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Stability Analysis of the Interacting Multiple Model Algorithm

Chze Eng Seah* and Inseok Hwang§

***Abstract*—**

The Interacting Multiple Model (IMM) algorithm is a well-known state estimation algorithm for hybrid systems. We derive a lower bound and an upper bound for the error covariance of the IMM algorithm for controllable and observable hybrid systems. We then derive sufficient conditions for the exponential stability of the IMM algorithm for a special class of hybrid systems by the Lyapunov approach.

I. INTRODUCTION

The Interacting Multiple Model (IMM) algorithm [1] has been used in many applications, such as target tracking and fault diagnosis. Several authors have considered performance analysis of the IMM algorithm under specific operating scenarios [2], [3], or for specific applications [4]. However, to the best of our knowledge, no conditions that guarantee the stability of the IMM algorithm in any application have been given in the control literature.

In this paper, we present a lower bound and an upper bound for the error covariance of the IMM algorithm for controllable and observable hybrid systems. We also derive sufficient conditions for the exponential stability of the IMM algorithm for a special class of hybrid systems. Our work is motivated by the work in [5], which derived sufficient conditions for the stability of the discrete-time Kalman filter. However, the IMM algorithm consists of a set of interacting Kalman filters whose means and covariance updates are coupled or mixed at each time step. Hence, it is a challenge to overcome the complexity due to this mixing to prove the the stability of the IMM algorithm.

The paper is organized as follows: In Section II, we present the filter equations of the IMM algorithm, and review the conditions for stability of the Kalman filter. A set of sufficient conditions for stability of the IMM algorithm are then derived in Sections III. Conclusions are given in Section IV.

II. BACKGROUND AND MOTIVATION

A. Review of the IMM Algorithm

The Interactive Multiple Model Estimation (IMM) algorithm uses a bank of Kalman filters, each matched to a mode of the following stochastic hybrid system:

$$x(k) = A(k)x(k-1) + B(k)w(k) \tag{1}$$

$$z(k) = C(k)x(k) + v(k) \tag{2}$$

*Graduate student, School of Aeronautics and Astronautics, Purdue University seah@purdue.edu

§Assistant professor, School of Aeronautics and Astronautics, Purdue University ihwang@purdue.edu

where $A(k) = A_{m(k)}$, $B(k) = B_{m(k)}$, $C(k) = C_{m(k)}$ are the system matrices correspond to a mode $m(k) \in \{1, 2, \ldots, r\}$ at time k; $w(k)$ and $v(k)$ are white zero-mean Gaussian noise vectors with covariance $Q_{m(k)}$ and $R_{m(k)}$ respectively. The evolution of the mode $m(k)$ is given by

$$\pi_{ij} = p[m(k) = j|m(k-1) = i] \quad \text{for} \quad i, j = 1, \ldots, r$$

where π_{ij} is a constant; $p[\cdot|\cdot]$ denotes a conditional probability. We assume that, for all $i, j = 1, \ldots, r$, A_j is non-singular and

$$0 < \xi_1 I \le Q_j \le \xi_2 I \qquad 0 < \xi_3 I \le R_j \le \xi_4 I \tag{3}$$

Let Z^k denote the set of measurements up to time k. The IMM algorithm computes the posterior mean $\hat{x}_j(k|k)$ and covariance $P_j(k|k)$ for each Kalman filter j, and the mode probability $\alpha_j(k) := p[m(k) = j|Z^k]$ recursively as follows:

1. Mixing: Compute the mixing probability

$$\begin{aligned}\gamma_{ji}(k-1) &:= p[m(k-1) = i|m(k) = j, Z^{k-1}] \\ &= \frac{1}{\sum_{l=1}^{r} \pi_{lj}\alpha_l(k-1)} \pi_{ij}\alpha_i(k-1)\end{aligned} \tag{4}$$

The initial conditions to Kalman filter j are given by

$$\hat{x}_{j0}(k-1) = \sum_{i=1}^{r} \gamma_{ji}(k-1)\hat{x}_i(k-1|k-1) \tag{5}$$

$$\begin{aligned}P_{j0}(k-1) = \sum_{i=1}^{r} \big\{ P_i(k-1|k-1) + [\hat{x}_i(k-1|k-1) - \\ \hat{x}_{j0}(k-1)][\hat{x}_i(k-1|k-1) - \hat{x}_{j0}(k-1)]^T \big\} \gamma_{ji}(k-1)\end{aligned} \tag{6}$$

