2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Robust H_∞ Control for a Class of Uncertain Switched Nonlinear Systems using Constructive Approach

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Abstract— **This paper deals with the problem of robust H_∞ control for a class of uncertain switched nonlinear systems via the multiple Lyapunov functions approach. Based on the explicit construction of Lyapunov functions, which avoids solving the Hamilton-Jacobi-Isaacs (HJI) inequalities, a condition for the solvability of the robust H_∞ control problem and design of both switching laws and controllers is presented. As an application, a hybrid state feedback strategy is proposed to solve the standard robust H_∞ control problem for nonlinear systems when no single continuous controller is effective.**

I. INTRODUCTION

In recent years, switched control systems have drawn considerable attention in the control community due to their theoretical significance and practical applications [1-6]. The motivation for studying switched systems arises from the fact that many systems encountered in practice, such as mechanical systems, the automotive industry, switching power converters and many other fields, exhibit switching between several subsystems depending on various environmental factors and from the fact that the methods of intelligent control design are mainly based on the idea of controller switching. Loosely speaking, a switched system belongs to an important class of hybrid systems, which consists of a family of continuous-time subsystems and a rule that specifies the switching among them. The interaction between continuous dynamics and discrete dynamics makes the behavior of switched systems much complicated. For instance, switching between stable subsystems may lead to instability. Regarding design of switched systems, the multiple Lyapunov function approach has been proven to be a powerful and effective tool [7,8].

On the other hand, H_∞ control problem for nonlinear systems has been extensively explored and results rely heavily on the solution of HJI inequalities [9,10]. However, so far no effective numerical methods are available for solving HJI inequalities. This motivated some attempts to look for methods for nonlinear systems which do not require solving HJI inequalities [11,12]. H_∞ control problem has been rarely addressed for switched systems, especially for the nonlinear case in which results mainly focus on special structures. Such a problem has been considered in [13] by using dwell time approach incorporated with a piecewise Lyapunov function for both switched linear and nonlinear systems. The problem of H_∞ control for switched nonlinear systems is addressed in [14] via the multiple Lyapunov functions approach. The aforementioned results are dependent on HJI inequalities. In [15], H_∞ control problem for a class of cascade switched nonlinear system is concerned by using common Lyapunov function and single Lyapunov function approach respectively and results do not rely on HJI inequalities.

In this paper, the robust H_∞ control problem for a class of uncertain switched nonlinear systems is considered. It is assumed that each subsystem of the switched system to be controlled is globally asymptotically stable with a known Lyapunov function. Based on this Lyapunov function and the output function of each subsystem, a new Lyapunov function candidate is constructed. By using the multiple Lyapunov functions technique, a sufficient condition for the switched nonlinear systems to be asymptotically stable with H_∞-norm bound and design of both switching laws and controllers is derived for all admissible uncertainties. Then, for a non-switched nonlinear system, when a single continuous feedback control law can not solve the standard robust H_∞ control problem, the problem is solved by controller switching among finite candidate controllers based on switching technique. Finally, a numerical example illustrates the effectiveness of the proposed approach. Compared with the existing results, our approach for robust H_∞ controllers design do not rely on solutions of HJI inequalities, which is highly desirable and of significant advantage due to the lack of efficient numerical methods for solving HJI inequalities.

II. PROBLEM FORMULATION

Consider switched nonlinear systems described by the state-space model of the form

$$\begin{aligned} \dot{x} &= f_\sigma(x)+\Delta f_\sigma(x)+(g_\sigma(x)+\Delta g_\sigma(x))\,u_\sigma \\ &\quad +(p_\sigma(x)+\Delta p_\sigma(x))\,\omega_\sigma, \\ z &= h_\sigma(x)+d_\sigma(x)u_\sigma, \end{aligned} \tag{1}$$

where $\sigma(t):\Re^+\to M=\{1,2,\dots,m\}$ is the right continuous piecewise constant switching signal to be designed, $x\in\Re^n$ is the state vector, $u_i\in\Re^{m_i}$ and $\omega_i\in\Re^{p_i}$ which belong to $L_2[0,\infty)$ denote the control input and disturbance input of the i-th subsystem respectively, $z\in\Re^{q_i}$ is the regulated

This work was supported by Dogus University Fund for Science and the NSF of P. R. China under the grant No. 60574013.

