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A Markovian Regime-Switching Stochastic Differential Game for Portfolio Risk Minimization

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Abstract— **A risk minimization problem is considered in a continuous-time Markovian regime-switching financial model modulated by a continuous-time, finite-state Markov chain. We interpret the states of the chain as different market regimes. A convex risk measure is used as a measure of risk and an optimal portfolio is determined by minimizing the convex risk measure of the terminal wealth. We explore the state of the art of the stochastic differential game to formulate the problem as a Markovian regime-switching version of a two-player, zero-sum stochastic differential game. A verification theorem for the Hamilton-Jacobi-Bellman (HJB) solution of the game is provided.**

I. INTRODUCTION

Risk management is an important issue in the modern banking and finance industries. Some recent financial crises, including the Asian financial crisis, the collapse of Long-Term Capital Management (LTCM), the turmoil at Barings and Orange Country, raise the concern of regulators about the risk taking activities of banks and financial institutions and their practice of risk management. Recently, Value at Risk (VaR) has emerged as a standard and popular tool for risk measurement and management. VaR tells us the extreme loss of a portfolio over a fixed time period at a certain probability (confidence) level. [1] develops a theoretical approach for developing measures of risk. They present a set of four desirable properties for measures of risk and introduce the class of coherent risk measures. They point out that VaR does not, in general, satisfy one of the four properties, namely, the subadditivity property. This motivates the quest for some theoretically consistent risk measures. [11] argues that the risk of a portfolio might increase nonlinearly with the portfolio's size due to the liquidity risk. They relax the subadditive and positive homogeneous properties and replace them with the convex property. They introduce the class of convex risk measures, which include the class of coherent risk measures. [7] (see Chapter 15 therein) provides a comprehensive account of coherent risk measures and convex risk measures.

In the past two decades or so, applications of regime-switching models in finance have received much attention. However, relatively little attention has been paid to the use of regime-switching models for quantitative risk management until recently. It is important to take the regime-switching effect into account in long-term financial risk management, such as managing the risk of pension funds, since there might be structural changes in the economic fundamentals over a long time period. Some recent works concerning the regime-switching effect on quantitative risk measurement include [8], [9], and others. However, these works mainly concern certain aspects of quantitative risk measurement and do not focus on risk management and control issues.

In this note, we explore the state of the art of a stochastic differential game for minimizing portfolio risk under a continuous-time Markovian regime-switching financial model. Stochastic differential games are an important topic in the interplay between mathematics and economics. Some early works on the mathematical theory of stochastic differential games include [3], [4], and others. Some recent works on stochastic differential games and their applications include [19], [15], [18], [13], [14], and others. Here, we suppose that an investor invests in a money market account and a stock whose price process follows a Markovian regime-switching geometric Brownian motion (GBM). The interest rate of the money market account, the drift and the volatility of the stock are modulated by a continuous-time, finite-state Markov chain. The states of the chain are interpreted as different market regimes. We adopt a convex risk measure introduced by [11] as a measure of risk, and our goal is to minimize the convex risk measure of the terminal wealth of the investor. Following the plan of [15], we formulate the problem as a Markovian regime-switching stochastic differential game with two players, namely, the investor and the market. We introduce a product of two density processes, one for the Brownian motion and one for the Markov chain process, to generate a family of real-world probability measures in the representation of the convex risk measure. So, the market has two control variables, namely, the market price of risk for the change of measures related to the Brownian motion and the Q-matrix of the Markov chain. We provide a verification theorem for the Markovian regime-switching HJB equation to the solution of the game corresponding to the risk minimization problem.

This note is based on part of [10]. We state results which will be published later in [10] without proofs.

II. ASSET PRICE DYNAMICS

We consider a continuous-time financial model consisting of two primitive assets, namely, a money market account and a stock. These assets are assumed to be tradable continuously on a fixed time horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$. We fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ represents a reference probability measure from which a family of absolutely continuous real-world probability measures are generated.

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Now, we introduce a continuous-time, finite-state Markov chain to describe the evolution of market regimes over time. Let $\mathbf{X} := \{X(t)\}_{t\in\mathcal{T}}$ denote a continuous-time, finite-state Markov chain on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $\mathcal{S} := \{s_1, s_2 \ldots, s_N\}$. The states of the chain represent different market regimes. Without loss of generality, we identify the state space of the chain to be a finite set of unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i \in \Re^N$ and the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, N$. $\mathcal{E}$ is called the canonical state space of $\mathbf{X}$.

Let $A(t) = [a_{ij}(t)]_{i,j=1,2,\ldots,N}$, $t \in \mathcal{T}$, denote a family of generators, or rate matrices, of the chain $\mathbf{X}$ under $\mathcal{P}$. Here, $a_{ij}(t)$ represents the instantaneous intensity of the transition of the chain $\mathbf{X}$ from state i to state j at time t. Note that for each $t \in \mathcal{T}$, $a_{ij}(t) \geq 0$, for $i \neq j$ and $\sum_{i=1}^N a_{ij}(t) = 1$, so $a_{ii}(t) \leq 0$. We assume that $a_{ij}(t) > 0$, for each $i, j = 1, 2, \ldots, N$ and each $t \in \mathcal{T}$. For any such matrix $A(t)$, write $a(t) := (a_{11}(t), \ldots, a_{ii}(t), \ldots, a_{NN}(t))^*$ and $A_0(t) := A(t) - diag(a(t))$, where $diag(y)$ is a diagonal matrix with the diagional elements given by the vector y. These notations are adopted in [2]. With the canonical representation of the state space of the chain, [6] provides the following semimartingale decomposition for $\mathbf{X}$:

$$X(t) = X(0) + \int_0^t A(u)X(u)du + M(t) \ ,$$

where $\{M(t)\}_{t\in\mathcal{T}}$ is an $\Re^N$-valued martingale with respect to the filtration generated by $\mathbf{X}$ under $\mathcal{P}$.

