2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Repetitive Model Predictive Control using Linear Matrix Inequalities

Pradeep Y. Tiwari and Mayuresh V. Kothare[1]

Abstract— **Repetitive systems can be characterized by two time variables, namely, the finite time within each repeating cycle and the cycle index, each embodying a distinct connotation of time. Conventional optimal control theory does not explicitly account for this two dimensional (2D) description of repetitive systems. We propose a new formulation for control of repetitive systems using Model Predictive Control (MPC) that explicitly incorporates a 2D representation of the system. The proposed formulation uses a 2D Lyapunov function and the stability requirements are established along each time dimension of the system. The resulting controller synthesis problem is expressed in convex form using Linear Matrix Inequalities (LMIs). The approach allows explicit incorporation of input/output constraints in the controller design. Two examples illustrate the applicability of the proposed approach.**

I. INTRODUCTION

Repetitive processes occupy a significant place in nature, technology and economics. In essence, a repetitive system is characterized by repeating dynamics over a fixed cycle period in response to repetitive operational tasks and disturbances [1]. Repetitive processes have also been variously referred to as cyclic or periodic systems [2]. The motion of a spacecraft and satellite, the change of seasons of the year, the corresponding cycles of farm work, biological rhythms, etc. provide examples of systems in this category. Repetitive processes also occur in a variety of manufacturing systems, e.g., Pressure Swing Adsorption (PSA), Simulated Moving Bed (SMB) chromatography [3], batch processes [4] and long-wall coal cutting and metal rolling operations [5].

Typically, a repetitive system can be described in terms of two distinct time scales, namely, the finite time scale within each repeating cycle and the cycle index. As a result, repetitive systems are also sometimes referred to as 2-dimensional (2D) systems [6]. Conventional control formulations do not explicitly incorporate and/or exploit this 2D representation of cyclic systems. Additionally, conventional one-dimensional (just one time dimension) control formulations have been shown to have limited success in controlling repetitive systems [1].

Existing techniques to control repetitive systems fall under the following broad categories, with considerable overlap amongst these categories:

- Repetitive control
- Iterative learning control
- Model-based predictive control

Repetitive control (RC) is a technique designed to make run-to-run improvement for a process that must track a periodic trajectory or reject periodic disturbances. The theoretical basis for the development of RC is the internal model principle (IMP) proposed in [7]. The basic theory for the continuous time RC was developed in [8] where sufficient conditions for stability of repetitive systems were developed using the small gain theorem and the stability theorem for time-lag systems.

Iterative Learning Control (ILC) is based on an approach similar to the one in RC, namely, run-to-run improvement, but differs in that ILC deals with systems whose states are reset at the start of each run or cycle. Many processes in the chemical and pharmaceutical industries are of a batch nature and fit in the framework of ILC. ILC was applied to an isothermal extruder to improve the operation of a cyclic extrusion process [9]. An extensive catalog of publications in the area of ILC is beyond the scope of our paper and can be found in the comprehensive survey [1].

Until recently, classical ILC and RC methods were limited in their capabilities due to their inability to handle constrained multivariable and non-square systems and also due to their inability to incorporate practically meaningful optimization objectives in the controller design.

On the other hand, Model Predictive Control (MPC) [10] is a technique that provides the flexibility to handle constrained multivariable and non-square systems with time delays and periodic disturbances while permitting the use of a range of model forms. A significant advantage of MPC is its ability to include economically relevant optimization objectives. Recognizing these strengths, recently, a number of new formulations and extensions of ILC based on MPC have been proposed that have greatly expanded its applicability. Along these lines, [11] proposed a linear quadratic (LQ) criterion based formulation for discrete-time ILC systems. Lee et. al. [12] later presented a more complete formulation that guaranteed attainment of the minimum achievable error using MPC coupled with iterative learning. A combination of repetitive control and MPC was applied to a simulated moving bed (SMB) chromatography system in [2], [3].