2. Filtering: Each Kalman filter j computes

$$\hat{x}_j(k|k) = A_j\hat{x}_{j0}(k-1) + K_j(k)r_j(k) \tag{7}$$

$$r_j(k) = z(k) - C_jA_j\hat{x}_{j0}(k-1) \tag{8}$$

$$K_j(k) = P_j(k|k-1)C_j^T S_j^{-1}(k) \tag{9}$$

$$P_j(k|k-1) = A_jP_{j0}(k-1)A_j^T + B_jQ_jB_j^T \tag{10}$$

$$S_j(k) = C_jP_j(k|k-1)C_j^T + R_j \tag{11}$$

$$P_j(k|k) = [P_j^{-1}(k|k-1) + C_j^T R_j^{-1} C_j]^{-1} \tag{12}$$

3. Mode Update: Compute the Likelihood function

$$\Lambda_j(k) := \mathcal{N}_q(r_j(k); 0, S_j(k)) \tag{13}$$

where q is the dimension of $r_j(k)$; $\mathcal{N}_q(\cdot; 0, \Sigma)$ denotes a q-dimensional multivariate Gaussian pdf with mean zero and covariance Σ. The mode probability is given by

$$\alpha_j(k) = \frac{1}{\sum_{l=1}^{r} \Lambda_l(k)\alpha_l^-(k)} \Lambda_j(k)\alpha_j^-(k) \tag{14}$$

where $$\alpha_j^-(k) = \sum_{i=1}^{r} \pi_{ij}\alpha_i(k-1) \tag{15}$$

4. Output: The combined mean and covariance are

$$\hat{x}(k) = \sum_{j=1}^{r} \alpha_j(k)\hat{x}_j(k)$$

$$P(k) = \sum_{j=1}^{r} \left\{P_j(k) + [\hat{x}_j(k) - \hat{x}(k)][\hat{x}_j(k) - \hat{x}(k)]^T\right\}\alpha_j(k)$$

B. State Estimation Errors of the IMM Algorithm

We define the estimation error for Kalman filter j as

$$e_j(k) := x(k) - \hat{x}_j(k) \tag{16}$$

Using (5), (7) and (16), we can show that [3]

$$e_j(k) = [I - K_j(k)C_j]A_j \sum_{i=1}^{r} \gamma_{ji}(k-1)e_i(k-1) + \phi(k) \tag{17}$$

where

$$\begin{aligned}\phi(k) =& [A_T(k) - A_j - K_j(k)(C_T(k)A_T(k) - C_jA_j)]x(k) \\ &+ [I - K_j(k)C_T(k)]B_T(k)w_T(k) - K_j(k)v_T(k)]\end{aligned}$$

and $A_T(k), B_T(k), C_T(k)$ are the system matrices corresponding to the true mode (or true system dynamics). We consider the following coupled systems

$$e_j(k) = [I - K_j(k)C_j]A_j \sum_{i=1}^{r}\gamma_{ji}(k-1)e_i(k-1) \qquad j = 1, 2, \ldots, r \tag{18}$$

which represents the homogeneous part of (17). Using (9)-(12), we can show that

$$P_j(k|k) = [I - K_j(k)C_j]P_j(k|k-1) \tag{19}$$

Hence, (18) can also be written as

$$e_j(k) = P_j(k|k)P_j^{-1}(k|k-1)A_j \sum_{i=1}^{r} \gamma_{ji}(k-1)e_i(k-1) \tag{20}$$

Let $\tilde{e}(k) = [e_1^T(k) \quad e_1^T(k) \quad \ldots \quad e_r^T(k)]^T$. The system of equations in (18) can be written as

$$\tilde{e}(k) = \tilde{\Psi}(k, k-1)\Gamma(k-1)\tilde{e}(k-1) \tag{21}$$

where $$\tilde{\Psi}(k, k-1) = diag\left\{[I - K_i(k)C_i]A_i\right\}_r$$
$$\Gamma(k-1) = \left[\gamma_{ji}(k-1)\right]_{r\times r}$$

We use $diag\{E_i\}_r$ to denote a block diagonal matrix consisting of matrices $E_1, E_2, \ldots, E_r$, and $[a_{ij}]_{r\times r}$ to denote a $r \times r$ matrix with entries a_{ij}. Note that

$$\sum_{i=1}^{r} \gamma_{ji}(k) = 1 \qquad \forall k \geq 0 \tag{22}$$

We say that the IMM algorithm is globally exponentially stable if the origin of system (21) is globally exponentially stable as defined below.

Definition 1: Exponential stability. The origin of system (21) is exponentially stable if there exist scalars $0 \leq \lambda < 1$, $\zeta > 0$, $\rho_0 > 0$, and an integer $k_0 \geq 0$ such that

$$\|\tilde{e}(k)\| \leq \zeta\|\tilde{e}(k_0)\|\lambda^{k-k_0} \quad \forall k \geq k_0, \|\tilde{e}(k_0)\| < \rho_0$$

It is globally exponentially stable if ρ_0 is arbitrarily large. Note that $\|e\| := \sqrt{e^Te}$ denotes the 2-norm of vector e. From Lyapunov's stability theorem, the system (21) is exponentially stable if there exists a finite positive integer N, positive scalars μ_1, μ_2, μ_3, and a scalar function $V(\tilde{e}(k), k)$ such that

$$\mu_1\|\tilde{e}(k)\|^2 \leq V(\tilde{e}(k), k) \leq \mu_2\|\tilde{e}(k)\|^2 \tag{23}$$

$$V(\tilde{e}(k), k) - V(\tilde{e}(k-N), k-N) \leq -\mu_3\|\tilde{e}(k)\|^2 \tag{24}$$

for all $\|\tilde{e}(k)\| < \rho, k \geq k_0, \rho > 0$ [6]. If ρ is arbitrarily large, then the system (21) is globally exponentially stable.

