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A Linear Control System Model for Risk Reserves

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***Abstract*— A discrete time, linear, stochastic control system is constructed to model the risk reserves for an insurance company. The model has the autoregressive form. A control is used to regulate the risk reserve. The sequence of controls is determined by two approximations, the normal power approximation of order two and a log normal approximation. These approximations use the first three moments which incorporate the skewness of the distributions that is important for these problems. An example of automobile insurance is considered to compare the two approximations for the stationary control law. It is shown that the two approximations are given stationary controls that closely agree.**

I. Introduction

The risk that is specific for the insurance companies is the underwriting risk. This risk originates in the insurance business to provide coverage against unfavorable events by aggregating a multitude of entities exposed to the same perils. It is generally accepted that insurance is practicable because of the Law of Large Numbers of probability theory. However, this statement should be made more precise. Insurance is possible because of the Law of Large Numbers and of the risk margin charged to the premiums. This risk margin is used to establish risk reserves. Insurance regulators require the insurance companies to hold sufficient capital to guarantee that the probability of not meeting their obligations to the insured is very small. In a mathematical formulation this means that the regulators impose shortfall constraints.

Linear control theory was apparently initially applied to the modelling of the reserve funds for a property/casualty insurance company by Martin-Löf ([4]). Even at that time an intensive development of computer simulation models of the insurance industry had been initiated. Most of the current large scale models concentrate on financial modelling and on testing deterministic scenarios. Nonetheless, as it is documented in [2] on workmen's compensation insurance, stochastic simulation models are required for the evaluation of the uncertainty in the claims' reserves. Therefore it is important to consider this facet of insurance models and to devote research to their mathematical aspects. It has been noted that the scope of the computational risk theory methods is wider than was expected when some approximations to probability distributions, that were a substantial part of risk theory until the 1980's, are employed.

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This paper is devoted to a description of risk reserves as linear systems. An extension of the linear, quadratic control model to a model using moments including the third order is described that seems to make it applicable to a larger class of situations.

Although a property/casualty insurance company is considered, the subsequent methods have wider applicability.

II. Reserve Fund

A *reserve fund* is understood to be a collection of reserves designated for the coverage of losses caused by a specified class of contingencies. It is considered to be part of the liabilities in the account balance of an insurer. The assets of the fund are considered as a part of the financial placements (investments) of the insurer. The fund is maintained by contributions representing in a special case the received insurance premiums after deduction for expenses, dividends, etc.

The evolution of the fund is considered in a sequence of periods (e.g., accounting years). An important feature is that the expenses from casualties occurring in one period are often paid during several subsequent periods due to the delays in claim settlement and claim evidence or because the payments are life annuities.

At the closing of the accounts at the end of each period, the fund is subdivided into three main parts:

1) Contributions reserve
2) Loss reserve
3) Risk reserve.

The contributions often contain a fixed amount of prepaid coverage. This liability is matched by the contributions reserve. The liabilities from the events that have occurred in the past are to be covered by the loss reserve. Finally the risk reserve is a provision for the discrepancies between the actual payments and their estimates. The values of these three reserves at the end of period t are denoted S_t, T_t, and U_t, respectively. The corresponding share of the company's assets, A_t, is

$$A_t = S_t + T_t + U_t \ . \tag{1}$$

The return on the invested assets is an important factor in the financial results of an insurer. This return is described by a constant rate of return i on the assets of the fund. The use of some time dependent rates can be included without difficulty.

Let B_t be the total contributions and X_t be the total paid loss during period t. If β and γ are the average dates of payments of the contributions and of the costs within the period, then $(A_t, t \in \mathbb{N})$ satisfies

$$A_t = A_{t-1}(1+i) + B_t(1+i)^{1-\beta} - X_t(1+i)^{1-\gamma} . \quad (2)$$

If both of these payments are evenly distributed within a period then

$$\beta = \gamma = \frac{1}{2} .$$

A detailed description of the various terms in (2) is given in the subsequent sections.

The modelling begins with the period 1 and continues for $t = 2, 3, \dots$ The claims paid and the contributions made in the preceding periods, denoted $t = 0, -1, -2, \dots$, determine the initial values of the respective reserves.