The previously reported MPC-based ILC techniques used the well-known "lifted-system" representation [2], [3], [12] which involves defining augmented variables consisting of all the samples of a variable within a period. The resulting "lifted-system" representation then relates the cycle-to-cycle dynamics of the repetitive system, effectively as a one-dimensional system. One drawback of this formulation is

Pradeep Y. Tiwari and Mayuresh V. Kothare are with the Chemical Process Modeling and Control Center, Department of Chemical Engineering, Lehigh University, Bethlehem, PA 18015, U.S.A., pyt2@lehigh.edu, mayuresh.kothare@lehigh.edu

[1]Corresponding author is M. V. Kothare

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

that instantaneous disturbances occurring within a cycle are not appropriately handled and various "unlifting" techniques have to be used to resolve this weakness [3].

Recently, Lyapunov-based stability analysis of 2D linear repetitive processes was reported in [13], using results on stability theory of 2D repetitive systems from [6]. A combination of feedback from the current cycle and feedforward from the past cycle was used to design a stabilizing control law for the repetitive system using a diagonally augmented Lypunov function [13]. The stabilizing control law was shown to be computable in convex form using Linear Matrix Inequalities (LMIs). This idea was later extended in [14] to design a stabilizing PI controller.

In this paper, we propose to use the basic approach proposed in these previous efforts [13], [15] to develop a novel MPC formulation for repetitive processes, using tools from LMI-based MPC [16]. Our proposed approach explicitly incorporates the two time scales of repetitive systems, as well as input and output constraints. Thus, the proposed formulation overcomes the shortcomings of previous MPC formulations that employ lifting techniques and at the same time, extends the basic formulation of [13] to a constrained predictive control setting. The resulting optimization problem is recast as a convex problem involving LMIs.

NOTATION $\Re$ is the set of real numbers. For matrix A, A^T denotes its transpose, A^{-1} its inverse (if it exists), $\sigma_{\max}(A)$ its maximum singular value. The matrix inequality $A > (\geq) B$ means that A, B are square Hermitian and A-B is positive (semi-)definite. For a set of scalars $\{a_i\}$ (or matrices $\{A_i\}$), a_i (or A_i) denotes the i^{th} scalar (or matrix). I denotes the identity matrix. For vector x, $\|x\|_P$, $P > 0$, denotes its weighted vector 2-norm, x_i its i^{th} component. $x(k)$ or $x(k|k)$ denotes the state measured at real time k; $x(k+i|k)$, $(i \geq 1)$ the predicted value of the state at time $k+i$ predicted using measurements at real time k.

II. Problem formulation

As discussed in the previous section, a repetitive system is a two dimensional (2D) system with time within a cycle and the cycle index forming the two dimensions of the system. The performance of such a system at any time is dependent on the information propagating along the two dimensions i.e., system dynamics are functions of measurements along the cycle and from cycle to cycle. Following [6], a 2D repetitive system can be defined using the following discrete time state space model equations:

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \tag{1}$$

where, $k \geq 0$ is the cycle (or pass) index, $0 \leq p \leq \alpha$ is discrete time within the cycle, $\alpha \geq 0$ is cycle length. $x_k(p) \in \Re^n$, $y_k(p) \in \Re^m$ and $u_k(p) \in \Re^l$ are the plant state vector, the plant output and input vectors respectively at discrete time p within cycle k. A, B, B_0, C, D, D_0 are the state space matrices. The matrices B_0 and D_0 characterize the effect of the previous cycle on the state evolution in the current cycle. To consider the general case, we define the initial conditions as $x_{k+1}(0) = d_{k+1}$, $k \geq 0$ and $y_0(p) = f(p)$ where $d_{k+1} \in \Re^n$ is a vector of constants and $f(p) \in \Re^m$ is a vector whose elements are functions of $\{p : 0 \leq p \leq \alpha\}$.

For repetitive systems, stability analysis must be carried out along both the dimensions i.e. stability within the cycle and stability from cycle to cycle. The stability theory for linear repetitive processes proposed in [6] refers to these two distinct stability requirements as stability along the cycle and asymptotic stability respectively. This theory requires that a bounded sequence of inputs gives rise to a bounded output profile along each cycle.