The main challenge in showing the exponential stability of this system comes from the coupling among the set of equations in (18) (also represented by the matrix $\Gamma(k-1)$ in (21)), and the coupling among the mixed initial conditions (6). Before we consider the stability of this system, we can gain much motivation by considering the exponential stability of a single Kalman filter.

C. Stability of the Kalman filter

Consider a discrete time stochastic system

$$x(k) = A(k)x(k-1) + B(k)w(k) \tag{25}$$

$$z(k) = C(k)x(k) + v(k) \tag{26}$$

where $A(k)$ is nonsingular; $w(k)$ and $v(k)$ are white zero-mean Gaussian noise vectors with covariance $Q(k)$ and $R(k)$ respectively. The covariance $P(k|k)$ of the Kalman filter for the system (25)-(26) is given by

$$\begin{aligned}P(k|k) = \big\{&[A(k)P(k-1|k-1)A^T(k) + \\ &B(k)Q(k)B^T(k)]^{-1} + C^T(k)R^{-1}(k)C(k)\big\}^{-1}\end{aligned} \tag{27}$$

We define the function $\Upsilon_k(P)$ as

$$\begin{aligned}\Upsilon_k(P) :=& \big\{[A(k)PA^T(k) + B(k)Q(k)B^T(k)]^{-1} + \\ &C^T(k)R^{-1}(k)C(k)\big\}^{-1}\end{aligned} \tag{28}$$

Then the covariance $P(k|k)$ is given by

$$P(k|k) = \Upsilon_k(\Upsilon_{k-1}(\ldots\Upsilon_1(P(0|0))\ldots) \tag{29}$$

Next, we present the following concepts of uniform controllability and uniform observability [7]:

Definition 2: Uniform Controllability. We define a transition matrix $\Phi(k, i)$ as follows:

$$\begin{aligned}\Phi(k, k) &:= I, \qquad \Phi(k, k-1) := A(k) \\ \Phi(k, i) &:= \Phi(k, k-1)\Phi(k-1, i) \qquad \text{for } k > i \\ \Phi(k, i) &:= \Phi^{-1}(i, k) \qquad \text{for } k < i\end{aligned}$$

The system (25)-(26) is uniformly controllable if there exist a positive integer N and scalars $\kappa_1 > 0$, $\kappa_2 < \infty$ such that

$$\kappa_1 I \leq \sum_{i=k-N+1}^{k} \Phi(k,i)B(i)Q(i)B^T(i)\Phi^T(k,i) \leq \kappa_2 I$$

Definition 3: Uniform Observability. The system (25)-(26) is uniformly observable if there exist a positive integer N and scalars $\kappa_3 > 0$, $\kappa_4 < \infty$ such that

$$\kappa_3 I \leq \sum_{i=k-N}^{k} \Phi^T(i,k)C^T(i+1)R^{-1}(i+1)C(i+1)\Phi(i,k) \leq \kappa_4 I$$

Deyst and Price [5] have then shown the following results:

Lemma 1: If the system (25)-(26) is uniformly controllable and uniformly observable, and if $P(0|0) > 0$, then the Kalman filter covariance given by (29) is uniformly bounded from below and from above for all $k \geq N$, that is

$$\frac{\kappa_1}{1+\kappa_1\kappa_4} I \leq P(k|k) \leq \left(\frac{1}{\kappa_3} + \kappa_2\right) I \qquad k \geq N$$

For stability (or convergence of state estimation error $e(k)$) of the Kalman filter, we consider the system (c.f. (20)) [5]

$$\begin{aligned} e(k) &= P(k|k)P^{-1}(k|k-1)A(k)e(k-1) \\ &= [I - K(k)C(k)]A(k)e(k-1) \end{aligned} \tag{30}$$

where $K(k)$ is the Kalman filter gain given by

$$K(k) = P(k|k-1)C^T(k)[C(k)P(k|k-1)C^T(k) + R(k)]^{-1} \tag{31}$$

We define a Lyapunov function

$$V(e(k),k) = e^T(k)P^{-1}(k|k)e(k)$$

From Lemma 1, we see that (c.f. (23)) [5]

$$\mu_1\|e(k)\|^2 \leq V(e(k),k) \leq \mu_2\|e(k)\|^2$$

Furthermore, Deyst and Price [5] has shown that

$$V(e(k),k) - V(e(k-N),k-N) \leq -\mu_3\|e(k)\|^2 < 0$$

Deyst and Price [5] then proved the following theorem:

Theorem 1: If the system (25)-(26) is uniformly controllable and uniformly observable, then the system (30) is globally exponentially stable.

III. Stability of the IMM algorithm

A. A lower bound for $P_j(k|k)$

We would like to use a similar approach as above to show stability of the IMM algorithm. First, we would like to use Lemma 1 to derive a lower bound for $P_j(k|k)$ of the IMM algorithm. However, although the system (1)-(2) can be considered as a special case of the time-varying system (25)-(26), we cannot directly use the result of Lemma 1. This is because the covariance update of the IMM algorithm is more complicated than that of the (single) Kalman filter due to the 'mixing' in step 1 of the IMM algorithm.