Let y' denote the transpose of a vector or a matrix y. $\langle\cdot,\cdot\rangle$ is the scalar product in $\Re^N$. The instantaneous market interest rate $r(t)$ of the money market account B is determined by the Markov chain as:

$$r(t) = \langle \mathbf{r}, X(t)\rangle \ ,$$

where $\mathbf{r} := (r_1, r_2, \ldots, r_N)' \in \Re^N$ with $r_i > 0$ for each $i = 1, 2, \ldots, N$.
Then, the evolution of the balance of the money market account follows:

$$B(t) = \exp\left(\int_0^t r(u)du\right) , \quad B(0) = 1 \ .$$

The chain $\mathbf{X}$ determines the appreciation rate $\mu(t)$ and the volatility $\sigma(t)$ of the stock, respectively, as:

$$\mu(t) = \langle \boldsymbol{\mu}, X(t)\rangle \ ,$$

and

$$\sigma(t) = \langle \boldsymbol{\sigma}, X(t)\rangle \ ,$$

where $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_N)' \in \Re^N$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \Re^N$ with $\mu_i > r_i$ and $\sigma_i > 0$, for each $i = 1, 2, \ldots, N$.

Let $\mathbf{W} := \{W(t)\}_{t\in\mathcal{T}}$ denote a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the $\mathcal{P}$-augmentation of its own natural filtration. We suppose that $\mathbf{W}$ and $\mathbf{X}$ are stochastically independent. The evolution of the price process of the stock follows a Markovian regime-switching GBM:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t) \ , \quad S(0) = s > 0 \ .$$

Now, we specify the information structure of our model. Let $\mathcal{F}^X$ and $\mathcal{F}^S$ denote the right-continuous, complete filtrations generated by the values of the Markov chain and the stock price process, respectively. Write, for each $t \in \mathcal{T}$, $\mathcal{G}(t) := \mathcal{F}^X(t) \vee \mathcal{F}^S(t)$, the enlarged σ-field generated by $\mathcal{F}^X(t)$ and $\mathcal{F}^S(t)$.

In the sequel, we describe the evolution of the wealth process of an investor who allocates his/her wealth between the money market account and the stock. Let $\pi(t)$ denote the proportion of the total wealth invested in the stock at time $t \in \mathcal{T}$. Then, $1 - \pi(t)$ represents the proportion of the total wealth invested in the money market account at time t. We suppose that $\pi := \{\pi(t)\}_{t\in\mathcal{T}}$ is $\mathcal{G}$-adapted and cadlag (i.e. right continuous with left limit, RCLL). We further assume that π is self-financing, (i.e. there is no income or consumption), and that

$$\int_0^T \pi^2(t)dt < \infty \ , \quad \mathcal{P}\text{-a.s.}$$

Write $\mathcal{A}$ for the set of all such processes π. We call $\mathcal{A}$ the set of admissible portfolio processes.

Let $V(t) := V^\pi(t)$ denote the total wealth of the portfolio π at time t. Then, the evolution of the wealth process $V := \{V(t)\}_{t\in\mathcal{T}}$ is governed by:

$$\begin{aligned} dV(t) &= V(t)\{[r(t) + \pi(t)(\mu(t) - r(t))]dt \\ &\quad + \pi(t)\sigma(t)dW(t)\} \\ V(0) &= v > 0 \ . \end{aligned}$$

Our goal is to find the portfolio π which minimizes the risk of the terminal wealth. Here, we use a convex risk measure introduced in [11] as a measure of risk.

III. RISK MINIMIZATION

In this section, we first describe the notion of convex risk measures. Then, we present the risk minimization problem of an investor with wealth process described in the last section and formulate the problem as a Markovian regime-switching version of a two-player, zero-sum stochastic differential game.

The concept of a convex risk measure provides a generalization of a coherent risk measure as introduced in [1]. Suppose $\mathcal{S}$ denote the space of all lower-bounded, $\mathcal{G}(T)$-measurable random variables. A convex risk measure ρ is a functional $\rho : \mathcal{S} \to \Re$ such that it satisfies the following three properties:

1) If $X \in \mathcal{S}$ and $\beta \in \Re$, then
$$\rho(X + \beta) = \rho(X) - \beta \ .$$
2) For any $X, Y \in \mathcal{S}$, if $X(\omega) \leq Y(\omega)$, for all $\omega \in \Omega$, then $\rho(X) \geq \rho(Y)$.
3) For any $X, Y \in \mathcal{S}$ and $\lambda \in (0, 1)$,
$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \ .$$

The first, second and third properties are the translation invariance, monotonicity and convexity, respectively.