Theorem 1: [6] A discrete linear repetitive process described by (1) is asymptotically stable if and only if,

$$r(D_0) < 1$$

where $r(.)$ denotes the spectral radius.
If this property holds, then a control input sequence $\{u_k(p)\}_k$ that converges to $u_\infty(p)$ as $k \to \infty$ will lead to an output pass profile sequence $\{y_k(p)\}_k$ that converges to $y_\infty(p)$ as $k \to \infty$. This pass profile $y_\infty(p)$ is called the limiting pass profile and can be represented over $0 \leq p \leq \alpha$ for $D = 0$ as:

$$x_\infty(p+1) = (A + B_0(I_m - D_0)^{-1}C)x_\infty(p) + Bu_\infty(p)$$

$$y_\infty(p) = (I_m - D_0)^{-1}Cx_\infty(p)$$

However, this does not guarantee that the limiting profile is stable in the normal sense. Asymptotic stability requires boundedness of the resulting profile over a finite pass length only. However stability along the pass requires boundedness to be guaranteed independent of pass length making it a stronger requirement.

Theorem 2: [6] A discrete linear repetitive process described by (1) is stable along the pass if and only if the 2D characteristic polynomial

$$C(z_1, z_2) := det \begin{bmatrix} I - z_1 A & -z_1 B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix} \neq 0, \ \forall (z_1, z_2) \in \bar{U}^2 \tag{2}$$

where $\bar{U}^2 = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$.
Stability along the pass clearly forces the limiting profile to be stable. Also, as a necessary condition, (2) implies $r(D_0) < 1$, thus ensuring asymptotic stability by Theorem 1. To address the controller synthesis problem, the authors in [15] express (2) as sufficient LMI conditions and then synthesize a stabilizing feedforward-feedback control law of the form

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) \tag{3}$$

from these LMI conditions through a linearizing change of variables.

Building on these developments, our goal is to synthesize a control law that guarantees stability within a pass (and hence asymptotic stability by necessity) while satisfying constraints and minimizing a meaningful objective function in a receding horizon fashion.

III. MPC FOR REPETITIVE SYSTEMS USING LMIs

In [16], the authors developed an LMI-based robust MPC approach for one dimensional systems. In this section, we will extend the basic formulation in [16] to propose a predictive control framework that can be applied to 2D systems that provides not only 2D stability but also performs optimally while incorporating input and output constraints.

A. Control Objective

Typical batch or repetitive systems are required to closely follow a particular output profile repeatedly over each cycle while maintaining process constraints. This leads to a cyclic/repeating steady state over each pass of the system. We require that the proposed control scheme tracks a repeating reference trajectory while satisfying all the constraints by specifically exploiting the 2D nature of the system. Mathematically such a control objective at any discrete time p within a cycle $k+1$ can be defined as:

$$J_p = \min_{\substack{u_{k+1}(p+i|p)\\ i=0,\ldots,\alpha-p}} \sum_{i=0}^{\alpha-p} \|y_{k+1}(p+i|p) - y^r_{k+1}(p+i)\|^2 \tag{4}$$

where the control and prediction horizon $(\alpha - p)$ is equal to the remaining time in the current cycle. $y_{k+1}(p+i|p)$ is the predicted value of $y_{k+1}(p+i)$ at time p+i in the (k+1)th cycle, based on information at the current discrete time p in cycle $k+1$ using the process model (1). $y^r_{k+1}(p)$ refers to the output reference trajectory at discrete time p within the cycle with index $k+1$. Note that the objective for MPC is to minimize the output error over the remaining part of the current cycle. The prediction horizon and the control horizon are assumed equal. It is important to note that the control/prediction horizon is finite and is shrinking as we move forward within a cycle. Most approaches based on MPC consider either an infinite horizon or a fixed/constant finite horizon.

B. Augmented error model with integral information

As discussed in the previous section, the system is required to track the output reference trajectory or in other words the process output/pass profile must converge in the pass-to-pass direction to a so-called steady state or limit profile y^r. Following [14], we integrate the error information propagating across the cycles. This integral information can be defined with the help of a new variable:

$$e_k(p) = \sum_{j=0}^{k} (y_j(p) - y^r_j(p)) \tag{5}$$

at discrete time p within cycle k. It is clear that e_k sums the error across all the cycles ($j = 0$ to k) at the same discrete time p within each cycle. Next we define an extended output vector $z_k(p) = [y_k^T \ \ e_k^T]^T$. We now see that:

$$\begin{aligned} e_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) \\ &\quad + [D_0 \ \ I]z_k(p) - y^r_{k+1}(p) \end{aligned}$$

Thus, the extended output vector can be represented as:

$$\begin{aligned} z_{k+1}(p) &= \begin{bmatrix} C \\ C \end{bmatrix} x_{k+1}(p) + \begin{bmatrix} D \\ D \end{bmatrix} u_{k+1}(p) \\ &\quad + \begin{bmatrix} D_0 & 0 \\ D_0 & I \end{bmatrix} z_k(p) + \begin{bmatrix} 0 \\ -I \end{bmatrix} y^r(p) \end{aligned}$$

Note that $y^r(p)$ is replaced with $y^r_{k+1}(p)$ as it is assumed that the target trajectory for each cycle remains the same.