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output, $f_i(x)$, $g_i(x)$, $p_i(x)$, $h_i(x)$ and $d_i(x)$ are known smooth nonlinear function matrices of appropriate dimensions with $f_i(0)=0$ and $h_i(0)=0$, the smooth unknown function matrices $\Delta f_i(x)$, $\Delta g_i(x)$ and $\Delta p_i(x)$, which express the uncertainties of the system, are described by

$$\begin{aligned}\Delta f_i(x) &= e_{1i}(x)\delta_{1i}(x),\\ \Delta g_i(x) &= e_{2i}(x)\delta_{2i}(x),\\ \Delta p_i(x) &= e_{3i}(x)\delta_{3i}(x)\end{aligned}$$

with known smooth function matrices $e_{ji}(x)$ and unknown smooth function matrices $\delta_{ji}(x)$ which are assumed to belong to sets defined by

$$\Omega_j = \left\{\delta_{ji}(x) \mid \|\delta_{ji}(x)\| \le \|n_{ji}(x)\|, i \in M\right\}, \; j = 1,2,3. \quad (2)$$

where $n_{ji}(x)$ are given function matrices, $\|\cdot\|$ represents either the Euclidean vector norm or the induced matrix 2-norm, $\delta_{1i}(0) = 0$, $i \in M$.

We shall adopt the following assumptions for system (1).

Assumption 1: There exist smooth functions $W_i : \Re^n \to \Re$, which are positive definite and radially unbounded, and smooth positive definite functions $\alpha_i : \Re^n \to \Re$ such that

$$\frac{\partial W_i(x)}{\partial x} f_i(x) \le -\alpha_i(x), \quad \forall x \in \Re^n. \quad (3)$$

Assumption 2: The function matrices $d_i(x)$ are of full-column rank for any $x \in \Re^n$.

Assumption 3: The function vectors $h_i(x)$ are such that

$$\frac{h_i^T(x)h_i(x)}{\alpha_i(x)} < \infty, \quad \text{as } x \to 0. \quad (4)$$

Remark 1: Note that Assumption 1 implies that each subsystem of (1) is globally asymptotically stable. Assumption 2 means that the H_∞ control problem is "nonsingular", which is a standard assumption in nonlinear H_∞ control problems [10]. Assumption 3 simply implies that $h_i^T h_i = O(\alpha_i)$. Observe that since $h_i(x)$ is smooth and $h_i(0) = 0$, Assumption 3 is automatically satisfied when $\alpha_i(x)$ are quadratic functions.

For convenience, we adopt the following notations [7] for switched system (1). Let

$$\Sigma = \{x_0; (i_0, t_0), (i_1, t_1), \cdots, (i_n, t_n), \cdots, | i_k \in M, k \in N\}$$

denote a switching sequence with the initial state x_0 and the initial time t_0, where (i_k, t_k) means that the i_k-th subsystem is active for $t_k \le t < t_{k+1}$.

Now, the robust H_∞ control problem for switched system (1) can be formulated as follows:

Given a constant $\gamma > 0$, design a continuous state feedback controller $u_i = u_i(x)$ for each subsystem and a switching law $i = \sigma(t)$ such that

(i) The closed-loop system is asymptotically stable when $\omega_i = 0$.

(ii) System (1) has finite robust L_2-gain γ from ω_i to z for all admissible uncertainties, i.e., there holds

$$\int_0^T z^T(t)z(t)\,\mathrm{d}t \le \gamma^2 \int_0^T \omega_i^T(t)\omega_i(t)\,\mathrm{d}t + \beta(x_0)$$

for all $T > 0$ and all admissible uncertainties, where $\beta(\cdot)$ is some real-valued function.

III. MAIN RESULTS

This section gives a condition for the robust H_∞ control problem of the system (1) to be solvable, and designs continuous controllers for subsystems and a switching law.

In view of Assumption 1 and 3 and the fact that $h_i(x)$ are smooth, we shall make the following assumption.