[11] provides an elegant representation for convex risk measures. One can generate any convex risk measure from this

representation by a suitable choice of a family of probability measures. Let $\mathcal{M}_a$ denote a family of probability measures $\mathcal{Q}$ which are absolutely continuous with respect to $\mathcal{P}$. That is, if $\mathcal{P}(A) = 0$, then $\mathcal{Q}(A) = 0$, for any $A \in \mathcal{F}$ and $\mathcal{Q} \in \mathcal{M}_a$. Define a function $\eta : \mathcal{M}_a \to \Re$ such that $\eta(\mathcal{Q}) < \infty$, for all $\mathcal{Q} \in \mathcal{M}_a$. Then, [11] provides the following representation of a convex risk measure $\rho(X)$ of $X \in \mathcal{S}$:

$$\rho(X) = \sup_{\mathcal{Q} \in \mathcal{M}_a} \{E_{\mathcal{Q}}[-X] - \eta(\mathcal{Q})\} ,$$

for some family $\mathcal{M}_a$ and some function η.
Here, $E_{\mathcal{Q}}[\cdot]$ represents expectation under $\mathcal{Q}$. The function $\eta(\cdot)$ is called a "penalty" function, (see [11]).

Following [12] and [15], we choose the penalty function $\eta(\mathcal{Q})$ to be the relative entropy of $\mathcal{Q}$ with respect to $\mathcal{P}$. That is,

$$\eta(\mathcal{Q}) := I(\mathcal{Q}, \mathcal{P}) = E\left[\frac{d\mathcal{Q}}{d\mathcal{P}} \ln\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)\right] ,$$

where $E[\cdot]$ denotes expectation under $\mathcal{P}$.

Now, we generate a family $\mathcal{M}_a$ of equivalent real-world probability measures by a product of two density processes, one for the Brownian motion $\mathbf{W}$ and one for the Markov chain $\mathbf{X}$.

Define a Markovian regime-switching process $\theta(t)$ as:

$$\theta(t) = \langle \boldsymbol{\theta}, X(t) \rangle ,$$

where $\boldsymbol{\theta} := (\theta_1, \theta_2, \ldots, \theta_N)' \in \Re^N$ with $\theta_{(N)} := \max_{1 \le i \le N} \theta_i < \infty$. Write Θ for the space of all such processes.
Consider a $\mathcal{G}$-adapted process $\boldsymbol{\Lambda}^\theta := \{\Lambda^\theta(t)\}_{t \in \mathcal{T}}$

$$\begin{aligned} &\Lambda^\theta(t) \\ := \;& \exp\left(-\int_0^t \theta(s) dW(s) - \frac{1}{2}\int_0^t \theta^2(s) ds\right) . \end{aligned}$$

Then, by Itô's differentiation rule,

$$d\Lambda^\theta(t) = -\Lambda^\theta(t)\theta(t) dW(t) , \quad \Lambda^\theta(0) = 1 .$$

So, $\boldsymbol{\Lambda}^\theta$ is a $(\mathcal{G}, \mathcal{P})$-local-martingale.
Note that

$$E\left[\exp\left(\frac{1}{2}\int_0^T \theta^2(t) dt\right)\right] \le \exp\left(\frac{1}{2}\theta^2_{(N)} T\right) < \infty .$$

So, the Novikov condition is satisfied. Hence, $\boldsymbol{\Lambda}^\theta$ is a $(\mathcal{G}, \mathcal{P})$-martingale, and

$$E[\Lambda^\theta(T)] = 1 .$$

Suppose $C(t) := \{c_{ij}(t)\}_{i,j=1,2,\ldots,N}$ is a second family of generators, or rate matrices, of the chain $\mathbf{X}$ such that for each $i, j = 1, 2, \ldots, N$,

$$c_{ij}(t) = \langle \mathbf{c}_{ij}, X(t) \rangle ,$$

where $\mathbf{c}_{ij} := (c^1_{ij}, c^2_{ij}, \ldots, c^N_{ij})' \in \Re^N$ with $c^k_{ij} \ge 0$, for $i \ne j$ and $\sum_{i=1}^N c^k_{ij} = 0$, so $c^k_{ii} < 0$, for each $k = 1, 2, \ldots, N$.

For each $k = 1, 2, \ldots, N$, write $C^k := [c^k_{ij}]_{i,j=1,2,\ldots,N}$. Then,

$$C(t) = \sum_{k=1}^N C^k \langle X(t), \mathbf{e}_k \rangle ,$$

and so the generator $C(t)$ of the chain $\mathbf{X}$ at time t depends on the current value of the chain.

We wish to introduce a new (real-world) probability measure under which C is a family of generators of the chain $\mathbf{X}$. We follow the method in [2]. First, we define some notations. Let $\mathcal{C}$ denote the space of any such family $C(t)$, $t \in \mathcal{T}$. For any two matrices $A(t)$, with $a_{ij}(t) \ne 0$, for any $t \in \mathcal{T}$ and $i, j = 1, 2, \ldots, N$, and $C(t)$, write $D(t) := C(t)/A(t)$ for the matrix defined by $D(t) = [c_{ij}(t)/a_{ij}(t)]$, for each $t \in \mathcal{T}$. Write $\mathbf{1} := (1, 1, \ldots, 1)' \in \Re^N$ and $\mathbf{I}$ for the $(N \times N)$-identity matrix.