To define the error model we use the following new variables:

$$\left.\begin{aligned} \bar{x}_{k+1}(p) &= x_{k+1}(p) - x^r_{k+1}(p) \\ \bar{y}_{k+1}(p) &= y_{k+1}(p) - y^r_{k+1}(p) \\ \bar{u}_{k+1}(p) &= u_{k+1}(p) - u^r_{k+1}(p) \\ \bar{z}_{k+1}(p) &= z_{k+1}(p) - z^r_{k+1}(p) \end{aligned}\right\} \tag{6}$$

Following [14], the error model can then be defined using these error variables as:

$$\begin{aligned} \bar{x}_{k+1}(p+1) &= A\bar{x}_{k+1}(p) + B\bar{u}_{k+1}(p) + \bar{B}_0\bar{z}_k(p) \\ \bar{z}_{k+1}(p) &= \bar{C}\bar{x}_{k+1}(p) + \bar{D}\bar{u}_{k+1}(p) + \bar{D}_0\bar{z}_k(p) \end{aligned} \tag{7}$$

where,

$$\bar{B}_0 = \begin{bmatrix} B_0 & 0 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} C \\ C \end{bmatrix}, \bar{D} = \begin{bmatrix} D \\ D \end{bmatrix}, \bar{D}_0 = \begin{bmatrix} D_0 & 0 \\ D_0 & I \end{bmatrix}$$

C. Control Law

It is important that any control law used for a 2D system provides control action along both time dimensions of the system. This would require use of information available along the discrete time within the cycle and across the cycle. To accomplish this we use a combination of state and output feedback. As discussed in [14], an appropriate choice for such a 2D control law is the following:

$$\bar{u}_{k+1}(p) = K_1\bar{x}_{k+1}(p) + K_2\bar{y}_k(p) + K_3\bar{e}_k(p) \tag{8}$$

If we define a new augmented state vector as,

$$X_{k+1}(p) = \begin{bmatrix} \bar{x}^T_{k+1}(p) & \bar{y}^T_k(p) & \bar{e}^T_k(p) \end{bmatrix}^T \tag{9}$$

(8) then can be written as,

$$\bar{u}_{k+1}(p) = [K_1 \ \ K_2 \ \ K_3]X_{k+1}(p) \tag{10}$$

A further substitution of (10) in (7) results in,

$$\begin{bmatrix} \bar{x}_{k+1}(p+1) \\ \bar{y}_{k+1}(p) \\ \bar{e}_{k+1}(p) \end{bmatrix} = \left\{ \begin{bmatrix} A & B_0 & 0 \\ C & D_0 & 0 \\ C & D_0 & I \end{bmatrix} + \begin{bmatrix} B \\ D \\ D \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \right\} X_{k+1}(p)$$

Define,

$$X^+_{k+1}(p) = \begin{bmatrix} \bar{x}_{k+1}(p+1) \\ \bar{y}_{k+1}(p) \\ \bar{e}_{k+1}(p) \end{bmatrix}, \Phi = \begin{bmatrix} A & B_0 & 0 \\ C & D_0 & 0 \\ C & D_0 & I \end{bmatrix}$$

$$R = \begin{bmatrix} B \\ D \\ D \end{bmatrix}, \ K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$$

Finally using these new matrix definitions, the augmented error model for the 2D cyclic system can be stated as:

$$X_{k+1}^{+}(p) = (\Phi + RK)X_{k+1}(p) \tag{11}$$

The objective function can be rewritten in terms of the augmented state as:

$$J_p = \min_{\substack{u_{k+1}(p+i|p) \\ i=0,\ldots,\alpha-p}} \sum_{i=0}^{\alpha-p} X_{k+1}^{+}(p+i|p)^T Q_1 X_{k+1}^{+}(p+i|p) \tag{12}$$

where, $Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{Q} \end{bmatrix}$ is a weighting matrix of appropriate dimensions with $\bar{Q}$ defining weights on the outputs $\bar{z}_{k+1}(p)$. Note that along with the output tracking, state tracking can also be incorporated in the proposed framework by simply replacing the diagonal zero in Q_1 with an appropriate state weighing matrix.