III. Contributions Reserve

For the sake of simplicity, it is assumed that the contributions are made for the coverage of risks during at most one period in advance. The contributions $(B_t, t = 0, 1, 2, \dots)$ can be split into two parts as

$$B_t = B_t^0 + B_t^1 \quad (3)$$

where B_t^0 is the part that is earned by risk coverage during period t and B_t^1 is the unearned part that is entirely consumed during the next period. If there is a stability in the distribution of the risks and the contributions during the periods, then

$$B_t^0 = (1-\alpha)B_t$$
$$B_t^1 = \alpha B_t .$$

The contributions reserve that occurs at the end of the period t coincides with B_t^1, that is, $S_t = B_t^1$.

The sequences $(B_t, t \in \mathbb{N})$, $(B_t^0, t \in \mathbb{N})$, and $(B_t^1, t \in \mathbb{N})$ are assumed to be deterministic scenarios because in contrast to the claims process there are no important reasons to use probabilistic methods.

IV. Loss Reserve

The loss payments $(Y(t,j))$ in the periods $1, 2, \dots, n$ can be expressed in a triangular array as

$$\begin{array}{l} Y(1,0), Y(1,1), \dots, Y(1,n-2), Y(1,n-1) \\ Y(2,0), Y(2,1), \dots, Y(2,n-2) \\ \quad\vdots \\ Y(n,1) . \end{array}$$

The first argument denotes the period of origin and the second argument denotes the delay in payment or the so-called development period, that is, $Y(t,j)$ is the amount paid in the period $t+j$ for the costs that were incurred in period t.

The stability in the loss payments is given by assuming that the payments in the development periods $1, 2, \dots,$ are, up to an error term, multiples of the payment $Y_t = Y(t,0)$. Thus the delayed payments are described by the following equation

$$Y(t,j) = d_j(Y_t + e(t,j)) \quad (4)$$

for $j = 1, 2, \dots$ where $(d_j, j = 1, 2, \dots)$ are constant proportionality factors such that $d_j = 0$ for $j > r$. The quantities Y_t and $e(t,j)$ for $t = 1, 2, \dots$ and $j = 1, 2, \dots, r$ are assumed to be mutually independent random variables such that

$$Ee(t,j) = 0$$

for $t = 1, 2, \dots$ and $j = 1, 2, \dots, r$.

Thus the total loss payments during period t, X_t, can be given as

$$X_t = Y_t + Y(t-1,1) + \dots + Y(t-r,r) . \quad (5)$$

Equation (2) can be rewritten using (5) as

$$\begin{aligned} A_t &= A_{t-1}(1+i) + B_t(1+i)^{1-\beta} \\ &- (Y_t + d_1 Y_{t-1} + \dots + d_r Y_{t-r})(1+i)^{1-\gamma} - Z_t(1+i)^{1-\gamma} \end{aligned} \quad (6)$$

where

$$Z_t = d_1 e(t,1) + \dots + d_r e(t-r,r) .$$

From (4) it follows that at the end of the period t the expected payments to be made for the losses that were incurred in the past can be expressed in the following triangular array

$$\begin{array}{lllll} d_1 Y_t, & d_2 Y_t, & \dots, & d_{r-1} Y_t, & d_r Y_t \\ d_2 Y_{t-1}, & d_3 Y_{t-1}, & \dots & d_{r-1} Y_{t-1} & \\ \vdots & & & & \\ d_r Y_{t-r+1} . & & & & \end{array}$$

The rows of this array contain the expected payments of loss originating in periods $t, t-1, \dots, t-r+1$, the columns contain the expected payments to be made in the periods $t+1, t+2, \dots, t+r$. To determine the loss reserve T_t the payments must be discounted at the rate i and with regard to the average payment date γ.

Let v be the discount factor given by

$$v = (1+i)^{-1} .$$

Thus T_t can be expressed as

$$\begin{aligned} T_t &= Y_t \left[d_1 v^\gamma + d_2 v^{1+\gamma} + \dots + d_{r-1} v^{r-2+\gamma} + d_r v^{r-1+\gamma} \right] \\ &+ Y_{t-1} \left[d_2 v^\gamma + d_3 v^{1+\gamma} + \dots + d_r v^{r-2+\gamma} \right] + \dots + Y_{t-r+1} d_r v^\gamma . \end{aligned} \quad (7)$$

V. Risk Reserve

From (6) and (7) it follows that

$$\begin{aligned} A_t - T_t &= (1+i)(A_{t-1} - T_{t-1}) + B_t(1+i)^{1-\beta} \\ &- Y_t(v^{\gamma-1} + d_1 v^\gamma + \dots + d_r v^{r-1+\gamma}) - Z_t(1+i)^{1-\gamma} . \end{aligned} \quad (8)$$