Assumption 4: Given a constant $\gamma > 0$, there exist positive definite functions $K_i(\cdot)$, nonnegative functions $\beta_{ij}(x)$ and positive real numbers $\eta_i < \gamma^2$ such that

$$\frac{h_i^T(x)h_i(x)}{\alpha_i(x)} + \sum_{j=1}^m \beta_{ij}(x)\Big[V_i(x) - V_j(x)\Big] \le K_i(W_i) \quad (5)$$

and

$$\begin{aligned}&\frac{1}{2\varepsilon_i}\left[\frac{\partial W_i}{\partial x}\Big(e_{1i}e_{1i}^T + e_{2i}e_{2i}^T + e_{3i}e_{3i}^T\Big)\frac{\partial^T W_i}{\partial x} + n_{1i}^T n_{1i}\right] + \frac{1}{2\varepsilon_i^2}K_i(W_i)\\ &\cdot \frac{\partial W_i}{\partial x} p_i \left[2\gamma^2 I - \frac{1}{\varepsilon_i}K_i(W_i)n_{3i}^T n_{3i}\right]^{-1} p_i^T \frac{\partial^T W_i}{\partial x} \le \eta_i \alpha_i(x) \quad (6)\end{aligned}$$

where ε_i are real numbers satisfying

$$0 < \varepsilon_i < \frac{1}{\eta_i + 1}, \; i \in M.$$

Based on the functions $K_i(W_i)$, we will construct the following Lyapunov function candidates $V_i(W_i)$ for the switched system (1):

$$V_i(W_i) = \frac{1}{\varepsilon_i}\int_0^{W_i} K_i(t)\,\mathrm{d}t. \quad (7)$$

Since the functions W_i and $K_i(W_i)$ are smooth, it is clear that $V_i(W_i) < \infty$ for all $x \in \Re^n$. Furthermore, as $K_i(W_i)$ are positive definite functions of W_i which do not vanish as $W_i \to \infty$ and $W_i(\cdot)$ are positive definite and radially unbounded functions, $V_i(W_i(x))$ are positive definite and radially unbounded functions of the argument x.

Theorem 1: Let a constant $\gamma > \max_{i \in M}\{\sqrt{\eta_i}\}$ be given. Consider the switched system (1) satisfying Assumptions 1-4. Then, the hybrid state feedback controllers

$$\begin{aligned}u_i = u_i(x) = &-\left[2d_i^T(x)d_i(x) + \frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^T(x)n_{2i}(x)\right]^{-1}\\ &\cdot\left[\frac{1}{\varepsilon_i}K_i(W_i)g_i^T(x)\frac{\partial^T W_i}{\partial x} + 2d_i^T(x)h_i(x)\right], \; i \in M \quad (8)\end{aligned}$$

and the switching law

$$\sigma(t) = \min_i \left\{ i : i = \arg\max_{j \in M} V_j(x)\right\} \quad (9)$$

solve the robust H_∞ control problem.

Proof: The time-derivative of $V_i(W_i)$ along the trajectory of the switched system (1) is

$$\begin{aligned}\dot{V}_i &= \frac{\mathrm{d}V_i(W_i)}{\mathrm{d}W_i}\frac{\partial W_i}{\partial x}\left[f_i + \Delta f_i + (g_i + \Delta g_i)\,u_i + (p_i + \Delta p_i)\,\omega_i\right]\\ &= \frac{1}{\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}\left[f_i + \Delta f_i + (g_i + \Delta g_i)\,u_i + (p_i + \Delta p_i)\,\omega_i\right] \quad (10)\end{aligned}$$

Define

$$H_i(x, u_i, \omega_i) = \dot{V}_i + \left(z^T z - \gamma^2 \omega_i^T \omega_i\right).$$

