

$$\begin{aligned} V(\bar{x}(t)) =& V_{\sigma(t)}(\bar{x}(t)) = \bar{x}^T(t)P_{\sigma(t)}\bar{x}(t) \\ &+ \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\bar{x}}^T(s)e^{-\alpha(t-s)}S_{\sigma(t)}\dot{\bar{x}}(s)dsd\theta, \end{aligned} \tag{39}$$

and let $\hat{A}_i = \bar{A}_i + \triangle\bar{A}_i, \hat{D}_i = \bar{D}_i + \triangle\bar{D}_i$. The result follows from Theorem 1 if it holds that

$$\begin{bmatrix} \hat{\varphi}_{11}^i+\bar{Q} & \hat{\varphi}_{12}^i & -Y_i & \hat{A}_i^T S_i & P_i \\ * & \hat{\varphi}_{22}^i & -T_i & \hat{D}_i^T S_i & 0 \\ * & * & -\tau^{-1}e^{-\alpha\tau}S_i & 0 & 0 \\ * & * & * & -\tau^{-1}S_i & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0,\ i\in\underline{N}, \tag{40}$$

where

$$\begin{aligned} \hat{\varphi}_{11}^i &= \hat{A}_i^T P_i + P_i\hat{A}_i + \alpha P_i + Y_i^T + Y_i, \\ \hat{\varphi}_{12}^i &= P_i\hat{D}_i + T_i^T - Y_i, \\ \hat{\varphi}_{22}^i &= -T_i^T - T_i. \end{aligned}$$

Now, we show that (40) hold. We rewrite (40) as follows

$$\begin{aligned} &\Theta_i + \begin{bmatrix} \triangle\bar{A}_i^T P_i + P_i\triangle\bar{A}_i & P_i\triangle\bar{D}_i & 0 & \triangle\bar{A}_i^T & 0 \\ * & 0 & 0 & \triangle\bar{D}_i^T & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} \\ &= \Theta_i + \begin{bmatrix} P_i\bar{E}_i \\ 0 \\ 0 \\ \bar{E}_i \\ 0 \end{bmatrix} \Gamma_i(t) \begin{bmatrix} \bar{F}_i \bar{L}_i\, 0\, 0\, 0 \end{bmatrix} \\ &+ \begin{bmatrix} \bar{F}_i^T \\ \bar{L}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \Gamma_i^T(t) \begin{bmatrix} \bar{E}_i^T P_i\, 0\, 0\, \bar{E}_i^T\, 0 \end{bmatrix} < 0,\ i\in\underline{N}, \end{aligned} \tag{41}$$

where Θ_i is defined in (10). Lemma 2 indicates that (41) hold if there exists a positive number β, such that the following inequalities hold

$$\begin{aligned} &\Theta_i + \beta \begin{bmatrix} P_i\bar{E}_i \\ 0 \\ 0 \\ \bar{E}_i \\ 0 \end{bmatrix} \begin{bmatrix} \bar{E}_i^T P_i\ 0\ 0\ \bar{E}_i^T\ 0 \end{bmatrix} \\ &+\beta^{-1} \begin{bmatrix} \bar{F}_i^T \\ \bar{L}_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{F}_i\ \bar{L}_i\, 0\, 0\, 0 \end{bmatrix} < 0,\ i\in\underline{N}. \end{aligned} \tag{42}$$

It is easy to show that (42) are equivalent to (38), consequently, (40) hold and the proof is end.

Remark 2. Due to the Lyapunov functional candidate we designed does not include the time-varying delays terms $d_i(t) (i\in\underline{N})$, the number of constraint conditions is reduced compared to existing results (e.g., [4], [12]), which in turn reduces the difficulties of designing switching control laws.