Define, for each $t \in \mathcal{T}$,

$$N(t) := \int_0^t (\mathbf{I} - diag(X(u-))) dX(u) .$$

Here, $\mathbf{N} := \{N(t)\}_{t \in \mathcal{T}}$ is a vector of counting processes, where its component $N_i(t)$ counts the number of times the chain $\mathbf{X}$ jumps to state $\mathbf{e}_i$ in the time interval $[0, t]$, for each $i = 1, 2, \ldots, N$. Then, we cite the following result from [2] without proof.

Lemma 3.1: Let

$$\tilde{N}(t) := N(t) - \int_0^t A_0(u) X(u) du , \quad t \in \mathcal{T} .$$

Then, $\tilde{\mathbf{N}} := \{\tilde{N}(t)\}_{t \in \mathcal{T}}$ is an $(\mathcal{F}^X, \mathcal{P})$-martingale.

Consider a process $\boldsymbol{\Lambda}^C := \{\Lambda^C(t)\}_{t \in \mathcal{T}}$, $C \in \mathcal{C}$, defined by:

$$\begin{aligned} \Lambda^C(t) \;&=\; 1 + \int_0^t \Lambda^C(u-)[D_0(u)X(u-) - \mathbf{1}]' \\ &\quad (dN(u) - A_0(u)X(u-)du) . \end{aligned}$$

Note that from Lemma 3.1, $\boldsymbol{\Lambda}^C$ is an $(\mathcal{F}^X, \mathcal{P})$-martingale.

Define, for each $(\theta, C) \in \Theta \times \mathcal{C}$, a $\mathcal{G}$-adapted process $\boldsymbol{\Lambda}^{\theta,C} := \{\Lambda^{\theta,C}(t)\}_{t \in \mathcal{T}}$ as the product of the two density processes $\boldsymbol{\Lambda}^\theta$ and $\boldsymbol{\Lambda}^C$:

$$\Lambda^{\theta,C}(t) := \Lambda^\theta(t) \cdot \Lambda^C(t) .$$

Lemma 3.2: $\boldsymbol{\Lambda}^{\theta,C}$ is a $(\mathcal{G}, \mathcal{P})$-martingale.
The detail of the proof will be published in [10].

Define, for each $(\theta, C) \in \Theta \times \mathcal{C}$, a real-world probability measure $\mathcal{Q}^{\theta,C} \sim \mathcal{P}$ on $\mathcal{G}(T)$ as:

$$\frac{d\mathcal{Q}^{\theta,C}}{d\mathcal{P}} := \Lambda^{\theta,C}(T) . \tag{1}$$

Then, we generate a family of $\mathcal{M}_a$ of real-world probability measures as follows:

$$\mathcal{M}_a := \mathcal{M}_a(\Theta, \mathcal{C}) = \{\mathcal{Q}^{\theta,C} | (\theta, C) \in \Theta \times \mathcal{C}\} .$$

The following result is from [2]. We cite it in the following lemma without giving the proof.

Lemma 3.3: Suppose $\mathcal{Q}^{\theta,C}$ is defined by (1), for each $(\theta, C) \in \Theta \times \mathcal{C}$. Let

$$\tilde{N}^C(t) := N(t) - \int_0^t C_0(u)X(u)du \ , \quad t \in \mathcal{T} \ , \quad C \in \mathcal{C} \ .$$

Then, $\tilde{\mathbf{N}}^C := \{\tilde{N}^C(t)\}_{t\in\mathcal{T}}$ is an $(\mathcal{F}^X, \mathcal{Q}^{\theta,C})$-martingale.

Theorem 3.4: For each $(\theta, C) \in \Theta \times \mathcal{C}$, $\mathbf{X}$ is a Markov chain with a family of generators $C(t)$, $t \in \mathcal{T}$ under $\mathcal{Q}^{\theta,C}$.

The result follows from the assumption that $\mathbf{W}$ and $\mathbf{X}$ are independent and adapting the proof of Lemma 2.3 in [2] to the case when the generators are time-dependent.

Let $\bar{\eta} : \Theta \times \mathcal{C} \to \Re$ denote a map induced by the penalty function, or the relative entropy, $\eta : \mathcal{M}_a \to \Re$ as below:

$$\bar{\eta}(\theta, C) = \eta(\mathcal{Q}^{\theta,C}) \ , \quad \text{for each } (\theta, C) \in \Theta \times \mathcal{C} \ .$$

Write $E_{(\theta,C)}$ for expectation under $\mathcal{Q}^{\theta,C}$. Then, the convex risk measure associated with $\mathcal{M}_a$ can be written as the one associated with $(\Theta, \mathcal{C})$ as follows:

$$\rho(X) = \sup_{(\theta,C)\in\Theta\times\mathcal{C}} \{E_{(\theta,C)}[-X] - \bar{\eta}(\theta, C)\} \ . \qquad (2)$$