Then considering (10) and (2), the above equalities can be rewritten as

$$
\begin{aligned}
&H_i(x,u_i,\omega_i)\\
&= \frac{1}{\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}\left[f_i+\Delta f_i+(g_i+\Delta g_i)u_i+(p_i+\Delta p_i)\omega_i\right]\\
&\quad +z^Tz-\gamma^2\omega_i^T\omega_i\\
&= \frac{1}{\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}\left[f_i+e_{1i}\delta_{1i}+(g_i+e_{2i}\delta_{2i})u_i+(p_i+e_{3i}\delta_{3i})\omega_i\right]\\
&\quad +h_i^Th_i+2h_i^Td_iu_i+u_i^Td_i^Td_iu_i-\gamma^2\omega_i^T\omega_i\\
&\le \frac{1}{\varepsilon_i}K_i(W_i)\left[\frac{\partial W_i}{\partial x}f_i+\frac{1}{2}\frac{\partial W_i}{\partial x}e_{1i}e_{1i}^T\frac{\partial^T W_i}{\partial x}+\frac{1}{2}\delta_{1i}^T\delta_{1i}\right.\\
&\quad +\frac{\partial W_i}{\partial x}g_iu_i+\frac{1}{2}\frac{\partial W_i}{\partial x}e_{2i}e_{2i}^T\frac{\partial^T W_i}{\partial x}+\frac{1}{2}u_i^T\delta_{2i}^T\delta_{2i}u_i\\
&\quad \left.+\frac{\partial W_i}{\partial x}p_i\omega_i+\frac{1}{2}\omega_i^T\delta_{3i}^T\delta_{3i}\omega_i+\frac{1}{2}\frac{\partial W_i}{\partial x}e_{3i}e_{3i}^T\frac{\partial^T W_i}{\partial x}\right]\\
&\quad +h_i^Th_i+2h_i^Td_iu_i+u_i^Td_i^Td_iu_i-\gamma^2\omega_i^T\omega_i\\
&\le \frac{1}{\varepsilon_i}K_i(W_i)\left[\frac{\partial W_i}{\partial x}f_i+\frac{1}{2}\frac{\partial W_i}{\partial x}e_{1i}e_{1i}^T\frac{\partial^T W_i}{\partial x}+\frac{1}{2}n_{1i}^Tn_{1i}\right.\\
&\quad +\frac{\partial W_i}{\partial x}g_iu_i+\frac{1}{2}\frac{\partial W_i}{\partial x}e_{2i}e_{2i}^T\frac{\partial^T W_i}{\partial x}+\frac{1}{2}u_i^Tn_{2i}^Tn_{2i}u_i\\
&\quad \left.+\frac{\partial W_i}{\partial x}p_i\omega_i+\frac{1}{2}\omega_i^Tn_{3i}^Tn_{3i}\omega_i+\frac{1}{2}\frac{\partial W_i}{\partial x}e_{3i}e_{3i}^T\frac{\partial^T W_i}{\partial x}\right]\\
&\quad +h_i^Th_i+2h_i^Td_iu_i+u_i^Td_i^Td_iu_i-\gamma^2\omega_i^T\omega_i\\
&\triangleq H_i^*(x,u_i,\omega_i).
\end{aligned}
$$

In view of Assumption 3, solving

$$
\begin{cases}
\dfrac{\partial H_i^*}{\partial \omega_i}=0,\\
\dfrac{\partial H_i^*}{\partial u_i}=0
\end{cases}
$$

for ω_i and u_i respectively leads to the saddle point of $H_i^*(x,u_i,\omega_i)$

$$
\omega_i^*=\frac{1}{\varepsilon_i}K_i(W_i)\left[2\gamma^2I-\frac{1}{\varepsilon_i}K_i(W_i)n_{3i}^Tn_{3i}\right]^{-1}\left(\frac{\partial W_i}{\partial x}p_i\right)^T, \tag{11}
$$

$$
u_i^*=-\left[2d_i^Td_i+\frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^Tn_{2i}\right]^{-1}\left[\frac{1}{\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}g_i+2h_i^Td_i\right]^T. \tag{12}
$$