IV. Numerical example

Consider the switched linear uncertain system (1) with time varying delays and the reference system (2) with

$$A_1 = \begin{bmatrix} -4 & -2.5 \\ 1.2 & -1.5 \end{bmatrix},\quad D_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0 \end{bmatrix},\quad B_1 = \begin{bmatrix} -0.1 \\ -0.3 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} -2 & 0.5 \\ -3.2 & -3.5 \end{bmatrix},\quad D_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0 \end{bmatrix},\quad B_2 = \begin{bmatrix} -0.1 \\ 0.7 \end{bmatrix};$$

$$A_r = \begin{bmatrix} -4.5 & -1.5 \\ 1.2 & -1.5 \end{bmatrix},\ F_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix},\ F_2 = \begin{bmatrix} 0.6 & 0 \\ 0 & -0.1 \end{bmatrix};$$

$$E_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.5 \end{bmatrix},\quad L_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},\quad H_1 = \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix};$$

$$E_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.1 \end{bmatrix},\quad L_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.4 \end{bmatrix},\quad H_2 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix};$$

$$\Pi_1 = \Pi_2 = M = diag\{1,1\};$$

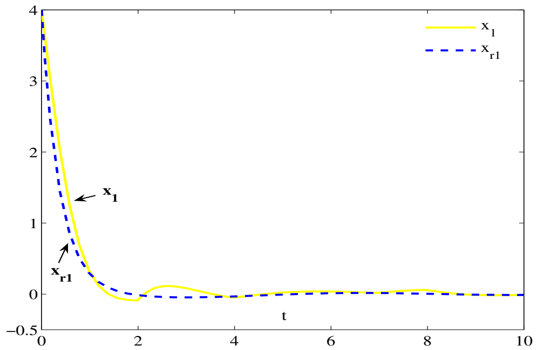

Fig. 1. State x_1 tracking the reference state x_{r1}.

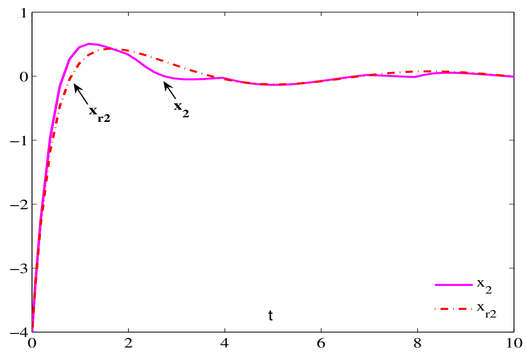

Fig. 2. State x_2 tracking the reference state x_{r2}.

and $d_\sigma(t) = 0.6 + 0.6 \sin t$. For $\alpha = 0.6,\ \tau = 1.2$, solving (38) gives piecewise Lyapunov functional (39) with

$$P_1 = \begin{bmatrix} \hat{P}_1^{-1} & 0 \\ 0 & \hat{P}_1^{-1} \end{bmatrix}, P_2 = \begin{bmatrix} \hat{P}_2^{-1} & 0 \\ 0 & \hat{P}_2^{-1} \end{bmatrix},$$

$$S_1 = \begin{bmatrix} \hat{S}_1 & 0 \\ 0 & \hat{S}_1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} \hat{S}_2 & 0 \\ 0 & \hat{S}_2 \end{bmatrix},$$

where $\hat{P}_1 = \begin{bmatrix} 0.1542 & -0.0153 \\ -0.0153 & 0.1494 \end{bmatrix}$, $\hat{P}_2 = \begin{bmatrix} 0.0897 & -0.0378 \\ -0.0378 & 0.0815 \end{bmatrix}$, $\hat{S}_1 = \begin{bmatrix} 2.68160.0027 \\ 0.00272.6775 \end{bmatrix}$, $\hat{S}_2 = \begin{bmatrix} 2.7482 & -0.0531 \\ -0.0531 & 2.7368 \end{bmatrix}$. Consequently, the controller gains are given as $K_1 = \begin{bmatrix} 0.52951.0927 \end{bmatrix}$ and $K_2 = \begin{bmatrix} 4.02921.8681 \end{bmatrix}$. Solving (38) gives $\mu = 2.8645$, and according (37), we have $\tau_\alpha^* = \frac{\ln \mu}{\alpha} = 1.7540$. By using average dwell time method provided by Theorem 1 and 2, we obtained that system (1) is with the weighted H_∞ model reference robust tracking performance, the simulation results are depicted in Fig.1-Fig.3.