Following [15], we define a vector process $\mathbf{Z} := \{Z(t)\}_{t\in\mathcal{T}}$ by

$$\begin{aligned} dZ(t) &= (dZ_0(t), dZ_1(t), dZ_2(t), dZ_3(t), dZ_4(t))' \\ &= (dZ_0(t), dZ_1^{\pi}(t), dZ_2^{\theta}(t), dZ_3^{C}(t), dZ_4(t))' \\ &= (dZ_0(t), dV^{\pi}(t), d\Lambda^{\theta}(t), d\Lambda^{C}(t), dX(t))' \ , \\ Z(0) &= z = (s, z_1, z_2, z_3, z_4)' \ , \end{aligned}$$

where under $\mathcal{P}$,

$$\begin{aligned} dZ_0(t) &= dt \ , \\ Z_0(0) &= s \in \mathcal{T} \ , \\ dZ_1(t) &= Z_1(t)\{[r(t) + (\mu(t) - r(t))\pi(t)]dt \\ &\quad +\sigma(t)\pi(t)dW(t)\} \ , \\ Z_1(0) &= z_1 > 0 \ , \\ dZ_2(t) &= -\theta(t)Z_2(t)dW(t) \ , \\ Z_2(0) &= z_2 > 0 \ , \\ dZ_3(t) &= Z_3(t-)(D_0(t)X(t-) - \mathbf{1})' \\ &\quad (dN(t) - A_0(t)X(t-)dt) \ , \\ Z_3(0) &= z_3 > 0 \ , \\ dZ_4(t) &= A(t)Z_4(t)dt + dM(t) \ , \\ Z_4(0) &= z_4 \ . \end{aligned}$$

Conditional on $Z(0) = z$, the penalty function, or the relative entropy, is given by:

$$\eta^z(\mathcal{Q}) := E^z\left[\frac{d\mathcal{Q}}{d\mathcal{P}} \ln\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)\right] \ ,$$

where $E^z[\cdot]$ represents expectation under $\mathcal{P}$ given that the initial value $Z(0) = z$. This notation is adopted in [16] and [15].

So, for each $(\theta, C) \in \Theta \times \mathcal{C}$, we define the induced penalty function $\bar{\eta}^z(\theta, C)$ as:

$$\begin{aligned} &\bar{\eta}^z(\theta, C) = \eta^z(\mathcal{Q}^{\theta,C}) \\ &= E^z\{Z_2^{\theta}(T)Z_3^{C}(T)[\ln(Z_2^{\theta}(T)) + \ln(Z_3^{C}(T))]\} \ . \end{aligned}$$

Now, conditional on $Z(0) = z$, the risk-minimizing problem is then to find the portfolio process $\pi \in \mathcal{A}$ in order to minimize the following conditional version of the convex risk measure associated with $\Theta \times \mathcal{C}$:

$$\sup_{(\theta,C)\in\Theta\times\mathcal{C}} \{E^z_{(\theta,C)}[-Z_1^{\pi}(T)] - \bar{\eta}^z(\theta, C)\} \ ,$$

where $E^z_{(\theta,C)}[\cdot]$ denotes expectation under $\mathcal{Q}^{\theta,C}$ given that $Z(0) = z$.

In other words, we need to solve the following problem:

$$\inf_{\pi\in\mathcal{A}} \sup_{(\theta,C)\in\Theta\times\mathcal{C}} \{E^z_{(\theta,C)}[-Z_1^{\pi}(T)] - \bar{\eta}^z(\theta, C)\} \ .$$

As in [15], we formulate the risk-minimizing problem as a zero-sum stochastic differential game between the investor and the market. The investor chooses a portfolio process $\pi \in \mathcal{A}$ so as to maximize the expected value of a monetary utility function, which is dual to the convex risk measure, of his/her terminal wealth. The market responds to this portfolio choice by selecting a real-world probability measure $\mathcal{Q}^{\theta,C} \in \mathcal{M}_a$, which minimizes the maximal expected utility.

Let $U : \mathcal{S} \to \Re$ such that $U(X) = -\rho(X)$, for each $X \in \mathcal{S}$. $U(\cdot)$ represents a monetary utility function and satisfies the concavity, monotonicity and translation invariance properties. From the representation of $\rho(X)$ in (2), $U(X)$ has the following representation:

$$\begin{aligned} &U(X) \\ = &\inf_{(\theta,C)\in\Theta\times\mathcal{C}} \{E_{(\theta,C)}[X] + \bar{\eta}(\theta, C)\} \ , \\ &\text{for each } X \in \mathcal{S}. \end{aligned}$$

Then, conditional on $Z(0) = z$, the risk-minimizing problem can be formulated as the zero-sum stochastic differential game between the investor and the market as:

$$\begin{aligned} \Phi(z) &= \inf_{(\theta,C)\in\Theta\times\mathcal{C}} \left(\sup_{\pi\in\mathcal{A}} E^z_{(\theta,C)}[Z_1^{\pi}(T)] + \bar{\eta}^z(\theta, C)\right) \\ &= E^z_{(\theta^*,C^*)}[Z_1^{\pi^*}(T)] + \bar{\eta}^z(\theta^*, C^*) \ . \end{aligned}$$

To solve the game, we need to find the value function $\Phi(z)$ and the optimal strategies $\pi^* \in \mathcal{A}$ and $(\theta^*, C^*) \in \Theta \times \mathcal{C}$ of the investor and the market, respectively.

IV. SOLUTION TO THE RISK-MINIMIZING PROBLEM

In this section, we present a vertification theorem for the Markovian regime-switching HJB solution of the stochastic differential game corresponding to the risk-minimizing problem.