Due to space limitations the proofs for some of the lemmas presented below are not given. The omitted proofs may be obtained from any of the authors. We first rewrite the covariance update equations of the IMM algorithm as follows:

Let

$$\begin{aligned} Q_j^e(k) := A_j\Big\{ &\sum_{i=1}^{r} \gamma_{ji}(k-1)\big[\hat{x}_i(k-1|k-1) - \\ &\hat{x}_{j0}(k-1)\big]\big[\hat{x}_i(k-1|k-1) - \hat{x}_{j0}(k-1)\big]^T\Big\} A_j^T \end{aligned} \tag{32}$$

Note that $Q_j^e(k)$ is a positive semi-definite matrix. Using (32), we replace (6) and (10) of the IMM algorithm by the following two equations:

$$P_j^0(k-1) = \sum_{i=1}^{r} \gamma_{ji}(k-1)P_i(k-1|k-1) \tag{33}$$

$$P_j(k|k-1) = A_jP_j^0(k-1)A_j^T + Q_j^e(k) + B_jQ_jB_j^T \tag{34}$$

Next, we use the following lemma to overcome the complexity in the covariance update due to the $\gamma_{ji}(k-1)$ terms.

Lemma 2: Given any positive definite matrices P_i, $i = 1, 2, \ldots, r$; nonnegative scalars γ_i with $\sum_{i=1}^{r} \gamma_i = 1$; and positive semi-definite matrices Q, R, we have

$$\Big\{\Big[\sum_{i=1}^{r}\gamma_i P_i + Q\Big]^{-1} + R\Big\}^{-1} \geq \sum_{i=1}^{r}\gamma_i\Big\{\big[P_i + Q\big]^{-1} + R\Big\}^{-1}$$

We now presents the following result for the lower bound of the covariance $P_j(k|k)$ of the IMM algorithm.

Lemma 3: Suppose the system (1)-(2) is uniformly controllable and uniformly observable, then

$$P_j(k|k) \geq \beta_1 I > 0 \qquad k \geq N$$

Proof: Substituting (33) and (34) into (12), we have with a change in notations of the subscripts,

$$\begin{aligned} P_{j_k}(k|k) = \Big\{\Big[&\sum_{j_{k-1}=1}^{r} \gamma_{j_k j_{k-1}}(k-1)A_{j_k}P_{j_{k-1}}(k-1|k-1)A_{j_k}^T \\ &+ Q_{j_k}^e(k) + B_{j_k}Q_{j_k}B_{j_k}^T\Big]^{-1} + C_{j_k}^T R_{j_k}^{-1}C_{j_k}\Big\}^{-1} \end{aligned}$$

Note that we use $j_k, j_{k-1}, \ldots$ to denote a sequence of modes $m(k) = j_k, m(k-1) = j_{k-1}, \ldots$. Using Lemma 2 and the fact that $Q_{j_k}^e(k) \geq 0$, we have

$$\begin{aligned} P_{j_k}(k|k) \geq &\sum_{j_{k-1}=1}^{r} \gamma_{j_k j_{k-1}}(k-1)\Big\{\Big[A_{j_k}P_{j_{k-1}}(k-1|k-1)A_{j_k}^T \\ &+ B_{j_k}Q_{j_k}B_{j_k}^T\Big]^{-1} + C_{j_k}^T R_{j_k}^{-1}C_{j_k}\Big\}^{-1} \end{aligned} \tag{35}$$

We define the function $\Upsilon_{k,j}(P)$ as

$$\Upsilon_{k,j}(P) := \big\{[A_jPA_j^T + B_jQ_jB_j^T]^{-1} + C_j^TR_j^{-1}C_j\big\}^{-1} \tag{36}$$

Comparing (28) and (36), $\Upsilon_{k,j}(P)$ can be considered as a special case of $\Upsilon_k(P)$ (which gives the covariance update for the Kalman filter of the system (25)-(26)) with $A(k) = A_j$, $B(k) = B_j$, etc. Using (35) and (36),

$$P_{j_k}(k|k) \geq \sum_{j_{k-1}=1}^{r} \gamma_{j_k j_{k-1}}(k-1)\Upsilon_{k,j_k}(P_{j_{k-1}}(k-1|k-1))$$

We can derive by iteration (using Lemma 2 repeatedly) that

$$P_{j_k}(k|k) \geq \sum_{j_{k-1}=1}^{r} \dots \sum_{j_0=1}^{r} \gamma_{j_k j_{k-1}}(k-1)\dots\gamma_{j_1 j_0}(0)P^*(k|k)$$

where

$$P^*(k|k) = \Upsilon_{k,j_k}\Big(\Upsilon_{k-1,j_{k-1}}\big(\dots\Upsilon_{1,j_1}(P_{j_0}(0|0))\big)\dots\Big) \tag{37}$$