Considering Assumption 1, (11) and (12), we obtain

$$
\begin{aligned}
&H_i^*(x,u_i^*,\omega_i^*)\\
&\le -\frac{1}{\varepsilon_i}K_i(W_i)\alpha_i+\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{1i}e_{1i}^T\frac{\partial^T W_i}{\partial x}\\
&\quad +\frac{1}{2\varepsilon_i}K_i(W_i)n_{1i}^Tn_{1i}+\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{2i}e_{2i}^T\frac{\partial^T W_i}{\partial x}\\
&\quad +\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{3i}e_{3i}^T\frac{\partial^T W_i}{\partial x}\\
&\quad +\frac{1}{2\varepsilon_i^2}K_i^2(W_i)\frac{\partial W_i}{\partial x}p_i\left[2\gamma^2I-\frac{1}{\varepsilon_i}K_i(W_i)n_{3i}^Tn_{3i}\right]^{-1}p_i^T\frac{\partial^T W_i}{\partial x}\\
&\quad -\frac{2}{\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}g_i\left[2d_i^Td_i+\frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^Tn_{2i}\right]^{-1}d_i^Th_i\\
&\quad -\frac{1}{2\varepsilon_i^2}K_i^2(W_i)\frac{\partial W_i}{\partial x}g_i\left[2d_i^Td_i+\frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^Tn_{2i}\right]^{-1}g_i^T\frac{\partial^T W_i}{\partial x}\\
&\quad +h_i^T\left\{I-2d_i\left[2d_i^Td_i+\frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^Tn_{2i}\right]^{-1}d_i^T\right\}h_i\\
&= -\frac{1}{\varepsilon_i}K_i(W_i)\alpha_i+\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{1i}e_{1i}^T\frac{\partial^T W_i}{\partial x}\\
&\quad +\frac{1}{2\varepsilon_i}K_i(W_i)n_{1i}^Tn_{1i}+\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{2i}e_{2i}^T\frac{\partial^T W_i}{\partial x}\\
&\quad +\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{3i}e_{3i}^T\frac{\partial^T W_i}{\partial x}-\frac{1}{2}r_iR_ir_i^T\\
&\quad +\frac{1}{2\varepsilon_i^2}K_i^2(W_i)\frac{\partial W_i}{\partial x}p_i\left[2\gamma^2I-\frac{1}{\varepsilon_i}K_i(W_i)n_{3i}^Tn_{3i}\right]^{-1}p_i^T\frac{\partial^T W_i}{\partial x}\\
&\quad +h_i^T\left\{I-2d_i\left[2d_i^Td_i+\frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^Tn_{2i}\right]^{-1}d_i^T\right\}h_i\\
&\quad +2h_i^Td_i\left[2d_i^Td_i+\frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^Tn_{2i}\right]^{-1}d_i^Th_i\\
&\le -\frac{1}{\varepsilon_i}K_i(W_i)\alpha_i+\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{1i}e_{1i}^T\frac{\partial^T W_i}{\partial x}\\
&\quad +\frac{1}{2\varepsilon_i}K_i(W_i)n_{1i}^Tn_{1i}+\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{2i}e_{2i}^T\frac{\partial^T W_i}{\partial x}\\
&\quad +\frac{1}{2\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}e_{3i}e_{3i}^T\frac{\partial^T W_i}{\partial x}+h_i^Th_i\\
&\quad +\frac{1}{2\varepsilon_i^2}K_i^2(W_i)\frac{\partial W_i}{\partial x}p_i\left[2\gamma^2I-\frac{1}{\varepsilon_i}K_i(W_i)n_{3i}^Tn_{3i}\right]^{-1}p_i^T\frac{\partial^T W_i}{\partial x},
\end{aligned}
$$

where

$$
R_i(x)=\left[2d_i^Td_i+\frac{1}{\varepsilon_i}K_i(W_i)n_{2i}^Tn_{2i}\right]^{-1},
$$

$$
r_i(x)=\frac{1}{\varepsilon_i}K_i(W_i)\frac{\partial W_i}{\partial x}g_i+2h_i^Td_i.
$$

It follows from Assumption 4 and the switching law (9) that

$$
H_i^*(x,u_i^*,\omega_i^*)\le\left(-\frac{1}{\varepsilon_i}+\eta_i+1\right)K_i(W_i)\alpha_i.
$$

Since ω_i^* is the maximum of ω_i, u_i^* is the maximum of u_i, then

$$
H_i(x,u_i,\omega_i)\le H_i^*(x,u_i,\omega_i)\le H_i^*(x,u_i^*,\omega_i)\le H_i^*(x,u_i^*,\omega_i^*).
$$

Hence, we have

$$
\dot V_i\le\left(-\frac{1}{\varepsilon_i}+\eta_i+1\right)K_i(W_i)\alpha_i-z^Tz+\gamma^2\omega_i^T\omega_i. \tag{13}
$$

Note that $0<\varepsilon_i<\dfrac{1}{\eta_i+1}$ and $K_i(W_i)\alpha_i$ are positive definite functions, (13) leads to

$$
\dot V_i+z^Tz-\gamma^2\omega_i^T\omega_i<0. \tag{14}
$$

Now, we introduce

$$
J_T=\int_0^T\left(z^Tz-\gamma^2\omega_i^T\omega_i\right)\mathrm{d}t.
$$

According to the switching sequence Σ, suppose $t_0=0$, $x(t_0)=x(0)$, when $T\in[t_k,t_{k+1})$, for any admissible uncertainties, we have