D. Stability condition

Any control strategy designed for a cyclic system must explicitly account for stability along the pass and asymptotic stability. Although there have been attempts to establish 2D stability for repetitive systems in the literature based on proportional and proportional integral action, none account for process constraints. Furthermore, no reported approach attempts to incorporate optimality within a stabilizing framework.

In this section, we approach the stability problem of 2D systems using a Lyapunov framework, which also allows the formulation of an optimal control problem. A 2D Lyapunov function for the system under consideration is defined as:

$$V(X_{k+1}(p)) = X_{k+1}^T P X_{k+1}(p), P > 0 \tag{13}$$

where $X_k(p)$ is the augmented state vector and P is a symmetric positive definite matrix. Note that the 2D nature of the augmented state X makes V inherit the multidimensionality of the processes. In the following theorem we state conditions that guarantees not only stability but also optimal performance.

Theorem 3: A cyclic system given by (11) is stable along the pass and stable asymptotically iff there exist matrix variables $\gamma > 0$, $\beta > 0$ Q and Y with diagonal symmetric matrix $P = (\gamma + \beta\bar{V})Q^{-1}$, $Y = KQ$ that are the solution to the following optimization problem:

$$\min\ (\gamma + \beta\bar{V})$$

subject to

$$\begin{bmatrix} 1 & X_{k+1}^T(p|p)H^T \\ HX_{k+1}(p|p) & Q \end{bmatrix} \geq 0 \tag{14}$$

$$\begin{bmatrix} Q & Q\Phi^T + Y^T R^T & Q\Phi^T Q_1^{\frac{1}{2}} + Y^T R^T Q_1^{\frac{1}{2}} \\ \Phi Q + RY & Q & 0 \\ Q_1^{\frac{1}{2}}\Phi Q + Q_1^{\frac{1}{2}} RY & 0 & (\gamma+\beta\bar{V})I \end{bmatrix} \geq 0 \tag{15}$$

$$Q \geq 0$$

which are LMIs in Q, Y, γ and β. $H = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is a matrix of appropriate dimensions.

Proof: The stability in the sense of Lyapunov requires that the Lyapuonv function decrease with time. To establish the 2D stability we require 2D lyapunov function (13) to satisfy:

$$\begin{aligned} &V(X_{k+1}^{+}(p+i|p)) - V(X_{k+1}(p+i|p)) \\ &\leq -X_{k+1}^{+}(p+i|p)^T Q_1 X_{k+1}^{+}(p+i|p) \end{aligned} \tag{16}$$

In addition, to formulate the problem in an optimization framework we require that the following condition holds:

$$V(X_{k+1}^{+}(\alpha|p)) \leq \beta V(X_k^{+}(\alpha|p)) \tag{17}$$

where $\beta \geq 0$. Since $X_k^{+}(\alpha)$ is known, $V(X_k^{+}(\alpha)) = \bar{V}$ is also known and remains constant over the cycle $k+1$. The condition in (17) is non-convex and difficult to formulate as a LMI. To address this issue, we begin with the assumption that (17) holds. Then, we sum up equation (16) from $i = 0, 1, \ldots \alpha - p$, using (17). It can be shown that contraction of the Lyapunov function across the cycles for terms involving z_k can be omitted leading to a more conservative upper bound. Subsequently, one can show that the problem reduces to (14), (15) using steps similar to those in [16] and skipped here for brevity. Once the solution is obtained, we check if condition (17) is satisfied. ■

IV. Input/Output Constraints

A large offset from the reference trajectory or the presence of a large disturbance generally leads to a large control action. This control action is implemented through actuators the majority of which are limited by physical bounds. As a result when the required control action exceeds the physical bounds of the actuator, the control system does not perform as expected. Such actuator limits may cause the controller to windup and the process may go unstable. It is essential for a practical control strategy to account for such constraints. Following [16], in this section we show that input and output constraints can be readily incorporated within the proposed optimal control strategy for 2D systems.