V. Conclusions

In this paper, we have investigated the robust tracking control problem for switched linear uncertain systems with time-varying delays. Sufficient conditions for the solvability of the robust tracking control problems are developed. Weighted H_∞ model reference robust tracking performance is given for the switched systems with time-varying delays; with average dwell time technique, satisfactory tracking control results are obtained. Meanwhile, by using free weighting matrix scheme, the conservativeness of designing switching control laws is reduced. Consequently, the difficulties of the stability analysis and control synthesis of switched system with time-varying delays are substantially reduced by relaxing some constrain conditions compared with the existing results.

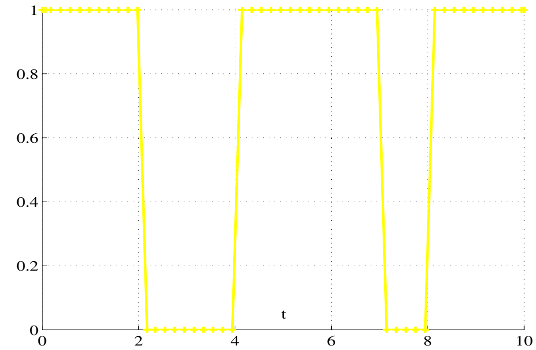

Fig. 3. Switching signal with average dwell time.

References

[1] Y. Cao, Y. Sun and C. Cheng, "Delay dependent robust stabilization of uncertain systems with multiple state delays," *IEEE Trans. Automat. Contr.*, vol.43, no.11, pp.1608-1612, 1998.

[2] E. Fridman and U. Shaked, "An improved stabilization method for linear time-delay systems," *IEEE Trans. Automat. Contr.*, vol.47, no.11, pp.1931-1937, 2002.

[3] Q. Han and K. Gu, "On robust stability of time delay systems with norm-bounded uncertainty," *IEEE Trans. Automat. Contr.*, vol.46, no.9, pp.1426-1431, 2001.

[4] Y. He, M. Wu, J. She and G. Liu, "Parameter-dependent Lyapunov functional for stability of time-delay systems with polytopic-type uncertainties". *IEEE Trans. Automat. Contr.*, vol.49,no.5, pp.828-832, 2004b.

[5] J. P. Hespanha, A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and control*, Phoenix, AZ, 1999, (pp.2655-2660).

[6] H. Ishii, B. A. Francis, "Stabilizing a linear system by switching control with dwell-time", *IEEE Trans. Automat. Contr.*, vol.47, no.12, pp.1962-1973, 2002.

[7] V. B. Kolmanovskii and J. P. Richard, "Stability of some linear system with delay," *IEEE Trans. Automat. Contr.*, vol.44, no.5, pp.984-989, 1999.

[8] D. Liberzon, *Switching in Systems and Control*. Birkhauser, Boston, 2003.

[9] D. Liberzon and A. S. Morse, "Basic problem in stability and design of switched systems," *IEEE Contr Syst Mag.*, vol. 19, no.5, pp.59-70, 1999.

[10] J. P. Richard, "Time-delay systems: on overview of some recent advances and open problems," *Automatica*, vol. 39, no.10, pp.1667-1694, 2003.

[11] W. E. Schmitendorf and B. R. Barmish, "Robust asymptotic tracking for linear systems with unknown parameters," *Automatica*, vol. 22, no.3, pp.355-360, 1986.

[12] X. M. Sun, J. Zhao and D. J. Hill, "Stability and L_2-gain analysis for switched delay systems: a delay-dependent method," *Automatica*, vol. 42, no.5, pp.1769-1774, 2006.

[13] Z. D. Sun and S. S. Ge, "Analysis and synthesis of switched linear control systems," *Automatica*, vol. 41, no.2, pp.181-195, 2005.

[14] L. Xie, "Output feedback H_∞ control of systems with parameter uncertainty," *International Journal of Control*, vol. 63, pp.741-750, 1996.

[15] G. S. Zhai, B. Hu, K. Yasuda and A. Michel, "Stability analysis of switched delayed systems with stable and unstable subsystems: An average dwell time approach," in *Proceedings of the American Control Conference*, Chicago, 2000, (pp.200-204).

[16] J. Zhao and D. J. Hill, "On stability, L_2 gain and H_∞ control for switched systems," *Automatica*, to appear, 2008.