$$\begin{aligned}
J_T &= \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\left(z^Tz-\gamma^2\omega_{i_j}^T\omega_{i_j}+\dot{V}_{i_j}(x(t))\right)\mathrm{d}t \\
&\quad -\sum_{j=0}^{k-1}\left(V_{i_j}(x(t_{j+1}))-V_{i_j}(x(t_j))\right) \\
&\quad +\int_{t_k}^{T}\left(z^Tz-\gamma^2\omega_{i_k}^T\omega_{i_k}+\dot{V}_{i_k}(x(t))\right)\mathrm{d}t \\
&\quad -\left(V_{i_k}(x(T))-V_{i_k}(x(t_k))\right) \\
&\le -\sum_{j=0}^{k-1}\left(V_{i_j}(x(t_{j+1}))-V_{i_j}(x(t_j))\right) \\
&\quad -\left(V_{i_k}(x(T))-V_{i_k}(x(t_k))\right) \\
&= V_{i_0}(x(0))-V_{i_k}(x(T)) \\
&\quad +\sum_{j=0}^{k-1}\left(V_{i_{j+1}}(x(t_{j+1}))-V_{i_j}(x(t_{j+1}))\right)
\end{aligned} \tag{15}$$

Since at switching time t_k,

$$V_{\sigma(t_{k-1})}(x(t_k))=V_{\sigma(t_k)}(x(t_k)),$$

then (15) leads to

$$\begin{aligned}
J_T &\le V_{i_0}(x(0))-V_{i_k}(x(T)) \\
&\quad +\sum_{j=0}^{k-1}\left(V_{i_{j+1}}(x(t_{j+1}))-V_{i_j}(x(t_{j+1}))\right) \\
&\le V_{i_0}(x(0))-V_{i_k}(x(T)) \\
&\le V_{i_0}(x(0))
\end{aligned}$$

Let

$$\beta(x(0))=\max_{i_0\in M}\{V_{i_0}(x(0))\}.$$

Therefore, we conclude that

$$\int_0^T z^T(t)z(t)\,\mathrm{d}t \le \gamma^2\int_0^T\omega_i^T(t)\omega_i(t)\,\mathrm{d}t+\beta(x(0))$$

holds for all admissible uncertainties and disturbance input ω_i, which means the switched system (1) has finite L_2-gain.

When $\omega_i=0$, it follows from (14) that

$$\dot{V}_i(x(t))\le\|z\|^2+\dot{V}_i(x(t))<0.$$

Asymptotical stability of the switched system (1) under switching law (9) follows. This completes the proof.

Remark 2: When $M=\{1\}$, the switched system (1) degenerates into a regular nonlinear system and the robust H_∞ control problem becomes the standard robust H_∞ control problem for nonlinear systems. Additionally, if $\Delta f(x)=0$, $\Delta g(x)=0$ and $\Delta p(x)=0$, this result is equivalent to the condition given in [12].

Remark 3: Let $\Delta g_i(x)=0$ and $\Delta p_i(x)=0$. If each subsystem of (1) is unstable, but there exist stabilizing controllers $u_i=v_{0i}(x)$ with $v_{0i}(0)=0$, smooth positive definite and proper functions $W_i(x)$ and smooth positive definite functions $\alpha_i(x)$ satisfying $v_{0i}^Tv_{0i}=O(\alpha_i)$ such that

$$\frac{\partial W_i(x)}{\partial x}\left[f_i(x)+g_i(x)v_{0i}(x)\right]\le-\alpha_i(x),\quad \forall x\in\Re^n,$$

Theorem 1 still works to solve the robust H_∞ control problem, providing that Assumption 1-4 hold. In fact, considering stabilizing state feedback $u_i(x)=v_{0i}(x)+v_i(x)$, where $v_i(x)$ are new controllers to be determined, the switched system (1) can be rewritten as

$$\begin{aligned}
\dot{x} &= \bar{f}_\sigma(x)+\Delta f_\sigma(x)+g_\sigma(x)u_\sigma+\left(p_\sigma(x)+\Delta p_\sigma(x)\right)\omega_\sigma, \\
z &= \bar{h}_\sigma(x)+d_\sigma(x)v_\sigma,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\bar{f}_\sigma(x)&=f_\sigma(x)+g_\sigma(x)v_{0\sigma}(x), \\
\bar{h}_\sigma(x)&=h_\sigma(x)+d_\sigma(x)v_{0\sigma}.
\end{aligned}$$

Note that system (16) satisfies Assumption 1-4. Therefore, Theorem 1 is still applicable.