Comparing (29) and (37), we see that $P^*(k|k)$ can be considered as the Kalman filter covariance of the system (25)-(26), with $A(k) = A_{j_k}$, $A(k-1) = A_{j_{k-1}}$, and so on. Thus, using Lemma 1, there exists a positive scalar β_1 such that

$$\Upsilon_{k,j_k}\Big(\Upsilon_{k-1,j_{k-1}}\big(\dots\Upsilon_{1,j_1}(P_{j_0}(0|0))\big)\dots\Big) \geq \beta_1 I \tag{38}$$

for any $j_{k-1}, \dots, j_0 \in \{1, 2, \dots, r\}$. From (22),

$$\sum_{j_{k-1}=1}^{r} \dots \sum_{j_0=1}^{r} \gamma_{j_k j_{k-1}}(k-1)\dots\gamma_{j_1 j_0}(0) = 1 \tag{39}$$

Thus
$$P_{j_k}(k|k) \geq \beta_1 I$$

■

B. An upper bound for $P_j(k|k)$

In this section, we derive an upper bound for the covariance $P_j(k|k)$. From (9) and (11), the set of gains $K_j(k)$ of the IMM algorithm are given by

$$K_j(k) = P_j(k|k-1)C_j^T[C_j P_j(k|k-1)C_j^T + R_j]^{-1} \tag{40}$$

Furthermore, from (40), we have

$$K_j(k)R_j K_j^T(k) = [I - K_j(k)C_j]P_j(k|k-1)C_j^T K_j^T(k) \tag{41}$$

From (33), we have

$$\begin{aligned} P_j^0(k) = &\sum_{i=1}^{r} \gamma_{ji}(k)P_i(k|k)[I - K_i(k)C_i]^T + \\ &\sum_{i=1}^{r} \gamma_{ji}(k)P_i(k|k)C_i^T K_i^T(k) \end{aligned} \tag{42}$$

Substituting (19) and (34) into (42), and then using (41), we have

$$\begin{aligned} P_j^0(k) = &\sum_{i=1}^{r} \gamma_{ji}(k)\Big\{[I - K_i(k)C_i]\big[A_i P_i^0(k-1)A_i^T + \\ &Q_i^e(k) + B_i Q_i B_i^T\big][I - K_i(k)C_i]^T + K_i(k)R_i K_i^T(k)\Big\} \end{aligned}$$

We define a function

$$\begin{aligned} \chi_{i,k}(P, K) := &[I - KC_i]\big[A_i P A_i^T + Q_i^e(k) + \\ &B_i Q_i B_i^T\big][I - KC_i]^T + KR_i K^T \end{aligned} \tag{43}$$

Hence

$$P_j^0(k) = \sum_{i=1}^{r} \gamma_{ji}(k)\chi_{i,k}(P_i^0(k-1), K_i(k)) \tag{44}$$

The following lemma establishes that the set of gains $K_j(k)$ in (40) "minimizes" the covariance $P_j^0(k)$ [8].

Lemma 4: Suppose $\gamma_{ji}(k)$ and $Q_j^e(k)$, for $k > 0$, $i, j = 1, \dots, r$, are given. Let $K_j^a(k)$ be an arbitrary sequence of gains. Define a sequence $T_j^0(k)$ with $T_j^0(0)$ given and

$$T_j^0(k) := \sum_{i=1}^{r} \gamma_{ji}(k)\chi_{i,k}(T_i^0(k-1), K_i^a(k)) \tag{45}$$

for $k > 0, j = 1, 2, \dots, r$. Let $P_j^0(k)$ be the sequence given by (44). Note that the gains $K_i^a(k)$ in (45) are arbitrary while those in (44) are the gains of the IMM algorithm given by (40). Then, if $P_j^0(0) \leq T_j^0(0)$, it follows that $P_j^0(k) \leq T_j^0(k)$ for all $k \geq 0$.

Proof: In [8] (or see also [4]), it has been shown that, for any $K_i^a(k)$,

$$\chi_{i,k}(P_i^0(k-1), K_i(k)) \leq \chi_{i,k}(P_i^0(k-1), K_i^a(k))$$

Now, suppose $P_i^0(k-1) \leq T_i^0(k-1)$, then

$$\begin{aligned} P_j^0(k) &= \sum_{i=1}^{r} \gamma_{ji}(k)\chi_{i,k}(P_i^0(k-1), K_i(k)) \\ &\leq \sum_{i=1}^{r} \gamma_{ji}(k)\chi_{i,k}(P_i^0(k-1), K_i^a(k)) \\ &\leq \sum_{i=1}^{r} \gamma_{ji}(k)\chi_{i,k}(T_i^0(k-1), K_i^a(k)) = T_j^0(k) \end{aligned}$$

Since $P_i^0(0) \leq T_i^0(0)$, by induction, $P_i^0(k) \leq T_i^0(k)$ for all $k \geq 0$. ■

We present the following corollary based on Theorem 1.