First, we describe the relationship between the control process of the game and the information structure. Since there are only two driving processes for the vector process $\mathbf{Z}$, namely, the standard Brownian motion $\mathbf{W}$ and the Markov chain $\mathbf{X}$, the vector process $\mathbf{Z}$ is adapted to the enlarged fitration $\mathcal{G} := \{\mathcal{G}(t)\}_{t\in\mathcal{T}}$. $\mathbf{Z}$ is also a Markovian process with respect to $\mathcal{G}$. Under mild technical conditions, the Markovian controls have essentially the same performance as the more general adapted controls in the classical stochastic optimal control theory (see, for example, [16] and [17]). It is also noted in [5] that the optimal control processes can be taken to be Markovian when the dynamics of the state processes are Markovian.

As in [15], we restrict ourselves to consider only Markovian controls for the risk-minimizing problem. Suppose $\mathcal{O} := (0,T)\times(0,\infty)\times(0,\infty)\times(0,\infty)$ representing our solvency region. Let K_1 denote the set such that $\pi(t)\in K_1$. To restrict ourselves to Markovian controls, we assume that

$$\pi(t) := \bar{\pi}(Z(t)) \ ,$$

for some functions $\bar{\pi} : \mathcal{O}\times\mathcal{E}\to K_1$.

Here, we do not distinguish notationally between π and $\bar{\pi}$. So, we can simply identify the control process with deterministic function $\pi(z)$, for each $z\in\mathcal{O}\times\mathcal{E}$. This is called a feedback control.

Note that $(\theta(t), C(t))$ is Markovian with respect to $\mathcal{F}^X$, and hence, it is also Markovian with respect to $\mathcal{G}$. So, the control processes $(\theta(t), C(t), \pi(t))$ are Markovian. They are also feedback control processes since they depend on the current value of the state process $Z(t)$.

Consider a process $\mathbf{Y} := \{Y(t)\}_{t\in\mathcal{T}}$ defined by:

$$dY(t) = (D_0(t)X(t-) - \mathbf{1})' dN(t) \ .$$

From (2),

$$dN(t) = (\mathbf{I} - diag(X(t-)))dX(t) \ ,$$

so

$$dY(t) = (D_0(t)X(t-) - \mathbf{1})'(\mathbf{I} - diag(X(t-)))dX(t) \ .$$

Let $\Delta Y(t)$ denote the jump of the process $\mathbf{Y}$ at time t. Then,

$$\begin{aligned}\Delta Y(t) &:= Y(t) - Y(t-)\\ &= (D_0(t)X(t-) - \mathbf{1})'(\mathbf{I} - diag(X(t-)))\Delta X(t)\\ &= (D_0(t)X(t-) - \mathbf{1})'\\ &\quad (\mathbf{I} - diag(X(t-)))(X(t) - X(t-)) \ .\end{aligned}$$

By some algebra,

$$\Delta Y(t) = \sum_{i,j=1}^{N} (d_{ji} - 1)\langle X(t), \mathbf{e}_j\rangle\langle X(t-), \mathbf{e}_i\rangle \ .$$

Define, for each $i = 1, 2, \ldots, N$, the set

$$\mathcal{Y}_i := \{d_{1i} - 1, d_{2i} - 1, \ldots, d_{Ni} - 1\} \ .$$

Consider a random set $\mathcal{Y}(X(t))$ defined by:

$$\mathcal{Y}(X(t)) = \sum_{i=1}^{N} \mathcal{Y}_i \langle X(t), \mathbf{e}_i\rangle \ , \quad t\in\mathcal{T} \ .$$

Let $\mathcal{Y} := \cup_{i=1}^{N}\mathcal{Y}_i$. Then,

$$\mathcal{Y} = \{d_{ji} - 1 | i, j = 1, 2, \ldots, N\} \ .$$

Clearly, $\mathcal{Y}(X(t))\subset\mathcal{Y}$, for each $t\in\mathcal{T}$.

Given $X(t-) = e_i$ $(i = 1, 2, \ldots, N)$, $\mathcal{Y}_i$ represents the set of all possible values of the jump $\Delta Y(t)$ at time t. The random set $\mathcal{Y}(X(t))$ represents the set of possible values of the jump $\Delta Y(t)$ conditional on the value of $X(t)$.

Suppose γ denote the random measure which selects the jump times and sizes of the process $\mathbf{Y}$. Let $\delta_a(\cdot)$ denote the Dirac measure or the point mass at $a\in\Re$. Then, for each $K\in\mathcal{Y}$, the random measure is:

$$\begin{aligned}&\gamma(t, K;\omega)\\ &= \sum_{0<u\leq t} I_{\{\Delta Y(u)\in K, \Delta Y(u)\neq 0\}}\\ &= \sum_{0<u\leq t} I_{\{\Delta Y(u)\neq 0\}}\delta_{(u,\Delta Y(u))}((0,t]\times K) \ .\end{aligned}$$

To simplify the notation, we suppress the subscript ω and write $\gamma(t, K) := \gamma(t, K;\omega)$.