A. Invariant Ellipsoid

First we establish the following lemma that provides an ellipsoid bounding on the future state estimates for a cyclic system given by (11).

Lemma 1: For a cyclic 2D system given by (11) if there exists a Q, Y, γ and β at some discrete time p within cycle $k+1$ such that $Y = KQ$ and

$$X_{k+1}^T(p|p)Q^{-1}X_{k+1}(p|p) \leq 1$$

or equivalently,

$$X_{k+1}^T(p|p)PX_{k+1}(p|p) \leq (\gamma+\beta\bar{V}) \tag{18}$$

with $P = (\gamma+\beta\bar{V})Q^{-1}$, then

$$\max_{(\alpha-p)\geq i\geq 1} X_{k+1}^{+}(p+i|p)^T Q^{-1} X_{k+1}^{+}(p+i|p) < 1 \tag{19}$$

equivalently,

$$\max_{(\alpha-p)\geq i\geq 1} X_{k+1}^{+}(p+i|p)^T P X_{k+1}^{+}(p+i|p) < (\gamma+\beta\bar{V})$$

In other words, $\mathscr{E} = \{z|\ z^T Q^{-1} z \leq 1\} = \{z|\ z^T P z \leq (\gamma+\beta\bar{V})\}$ is an invariant ellipsoid for the predicted states of the cyclic system given by (11).

Proof: It is simple to show that above lemma always holds if *Theorem 3* holds or in other words Lyapunov function is decreasing along the cycle. ■

B. Input Constraint

Extending results from [17] for continuous time, in this section we incorporate LMI based sufficient conditions accounting for the bounds on control action u into the proposed discrete MPC based approach for 2D systems.

Consider the following Euclidean norm constraint on the control action:

$$\|u_{k+1}(p+i|p)\|_2 \leq u_{max}, \quad 0 \leq i \leq (\alpha-p) \qquad (20)$$

The above constraint is imposed on the entire future horizon (remaining part of the cycle) while calculating the entire future control sequence.

The following Lemma states conditions for the above constraint to be satisfied.

Lemma 2: For a cyclic system given by (11) with control action defined by (10), if there exist matrix variables Q and $Y = KQ$ that satisfy (14), (15) and the following LMI condition at discrete time p within cycle $k+1$

$$\begin{bmatrix} u_{max}^2 I & Y \\ Y^T & Q \end{bmatrix} \geq 0 \qquad (21)$$

then the euclidean norm of the control input sequence over the remaining cycle is bounded by u_{max}.

Proof: Follows along the same lines as the proof in [16] and is skipped for brevity.

We now consider constraints on the peak value of individual components of control input to a Multi-Input Multi-Output (MIMO) system. Such a constraint is encountered more frequently in practice than a Euclidean norm constraint. Bounds on the peak value of each component can be expressed as:

$$|u_{k+1}^j(p+i|p)| \leq u_{max}^j, \ 0 \leq i \leq (\alpha-p), \ j = 1..l \qquad (22)$$

The following Lemma provides a LMI feasibility condition that guarantees that the above constraint is satisfied.

Lemma 3: For a cyclic system given by (11) with control action defined by (10), if there exist matrix variables Q, X, and $Y = KQ$ that satisfy (14), (15) and the following LMI condition at discrete time p within cycle $k+1$

$$\begin{bmatrix} Z & Y \\ Y^T & Q \end{bmatrix}, \ Z_{jj} \leq (u_{max}^j)^2 \qquad (23)$$

where $j = 1..l$, then the control input u^j is bounded by u_{max}^j over the remaining cycle.

Proof: Follows along the same lines as the proof in [16] and is skipped for brevity.

C. Output Constraint

Performance based criteria for industrial processes are often stated as bounds on purity, temperature, speed, etc. which can be typically expressed as output constraints at discrete time p within cycle $k+1$:

$$\max_{1\leq i\leq(\alpha-p)} \|y_{k+1}(p+i|p)\|_2 \leq y_{max}, \ 0 \leq i \leq (\alpha-p) \qquad (24)$$

The above condition is further formulated as a LMI using the following Lemma.