Corollary 1: Let $F(k) = [I - K(k)C(k)]A(k)$. Suppose the system (25)-(26) is uniformly controllable and uniformly observable, then there exist gains $K(k)$ such that $\forall k \geq k_0$,

$$\|F(k)F(k-1)\dots F(k_0)\| \leq c_0\lambda_0^{k-k_0} \quad c_0 > 0, 0 \leq \lambda_0 < 1$$

We then have the following lemma that establishes an upper bound for $P_j(k|k)$.

Lemma 5: Suppose the system (1)-(2) is uniformly controllable and uniformly observable, then

$$P_j(k|k) \leq \beta_2 I, \qquad \beta_2 < \infty, k \geq N$$

Proof: Substituting $F_i = [I - K_i^a C_i]A_i$, $G_i(k) = F_i A_i^{-1}[Q_i^e(k) + B_i Q_i B_i^T]A_i^{-T}F_i^T + K_i^a R_i K_i^{a^T}$ into (45) yields

$$T_j^0(k) = \sum_{i=1}^{r} \gamma_{ji}(k)\Big\{F_i T_i^0(k-1)F_i^T + G_i(k)\Big\}$$

By iterations, we have (with a change in the subscript notations)

$$
\begin{aligned}
T^0_{j_{k+1}}(k) = &\sum_{j_k=1}^{r} \dots \sum_{j_s=1}^{r} \gamma_{j_{k+1}j_k}(k) \dots \gamma_{j_{s+1}j_s}(s) \\
&\big[F_{j_k} \dots F_{j_1} T^0_{j1}(0) F^T_{j_1} \dots F^T_{j_k} + \\
&\sum_{s=1}^{k} F_{j_k} \dots F_{j_s} G_{j_{s-1}}(s-1) F^T_{j_s} \dots F^T_{j_k} + G_{j_k}(k)\big]
\end{aligned} \tag{46}
$$

Using Corollary 1, and considering $A(k) = A_{j_k}$, $K(k) = K^a_{j_k}(k)$, etc, there exist gains $K^a_{j_k}(k)$, such that

$$
\|F_{j_k} F_{j_{k-1}} \dots F_{j_{k_0}}\| \le c_0 \lambda_0^{k-k_0} \quad \forall k \ge k_0, c_0 > 0, \lambda_0 < 1 \tag{47}
$$

By taking the 2-norm on (46), and utilizing (22) and (47), we see that $\|T^0_{j_k}(k)\| \le c_2 < \infty$ for all $k > 0$. Hence, $T^0_{j_k}(k)$ is uniformly bounded from above, i.e. $T^0_{j_k}(k) \le c_2 I$. From Lemma 4, it follows that $P^0_{j_k}(k) \le T^0_{j_k}(k) \le c_2 I$. Then, from (34) and (12), we see that $P_j(k|k)$ is uniformly bounded from above.

■

C. Stability conditions for a special case

We define a Lyapunov function for the system (21) as

$$
V(\tilde{e}(k), k) = \max_{j \in \mathcal{Q}} e^T_j(k) P^{-1}_j(k|k) e_j(k) \tag{48}
$$

where $\mathcal{Q} := \{1, 2, \dots, r\}$. From Lemmas 3 and 5, there exist scalars $\kappa_5 > 0$, $\kappa_6 < \infty$ such that

$$
\kappa_5 \max_{j \in \mathcal{Q}} \|e_j(k)\|^2 \le V(\tilde{e}(k), k) \le \kappa_6 \max_{j \in \mathcal{Q}} \|e_j(k)\|^2
$$

Furthermore, it can be shown that [9]

$$
\max_{j \in \mathcal{Q}} \|e_j(k)\|^2 \le \|\tilde{e}(k)\|^2 \le r \max_{j \in \mathcal{Q}} \|e_j(k)\|^2 \tag{49}
$$

Hence, $V(\tilde{e}(k), k)$ satisfies (23). For exponential stability, we need to show that $V(\tilde{e}(k), k)$ also satisfies (24). We first present the following lemmas which would be used later:

Lemma 6: Given any vectors y_i and any positive definite matrices M_i, $i = 1, 2, \dots, r$, of appropriate dimensions; and nonnegative scalars γ_i such that $\sum_{i=1}^{r} \gamma_i = 1$, we have

$$
\Big(\sum_{i=1}^{r} \gamma_i y_i\Big)^T \Big(\sum_{i=1}^{r} \gamma_i M_i\Big)^{-1} \Big(\sum_{i=1}^{r} \gamma_i y_i\Big) \le \sum_{i=1}^{r} \gamma_i y_i^T M_i^{-1} y_i \tag{50}
$$

Lemma 7: For any positive semi-definite matrix M, any nonnegative scalars γ_i such that $\sum_{i=1}^{r} \gamma_i = 1$, and vectors y_i, we have

$$
\sum_{i=1}^{r} \gamma_i y_i^T M y_i \ge \Big[\sum_{i=1}^{r} \gamma_i y_i\Big]^T M \Big[\sum_{i=1}^{r} \gamma_i y_i\Big]
$$