Let $\gamma(dt, dy)$ denote the differential form of $\gamma(t, K)$. Define, for each $i = 1, 2, \ldots, N$, a probability mass function $n_i(\cdot, t)$ on $\mathcal{Y}_i$ as:

$$n_i(d_{ji} - 1, t) = a_{ji}(t) \ .$$

Then, the predictable compensator of $\gamma(dt, dy)$ is:

$$\nu_{X(t-)}(dt, dy) = \sum_{i=1}^{N} n_i(dy, t-)\langle X(t-), \mathbf{e}_i\rangle\, dt \ .$$

Write $\tilde{\gamma}(dt, dy)$ for the compensated version of the random measure $\gamma(dt, dy)$. That is,

$$\tilde{\gamma}(dt, dy) := \gamma(dt, dy) - \nu_{X(t-)}(dt, dy) \ .$$

Let $\mathcal{H}$ denote the space of functions $h(\cdot,\cdot,\cdot,\cdot,\cdot) : \mathcal{T}\times(\Re^+)^3\times\mathcal{E}\to\Re$ such that for each $z_4\in\mathcal{E}$, $h(\cdot,\cdot,\cdot,\cdot,x)$ is $\mathcal{C}^{1,2,1}(\mathcal{T}\times(\Re^+)^3)$. Write

$$\begin{aligned}&\mathbf{H}(s, z_1, z_2, z_3)\\ &:= (h(s, z_1, z_2, z_3, \mathbf{e}_1), \ldots, h(s, z_1, z_2, z_3, \mathbf{e}_N))'\in\Re^N \ .\end{aligned}$$

Define the Markovian regime-switching generator $\mathcal{L}^{\theta,\pi}$ acting on a function $h\in\mathcal{H}$ for a Markov process $\{Z^{\theta,C,\pi}(t)\}_{t\in\mathcal{T}}$ as:

$$\begin{aligned}&\mathcal{L}^{\theta,C,\pi}[h(s, z_1, z_2, z_3, z_4)]\\ &= \frac{\partial h}{\partial s} + z_1[r(s) + (\mu(s) - r(s))\pi(z)]\frac{\partial h}{\partial z_1}\\ &\quad + \frac{1}{2}\theta^2(s)z_2^2\frac{\partial^2 h}{\partial z_2^2} + \frac{1}{2}z_1^2\pi^2(z)\sigma^2(t)\frac{\partial^2 h}{\partial z_1^2}\\ &\quad - \theta(s)\pi(z)z_1 z_2\sigma(s)\frac{\partial^2 h}{\partial z_1\partial z_2}\\ &\quad + \int_{\mathcal{Y}(x)}\Big(h(s, z_1, z_2, z_3(1+y), z_4)\\ &\quad - h(s, z_1, z_2, z_3, z_4) - \frac{\partial h}{\partial z_3}z_3 y\Big)\nu_x(ds, dy)\\ &\quad + \langle\mathbf{H}(s, z_1, z_2, z_3), A(s)x\rangle \ .\end{aligned}$$

Then, we need the following lemma for the development of a verification theorem of the HJB solution to the stochastic differential game. This lemma can be proof by using the generalized Itô's formula in [5] and conditioning on $Z(0) = z$

under $\mathcal{P}$. The detail of the proof will be published in [10] later.

Lemma 4.1: Let $\tau < \infty$ be a stopping time. Assume further that $h(Z(t))$ and $\mathcal{L}^{\theta,C,\pi}[h(Z(t))]$ are bounded on $t \in [0, \tau]$. Then,

$$\begin{aligned} & E[h(Z(\tau))|Z(0) = z] \\ = \quad & h(z) + E\left[\int_0^\tau \mathcal{L}^{\theta,C,\pi}[h(Z(t))]dt|Z(0) = z\right] . \end{aligned}$$

We now describe the solution of the stochastic differential game between the investor and the market by the following verification theorem.

Theorem 4.2: Suppose $\bar{\mathcal{O}}$ is the closure of $\mathcal{O}$. Suppose there exists a function ϕ such that for each $x \in \mathcal{E}$, $\phi(\cdot,\cdot,\cdot,\cdot,x) \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$ and a Markovian control $(\hat{\theta}(t), \hat{C}(t), \hat{\pi}(t)) \in \Theta \times \mathcal{C} \times \mathcal{A}$, such that:

1) $\mathcal{L}^{\theta,C,\hat{\pi}}[\phi(s, z_1, z_2, z_3, x)] \geq 0$, for all $(\theta, C) \in \Theta \times \mathcal{C}$ and $(s, z_1, z_2, z_3, x) \in \mathcal{O} \times \mathcal{E}$,
2) $\mathcal{L}^{\hat{\theta},\hat{C},\pi}[\phi(s, z_1, z_2, z_3, x)] \leq 0$, for all $\pi \in \mathcal{A}$ and $(s, z_1, z_2, z_3, x) \in \mathcal{O} \times \mathcal{E}$,
3) $\mathcal{L}^{\hat{\theta},\hat{C},\hat{\pi}}[\phi(s, z_1, z_2, z_3, x)] = 0$, for all $(s, z_1, z_2, z_3, x) \in \mathcal{O} \times \mathcal{E}$,
4) for all $(\theta, C, \pi) \in \Theta \times \mathcal{C} \times \mathcal{A}$,
$$\begin{aligned} & \lim_{t \to T^-} \phi(t, Z_1^\pi(t), Z_2^\theta(t), Z_3^C(t), X(t)) \\ = \quad & Z_2^\theta(T) Z_3^C(T)[Z_1^\pi(T) + \ln(Z_2^\theta(T)) \\ & + \ln(Z_3^C(T))] \ , \end{aligned}$$
5) let $\mathcal{K}$ denote the set of stopping times $\tau \leq T$. The family $\{\phi(Z^{\theta,C,\pi}(\tau))\}_{\tau \in \mathcal{K}}$ is uniformly integrable.