Lemma 4: For a cyclic system given by (11) with control action defined by (10), if there exist matrix variables Q and $Y = KQ$ satisfying (14), (15) and the following LMI at discrete time p within cycle $k+1$

$$\begin{bmatrix} Q & (\Phi Q + RY)^T H^T \\ H(\Phi Q + RY) & y_{max}^2 I \end{bmatrix} \geq 0 \qquad (25)$$

then the euclidean norm of the plant outputs over the remaining cycle is bounded by y_{max}.

Proof: Follows along the same lines as the proof in [16].

V. Numerical Examples

A. Example 1

In this section we demonstrate the performance of the proposed results using a MIMO numerical example studied in [14]. We use LMI toolbox from *Matlab*® (version 7.1.0.183 R14 Service Pack 3) for all the calculations. Consider a system defined by (1) with the state space parameters given by:

$$A = \begin{bmatrix} 0.92 & 0.14 & -0.98 & 0.41 \\ -0.76 & -0.93 & -0.62 & 0.13 \\ 0.68 & -0.65 & 1.02 & -0.81 \\ 0.94 & 0.04 & 0.83 & 0.2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.99 & -0.99 & 0.07 \\ 0.07 & -0.94 & -0.63 \\ 0.98 & -0.73 & 0.02 \\ -0.37 & 0.19 & -0.65 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.75 & 0.75 & 0.31 & 0.84 \\ -0.86 & 0.99 & 0.33 & -0.84 \end{bmatrix}$$

$$D = \begin{bmatrix} -0.33 & -0.14 & 0.59 \\ -0.18 & 0.94 & -0.17 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} -0.01 & -0.43 \\ 0.29 & -0.13 \\ 0.98 & 1.09 \\ 1.09 & 0.17 \end{bmatrix}, \ D_0 = \begin{bmatrix} 1.11 & -0.66 \\ 0.46 & 1.23 \end{bmatrix}$$

The system outputs are required to track values of [10,-3] in the first half and [-10,3] in the second half over each pass with pass length equal to 20 (discrete time). We use Theorem 3 to solve for K that establishes both across the pass and along the pass stability. In addition, we require that the euclidian norm of the inputs over the control horizon is bounded by 450. We use Lemma 2 in addition to the Theorem 3 to evaluate the control input sequence $u_{k+1}(p+i|p)$ to be applied over the control horizon but apply only the

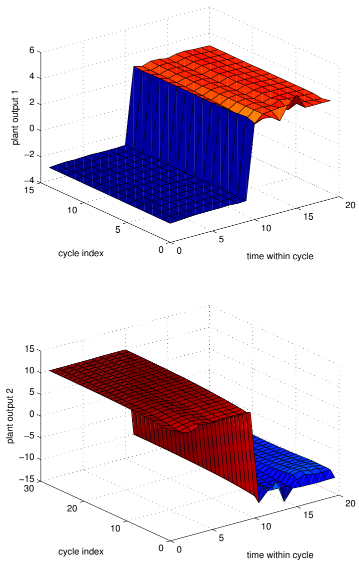

Fig. 1. Plant output 1 and 2 when input constraints are active (Lemma 2)

current control input $u_{k+1}(p|p)$ before moving onto the next discrete time to repeat the calculations. The system response is given in Figure 1. We can clearly see that the proposed approach not only performs significantly better but is also quick when compared to [14]. It is important to note that we have used a cycle length of 20, significantly smaller than 100 which has been used in [14]. A small pass length makes it difficult for the control action to stabilize the system within one pass and may lead to the requirement of high number of passes to finally bring the system to desired trajectory. We can clearly see that MPC based approach performs quicker inspite of having a smaller pass length.

VI. Conclusion

The two dimensional dynamics of a cyclic process makes it difficult to control using conventional control techniques which use only one dimensional system information. In this paper, we have proposed a novel and systematic predictive control framework that exploits the 2D information associated with cyclic processes to develop an efficient optimal predictive control strategy.

Following [15], we have used a 2D Lyapunov function to establish stability criteria for 2D processes that extends naturally to an optimal model-based controller synthesis formulation for optimally tracking cyclic reference trajectories. It is important to notice that the prediction horizon for 2D processes is finite along a cycle and infinite across the cycles. It is well established in the literature that the presence of a finite horizon limits the applicability of many of the well developed predictive control strategies and requires special care. We have shown in this paper that the 2D information associated with cyclic processes can be used to not only overcome this problem but also to use it to guarantee cycle-to-cycle improvement.