Lemma 8: Consider the system (21). For $j = 1, 2, \dots, r$, we define the functions

$$
V_j(e_j(k), k) := e^T_j(k) P^{-1}_j(k|k) e_j(k) \tag{51}
$$

$$
y_j(k) := A_j \sum_{i=1}^{r} \gamma_{ji}(k-1) e_i(k-1) \tag{52}
$$

$$
u_j(k) := \big[P_j(k|k) P^{-1}_j(k|k-1) - I\big] y_j(k) \tag{53}
$$

Then, we have

$$
\begin{aligned}
V_j(e_j(k), k) \le &\sum_{i=1}^{r} \gamma_{ji}(k-1) V_i(e_i(k-1), k-1) - \\
&e^T_j(k) C^T_j R^{-1}_j C_j e_j(k) - u^T_j(k) P^{-1}_j(k|k-1) u_j(k)
\end{aligned} \tag{54}
$$

for all $k > 0$, $j = 1, 2, \dots, r$.

Proof: From (12) and (51),

$$
V_j(e_j(k), k) = e^T_j(k) \big[P^{-1}_j(k|k-1) + C^T_j R^{-1}_j C_j\big] e_j(k)
$$

Using (33), (34), (52) and (53), we have (see [5] for details)

$$
\begin{aligned}
V_j(e_j(k), k) = &\Big[A_j \sum_{i=1}^{r} \gamma_{ji}(k-1) e_i(k-1)\Big]^T \Big[A_j \\
&\sum_{i=1}^{r} \gamma_{ji}(k-1) P_i(k-1|k-1) A^T_j + Q^e_j(k) + B_j Q_j B^T_j\Big]^{-1} \\
&\Big[A_j \sum_{i=1}^{r} \gamma_{ji}(k-1) e_i(k-1)\Big] - e^T_j(k) C^T_j R^{-1}_j C_j e_j(k) \\
&- u^T_j(k) P^{-1}_j(k|k-1) u_j(k)
\end{aligned}
$$

The inequality (54) can then be proved using Lemma 6 and the fact that $Q^e_j(k) + B_j Q_j B^T_j \ge 0$.

■

In the following, we will show that the IMM algorithm is globally exponentially stable for hybrid systems which satisfy the following conditions:

1) The system (1)-(2) is uniformly controllable.
2) The system (1)-(2) is uniformly observable and satisfies the observability condition in Definition 3 with $N \le 2$.
3) The observation model (2) is the same in all modes, i.e. $C_i = C$ and $R_i = R$ for all $i = 1, 2, \dots, r$.

Condition 3 above is common in hybrid state estimation applications such as target tracking. Condition 2 is more restrictive but it is still applicable in some applications such as that in [10]. We conjecture that Condition 2 could be relaxed to include general controllable hybrid systems which satisfy the observability condition with any finite N, and we hope to extend the result here to the more general case in future.

From Lemma 8, we can deduce the following:

$$
\begin{aligned}
&V_j(e_j(k), k) \le \\
&\sum_{i=1}^{r} \sum_{l=1}^{r} \gamma_{ji}(k-1) \gamma_{il}(k-2) V_l(e_l(k-2), k-2) - J_j(k)
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
J_j(k) = &\sum_{i=1}^{r} \gamma_{ji}(k-1) e^T_i(k-1) C^T R^{-1} C e_i(k-1) + \\
&\sum_{i=1}^{r} \gamma_{ji}(k-1) u^T_i(k-1) P^{-1}_i(k-1|k-2) u_i(k-1) \\
&+ e^T_j(k) C^T R^{-1} C e_j(k) + u^T_j(k) P^{-1}_j(k|k-1) u_j(k)
\end{aligned} \tag{56}
$$

Using Lemma 7 and (52), we can write (56) as

$$
\begin{aligned}
J_j(k) \geq\, & y_j^T(k)A_j^{-T}C^TR^{-1}CA_j^{-1}y_j(k)+ \\
& \sum_{i=1}^{r}\gamma_{ji}(k-1)u_i^T(k-1)P_i^{-1}(k-1|k-2)u_i(k-1) \\
& + e_j^T(k)C^TR^{-1}Ce_j(k) + u_j^T(k)P_j^{-1}(k|k-1)u_j(k)
\end{aligned}
\tag{57}
$$

Let

$$
M_j := A_j^{-T}C^TR^{-1}CA_j^{-1} \tag{58}
$$

From (20), (52) and (53), we have

$$
y_j(k) = e_j(k) - u_j(k) \tag{59}
$$

Substituting (58) and (59) into (57), we have (after simplifying the notations by $e_j = e_j(k)$, $u_{io} = u_i(k-1)$ etc.)

$$
\begin{aligned}
J_j \geq & (e_j-u_j)^TM_j(e_j-u_j) + \sum_{i=1}^{r}\gamma_{ji}u_{i0}^TP_{i0}^{-1}u_{i0} \\
& + e_j^TC^TR^{-1}Ce_j + u_j^TP_j^{-1}u_j := \underline{J}_j
\end{aligned}
\tag{60}
$$