Write, for each $z \in \mathcal{O} \times \mathcal{E}$ and $(\theta, C, \pi) \in \Theta \times \mathcal{C} \times \mathcal{A}$,

$$\begin{aligned} & J^{\theta,C,\pi}(z) \\ := \quad & E^z_{(\theta,C)}\{Z_1^\pi(T-s)[\ln(Z_2^\theta(T-s)) \\ & + \ln(Z_3^C(T-s))]\} \ . \end{aligned}$$

Then,

$$\begin{aligned} \phi(z) &= \Phi(z) \\ &= \inf_{(\theta,C) \in \Theta \times \mathcal{C}} \left(\sup_{\pi \in \mathcal{A}} J^{\theta,C,\pi}(z) \right) \\ &= \sup_{\pi \in \mathcal{A}} \left(\inf_{(\theta,C) \in \Theta \times \mathcal{C}} J^{\theta,C,\pi}(z) \right) \\ &= \sup_{\pi \in \mathcal{A}} J^{\hat{\theta},\hat{C},\pi}(z) = \inf_{(\theta,C) \in \Theta \times \mathcal{C}} J^{\theta,C,\hat{\pi}}(z) \\ &= J^{\hat{\theta},\hat{C},\hat{\pi}}(z) \ , \end{aligned}$$

and $(\hat{\theta}, \hat{C}, \hat{\pi})$ is an optimal Markovian control.

The proof is adapted from the proof of Theorem 3.2 in [15] and uses Lemma 4.1 here.

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REFERENCES

[1] P. Artzner, F. Delbaen, J. Eber and D. Heath, Coherent Measures of Risk, *Mathematical Finance*, vol. 9, 1999, pp 203-228.

[2] F. Dufour and R.J. Elliott, Filtering with Discrete State Observations, *Applied Mathematics and Optimization*, vol. 40, 1999, pp 259-272.

[3] R.J. Elliott, The Existence of Value in Stochastic Differential Games, *S.I.A.M. Journal of Control and Optimization*, vol. 14, 1976, pp 85-94.

[4] R.J. Elliott and M.H.A. Davis, Optimal Play in a Stochastic Differential Game, *S.I.A.M. Journal of Control and Optimization*, vol. 19, 1981, pp 543-555.

[5] R.J. Elliott, *Stochastic Calculus and Applications*, Springer, Berlin-Heidelberg-New York; 1982.

[6] R.J. Elliott, L. Aggoun and J.B. Moore, *Hidden Markov Models: Estimation and Control*, Springer, Berlin-Heidelberg-New York; 1994.

[7] R.J. Elliott and P.E. Kopp, *Mathematics of Financial Markets*, Springer, Berlin-Heidelberg-New York; 2005.

[8] R.J. Elliott, L.L. Chan and T.K. Siu, Risk Measures for Derivatives with Markov-modulated Pure Jump Processes. *Asia Pacific Financial Markets*, vol. 13, 2006, pp 129-149.

[9] R.J. Elliott, T.K. Siu and L.L. Chan, *A P.D.E. Approach for Risk Measures for Derivatives with Regime Switching*, Annals of Finance, 2007, Accepted.

[10] R.J. Elliott and T.K. Siu, *On Risk Minimizing Portfolios Under a Markovian Regime-Switching Black-Scholes Economy*, Preprint, Haskayne School of Business, University of Calgary, 2007.

[11] H. Föllmer and A. Schied, Convex Measures of Risk and Trading Constraints, *Finance and Stochastics*, vol. 6, 2002, pp 429-447

[12] P, Grandits and T. Rleinländder, On the Minimal Entropy Martingale Measure, *Annals of Probability*, vol. 30, 2002, pp 1003-1038.

[13] I. Karatzas and W.D. Sudderth, "Stochastic Games of Control and Stopping for a Linear Diffusion", *in Random Walk, Sequential Analysis and Related Topics: A Festschrift in Honor of Y.S. Chow*, 2007, To appear.

[14] I. Karatzas and M. Zamfirescu, Martingale Approach to Stochastic Differential Games of Control and Stopping, Preprint, Department of Mathematics, Columbia University, 2007.

[15] S. Mataramvura and B. Øksendal, Risk Minimizing Portfolios and HJB Equations for Stochastic Differential Games, Department of Mathematics, University of Oslo, Preprint, 2007. `http://www.math.uio.no/eprint/pure_math/2005/40-05.html`

[16] B. Øksendal, *Stochastic Differential Equations: an Introduction with Applications*, Springer, Berlin-Heidelberg-New York; 2003.

[17] B. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Springer, Berlin-Heidelberg-New York; 2004.

[18] B. Øksendal and A. Sulem, Risk Indifference Pricing in Jump Diffusion Markets, Department of Mathematics, University of Oslo, Preprint, 2007. `http://www.math.uio.no/eprint/pure_math/2007/17-07/index.html`

[19] D.W.K. Yeung and L.A. Petrosyan, *Cooperative Stochastic Differential Games*, Springer Series in Operations Research and Financial Engineering, Springer, New York; 2006.