We note that the prediction horizon used in the proposed approach shrinks as we move forward within a cycle. Such an approach is the only consistent way to capture the finite length of each cycle before the cycle index is updated. Moreover, this also makes the formulation computationally more efficient, eliminating predictions beyond the end of the cycle. The entire control strategy is recast in an LMI framework making it convex and hence computationally tractable. We have extended the proposed control strategy to incorporate the most frequently occurring nonlinearities in control applications,namely, input saturation/output constraints. We incorporate these constraints into our approach as LMIs of appropriate system variables.

The proposed approach can be extended to include model uncertainties as well and remains part of our current research effort.

References

[1] H.-S. Ahn, Y. Chen, and K. L. Moore, "Iterative Learning Control: Brief survey and categorization," *IEEE Transactions on Systems, Man and Cybernetics – Part C: Applications and Reviews*, vol. 37, no. 6, pp. 1099–1121, November 2007.

[2] S. Natarajan and J. H. Lee, "Repetitive model predictive control applied to a simulated moving bed chromatography system," *Comput. Chem. Eng.*, vol. 24, pp. 1127–1133, 2000.

[3] G. Erdem, S. Abel, M. Morari, M. Mazzotti, M. Morbidelli, and J. H. Lee, "Automatic control of Simulated Moving Beds," *Ind. Eng. Chem. Res.*, vol. 43, pp. 405–421, 2004.

[4] J. H. Lee and K. S. Lee, "Iterative learning control applied to batch processes: An overview," *Control Engineering Practice*, vol. 15, no. 10, pp. 1306–1318, October 2007.

[5] J. P. Edwards, "Stability problems in the control of multipass processes," *Proc. Inst. Elect. Eng.*, vol. 121, no. 11, pp. 1425–1431, 1974.

[6] E. Rogers and D. H. Owens, "Stability Analysis for Linear Repetitive processes," *ser. Lecture Notes for Control and Information Sciences, Springer-Verlag*, vol. 175, 1992.

[7] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," *Appl. Math. Opt.*, vol. 2, pp. 170–194, 1975.

[8] S. Hara, Y. Yamamoto, T. Omata, and N. Nakano, "Repetitive control system: A new type of servo system for periodic exogeneous signal," *IEEE Trans. Aut. Control*, vol. 33, pp. 659–668, 1988.

[9] M. Pandit and K. Buchheit, "Optimizing iterative learning control of cyclic production processes with application to extruders," *IEEE Trans. Aut. Control*, vol. 7, no. 3, pp. 382–390, MAY 1999.

[10] E. F. Camacho and C. Bordons, *Model Predictive Control*. New York: Springer, 1999.

[11] K. M. Tao, R. L. Kosut, and G. Aral, "Learning feedforward control," in *Proceedings of the 1994 American Control Conference*, Baltmore, MD, USA, 1994, pp. 2575–2579.

[12] J. H. Lee, K. S. Lee, and W. C. Kim, "Model-based iterative learning control with a quadratic crietrion for time-varying linear systems," *Automatica*, vol. 36, pp. 641–657, 2000.

[13] K. Galkowski, E. Rogers, S. Xu, J. Lam, and D. H. Owens, "LMIs-A fundamental tool in analysis and controller design for discrete linear repetitive processes," *IEEE Transactions on Circuits and Systems*, vol. 49, no. 6, pp. 768–778, June 2002.

[14] B. Sulikowski, K. Galkowski, E. Rogers, and D. Owens, "PI control of discrete linear repititive processes," *Automatica*, vol. 42, pp. 877–880, 2006.

[15] K. Galkowski, J. Lam, E. Rogers, S. Xu, B. Sulikowski, W. Paszke, and D. Owens, "LMI based stability analysis and robust controller design for discrete linear repetitive processes," *Int. J. of Robust and nonlinear control*, vol. 13, pp. 1195–1211, 2003.

[16] M. V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, Oct. 1996.

[17] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. Philadelphia, PA: SIAM, June 1994, vol. 15.