We now consider the problem of minimizing $\underline{J}_j$ with respect to the variables u_{io} and u_j. By differentiation,

$$
\frac{\partial \underline{J}_j}{\partial u_{io}} = 2\gamma_{ji}u_{i0}^TP_{i0}^{-1} \qquad i = 1,2,\ldots,r
$$

$$
\frac{\partial \underline{J}_j}{\partial u_j} = -2(e_j-u_j)^TM_j + 2u_j^TP_j^{-1}
$$

Putting the first derivatives to zero, we have

$$
\gamma_{ji}u_{i0}^* = 0 \qquad i = 1,2,\ldots,r \tag{61}
$$

$$
u_j^* = (M_j + P_j^{-1})^{-1}M_je_j \tag{62}
$$

Substituting (61) and (62) into (60), the minimum $\underline{J}_j$ is

$$
\begin{aligned}
\underline{J}_j^* =& e_j^T[I-(M_j+P_j^{-1})^{-1}M_j]^TM_j \\
& [I-(M_j+P_j^{-1})^{-1}M_j]e_j + e_j^TC^TR^{-1}Ce_j+ \\
& e_j^TM_j(M_j+P_j^{-1})^{-1}P_j^{-1}(M_j+P_j^{-1})^{-1}M_je_j \\
=& e_j^T[M_j - M_j(M_j+P_j^{-1})^{-1}M_j]e_j + e_j^TC^TR^{-1}Ce_j
\end{aligned}
\tag{63}
$$

Using (58) and the matrix inversion lemma,

$$
\begin{aligned}
& [M_j - M_j(M_j+P_j^{-1})^{-1}M_j] \\
&= A_j^{-T}C^T[R^{-1} - R^{-1}CA_j^{-1}(A_j^{-T}C^TR^{-1}CA_j^{-1}+ \\
& \quad P_j^{-1})^{-1}A_j^{-T}C^TR^{-1}]CA_j^{-1} \\
&= A_j^{-T}C^T[R + CA_j^{-1}P_jA_j^{-T}C^T]^{-1}CA_j^{-1}
\end{aligned}
$$

Using Lemma 5, we can show that $P_j = P_j(k|k-1)$ is uniformly bounded from above, i.e. $P_j \leq \beta_3 I$ where $0 < \beta_3 < \infty$. The matrices C, R and A_j are constant and bounded. Thus, it can be shown that $[R + CA_j^{-1}P_jA_j^{-T}C^T]^{-1} \geq \beta_4 I$ where $\beta_4 > 0$.Therefore, from (63),

$$
\underline{J}_j^* \geq \beta_4 e_j^TA_j^{-T}C^TCA_j^{-1}e_j + e_j^TC^TR^{-1}Ce_j
$$

From (3), $A_j^{-T}C^TR^{-1}CA_j^{-1} \leq \xi_3^{-1}A_j^{-T}C^TCA_j^{-1}$. Hence

$$
\begin{aligned}
\underline{J}_j^* &\geq \beta_4\xi_3 e_j^TA_j^{-T}C^TR^{-1}CA_j^{-1}e_j + e_j^TC^TR^{-1}Ce_j \\
&\geq \beta_5 e_j^T[A_j^{-T}C^TR^{-1}CA_j^{-1} + C^TR^{-1}C]e_j
\end{aligned}
$$

where $\beta_5 = \min(\beta_4\xi_3, 1) > 0$. If Condition 2 holds, then

$$
A_j^{-T}C^TR^{-1}CA_j^{-1} + C^TR^{-1}C \geq \kappa_3 I
$$

where $\kappa_3 > 0$. Thus,

$$
\underline{J}_j^* \geq \beta_5\kappa_3\|e_j\|^2
$$

From (55), we have

$$
V_j(e_j(k),k) \leq \max_{l\in\mathcal{Q}} V_l(e_l(k-2),k-2) - \beta_5\kappa_3\|e_j\|^2
$$

Using Lemma 3, we have

$$
V_j(e_j(k),k) \leq \max_{l\in\mathcal{Q}} V_l(e_l(k-2),k-2) - c_1V_j(e_j(k),k)
$$

where $c_1 = \beta_1\beta_5\kappa_3 > 0$. Hence,

$$
\begin{aligned}
& \max_{j\in\mathcal{Q}} V_j(e_j(k),k) \leq \\
& \quad \max_{l\in\mathcal{Q}} V_l(e_l(k-2),k-2) - \max_{j\in\mathcal{Q}} c_1V_j(e_j(k),k) \\
\Rightarrow\quad & \tilde{V}(\tilde{e}(k),k) - \tilde{V}(\tilde{e}(k-2),k-2) \leq -c_1\tilde{V}(\tilde{e}(k),k) \\
& \qquad\qquad\qquad\qquad\qquad\qquad \leq -c_2\|\tilde{e}(k)\|^2
\end{aligned}
$$

Thus, if Conditions 1-3 holds, the Lyapunov function $\tilde{V}(\tilde{e}(k),k)$ satisfy (23) and (24). By Lyapunov's stability theorem, the IMM algorithm is globally exponentially stable.

IV. Conclusions

We have derived bounds on the error covariance and presented sufficient conditions for the exponential stability of the IMM algorithm. We are currently working on the possible extension of the current stability conditions to general controllable and observable hybrid systems.

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