Next, we consider how to apply the obtained result to non-switched nonlinear systems by controller switching. For a nonlinear system, a continuous robust H_∞ controller may not exist or may be sometimes too complex to implement. Thus, in some control problems, control actions are decided by switching between finite candidate controllers. Subsequently, we try to use hybrid state feedback strategy to solve the robust H_∞ control problem for uncertain nonlinear systems.

Consider the following nonlinear system

$$\begin{aligned}
\dot{x} &= f(x)+\Delta f(x)+(g(x)+\Delta g(x))u \\
&\quad +(p(x)+\Delta p(x))\,\omega, \\
z &= h(x)+d(x)u,
\end{aligned} \tag{17}$$

where $x\in\Re^n$ is the state vector, $u\in\Re^m$ and $\omega\in\Re^p$ which belong to $L_2[0,\infty)$ denote the control input and disturbance input respectively, $z\in\Re^q$ is the regulated output, $f(x)$, $g(x)$, $p(x)$, $h(x)$ and $d(x)$ are known smooth nonlinear function matrices of appropriate dimensions with $f(0)=0$ and $h(0)=0$, the smooth unknown function matrices $\Delta f(x)$, $\Delta g(x)$ and $\Delta p(x)$, which express the uncertainties of the system, are described by

$$\begin{aligned}
\Delta f(x) &= e_1(x)\delta_1(x), \\
\Delta g(x) &= e_2(x)\delta_2(x), \\
\Delta p(x) &= e_3(x)\delta_3(x)
\end{aligned}$$

with known smooth function matrices $e_j(x)$ and unknown smooth function matrices $\delta_j(x)$ which are assumed to belong to sets defined by

$$\Omega_j=\left\{\delta_j(x)\,|\,\|\delta_j(x)\|\le\|n_j(x)\|\right\},\ j=1,2,3. \tag{18}$$

where $n_j(x)$ are given function matrices, $\delta_1(0)=0$.

We shall adopt the following assumptions for system (17).

Assumption 5: There exist smooth functions $W_i:\Re^n\to\Re$, which are positive definite and radially unbounded, and smooth positive definite functions $\alpha_i:\Re^n\to\Re$ such that

$$\frac{\partial W_i(x)}{\partial x}f(x)\le-\alpha_i(x),\quad \forall x\in\Re^n. \tag{19}$$

Assumption 6: The function matrices $d(x)$ is of full-column rank for any $x\in\mathfrak{R}^n$.

Assumption 7: The function vector $h(x)$ is such that

$$\frac{h^T(x)h(x)}{\alpha_i(x)}<\infty, \quad \text{as } x\to 0. \tag{20}$$

Assumption 8: Given a constant $\gamma>0$, there exist positive definite functions $K_i(\cdot)$, nonnegative functions $\beta_{ij}(x)$ and positive real numbers $\eta_i<\gamma^2$ such that

$$\frac{h^T(x)h(x)}{\alpha_i(x)}+\sum_{j=1}^{m}\beta_{ij}(x)\Big[V_i(x)-V_j(x)\Big]\leq K_i(W_i) \tag{21}$$

and

$$\frac{1}{2\varepsilon_i}\left[\frac{\partial W_i}{\partial x}\Big(e_1e_1^T+e_2e_2^T+e_3e_3^T\Big)\frac{\partial^T W_i}{\partial x}+n_1^Tn_1\right]+\frac{1}{2\varepsilon_i^2}K_i(W_i)$$
$$\cdot\frac{\partial W_i}{\partial x}p\left[2\gamma^2I-\frac{1}{\varepsilon_i}K_i(W_i)n_3^Tn_3\right]^{-1}p^T\frac{\partial^T W_i}{\partial x}\leq\eta_i\alpha_i(x) \tag{22}$$

where ε_i are real numbers satisfying

$$0<\varepsilon_i<\frac{1}{\eta_i+1}, i\in M.$$

For system (17), suppose that there exists the following class of finite candidate state feedback controllers

$$u_i=u_i(x)=-\left[2d^T(x)d(x)+\frac{1}{\varepsilon_i}K_i(W_i)n_2^T(x)n_2(x)\right]^{-1}$$
$$\cdot\left[\frac{1}{\varepsilon_i}K_i(W_i)g^T(x)\frac{\partial^T W_i}{\partial x}+2d^T(x)h(x)\right], \tag{23}$$

the control law u is generated by switching among them.

Theorem 2: Let a constant $\gamma>\sqrt{\eta_i}$ be given. Consider the switched system (17) satisfying Assumptions 5-8. Then, the hybrid state feedback controllers (23) and the switching law (9) solve the robust H_∞ control problem for system (17).

Proof: Substituting the designed controllers (23) into the system (17) results in a switched nonlinear system. Then, applying Theorem 1 yields the result.

IV. EXAMPLE

In this section, we give an example to demonstrate the effectiveness of the proposed design method.

Example: Consider the following uncertain switched nonlinear system

$$\begin{aligned}\dot{x} &= f_i(x)+\Delta f_i(x)+(g_i(x)+\Delta g_i(x))\,u_i\\ &\quad+(p_i(x)+\Delta p_i(x))\,\omega_i,\\ z &= h_i(x)+d_i(x)u_i, \qquad i=1,2,\end{aligned} \tag{24}$$

where

$$f_1(x)=-6x,\ g_1(x)=x,\ p_1(x)=1,\ h_1(x)=x\sin x,\ d_1(x)=1,$$

$$f_2(x)=-6x^3,\ g_2(x)=-1,\ p_2(x)=1,\ h_2(x)=x^3,\ d_2(x)=-1,$$

$$\Delta f_1(x)=\frac{1}{2}a_1x\sin x,\ e_{11}=\frac{1}{2},\ \delta_{11}(x)=a_1x\sin x,\ n_{11}=x,$$

$$\Delta f_2(x)=\frac{1}{2}a_2x^3\cos x,\ e_{12}=\frac{1}{2},\ \delta_{12}(x)=a_2x^3\cos x,\ n_{12}=x^3,$$

$$\Delta g_1(x)=\frac{1}{2}b_1\sin x,\ e_{21}=\frac{1}{2},\ \delta_{21}(x)=b_1\sin x,\ n_{21}=1,$$

$$\Delta g_2(x)=\frac{1}{2}b_2\cos x,\ e_{22}=\frac{1}{2},\ \delta_{22}(x)=b_2\cos x,\ n_{22}=1,$$

$$\Delta p_1(x)=0,\ \Delta p_2(x)=0,$$

and a_i, b_i are unknown constants belonging to $[0,1]$.

Obviously, $d_i(x)$ satisfy Assumption 2. In view of Assumption 1 and Assumption 3, we can choose

$$\alpha_1(x)=4x^2, \alpha_2(x)=4x^6$$

and

$$W_1(x)=\frac{1}{2}x^2, W_2(x)=\frac{1}{4}x^4.$$

Taking $\varepsilon_i=0.5$ and $\gamma=2$, from Assumption 4, we select $\eta_i=0.5$. Then from (6), we obtain $K_i(W_i)\leq 2$. Choosing $K_i(W_i)=1$, we get

$$V_1(W_1)=2W_1(x)=x^2,$$

$$V_2(W_2)=2W_2(x)=\frac{1}{2}x^4.$$

Let

$$\beta_1(x)=\sin^2 x \quad \text{and} \quad \beta_2(x)=\left(x^2+1\right)^{-2},$$

then (5) is satisfied.

The switching law

$$\sigma(t)=\begin{cases}1 & \text{if} \quad -\sqrt{2}\leq x\leq\sqrt{2},\\ 2 & \text{otherwise.}\end{cases}$$

and the hybrid controllers

$$u_1=-\frac{1}{2}\left(x^2+x\sin x\right),$$
$$u_2=x^3$$

solve the robust H_∞ problem.

V. CONCLUSIONS

In this paper, we have investigated the problem of robust H_∞ control for a class of uncertain switched nonlinear systems. Based on the multiple Lyapunov functions approach, a sufficient condition has been derived by designing a switching law and hybrid state feedback controllers. Furthermore, a hybrid state feedback strategy is proposed to solve the robust H_∞ control problem for uncertain nonlinear system. The proposed controller design method is based on the explicit construction of Lyapunov functions of the switched system, which avoids the need for solving HJI inequalities.

VI. ACKNOWLEDGMENTS

This work was supported by Dogus University Fund for Science and the NSF of P. R. China under the grant No. 60574013.

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