Since U_t can be expressed as

$$U_t = A_t - T_t - S_t$$

and

$$S_t = B_t^1$$

it follows that

$$U_t = (1+i)U_{t-1} + (1+i)\left(B_t v^{\beta} + B^1_{t-1} - B^1_t v\right) \\ - Y_t\left[v^{\gamma-1} + d_1 v^{\gamma} + \cdots + d_r v^{r-1+\gamma}\right] - Z_t(1+i)^{1-\gamma} . \quad (9)$$

The second term on the right hand side is not random and the other terms are mutually independent random variables. The equation (9) for U_t is a linear autoregressive model for $(U_t, t \in \mathbb{N})$. Let b_t and η_t be given by

$$b_t = B_t(v^{\beta-1} - \alpha) + B_{t-1}\alpha v^{-1}$$

and

$$\eta_t = Y_t\left[v^{\gamma-1} + d_1 v^{\gamma} + \cdots + d_r v^{r-1+\gamma}\right] + Z_t(1+i)^{1-\gamma} .$$

The sequence of random variables $(\eta_t, t \in \mathbb{N})$ is mutually independent.

Let

$$U_0 = u$$

be the initial risk reserve. The risk reserve provides protection against the adverse developments in the cost. Clearly it is important that the family of probabilities

$$P(U_t < 0)$$

should be sufficiently small for a given number of periods. Let the number of periods be n and let the smallness be described by $\varepsilon > 0$, that is, $P(U_t < 0) < \varepsilon$.

A special property of the distributions encountered in risk theory is their nonnegligible skewness. It is proposed to estimate the probability $P(U_t < 0)$ using the first three moments of U_t, that is,

$$E(U_t), \quad \mathrm{Var}(U_t), \quad \mathrm{Thr}(U_t) = E[(U_t - EU_t)^3]$$

for $t = 1, \ldots, n$. It is an elementary property that if φ and ψ are independent random variables with finite third moments then

$$\mathrm{Var}(\varphi + \psi) = \mathrm{Var}(\varphi) + \mathrm{Var}(\psi)$$
$$\mathrm{Thr}(\varphi + \psi) = \mathrm{Thr}(\varphi) + \mathrm{Thr}(\psi) .$$

Thus linear recursions for the first three moments of U_t can be obtained from the corresponding moments of η_t for $t = 1, \ldots, n$.

VI. Composite Risk Reserve

Equation (9) is also valid for composite reserves. Let U^m_t for $m = 1, 2, \ldots, p$ satisfy

$$U^m_t = (1+i)U^m_{t-1} + b^m_t - \eta^m_t \quad (10)$$

for $t = 1, 2, \ldots$ Then U_t is given by

$$U_t = \sum_{j=1}^{p} U^j_t .$$

The equation (9) is obtained by adding the family of equations (10) for $m = 1, \ldots, p$.

VII. Approximations of Skew Distributions

A common method in risk theory to approximate the solution u_ε of the equation

$$P(U < u_\varepsilon) = \varepsilon \quad (11)$$

is the normal power approximation of order 2 (the NP-2 approximation). To explain briefly the NP-2 approximation let ξ be the standardized random variable given by

$$\xi = \frac{U - E(U)}{\sqrt{\mathrm{Var}(U)}}$$

and let F be the distribution function of ξ and χ_ε be the solution of $F(\chi_\varepsilon) = \varepsilon$. Let Φ be the standard normal distribution and $\varphi = \Phi'$ be the corresponding probability density function. Let $\delta(\cdot)$ be a function such that

$$F(x + \delta(x)) = \Phi(x) \quad (12)$$

in the neighborhood of the ε-quantile z_ε of Φ. Assuming that δ is slowly varying it follows that the following approximate equality is valid

$$F(x) \approx \Phi(x - \delta(x)) \approx \Phi(x) - \varphi(x)\delta(x). \quad (13)$$

The two term Gram-Charlier expansion of F implies that

$$F(x) \approx \Phi(x) - \frac{\gamma_1}{6}(x^2 - 1)\varphi(x) \quad (14)$$

where

$$\gamma_1 = E\xi^3 .$$

Comparing the two approximations for F it follows that δ can be chosen as

$$\delta(x) = \frac{\gamma_1}{6}(x^2 - 1) \quad (15)$$

and hence

$$F\left(z_\varepsilon + \frac{\gamma_1}{6}(z_\varepsilon^2 - 1)\right) \approx \Phi(z_\varepsilon) = \varepsilon \quad (16)$$

$$\chi_\varepsilon \approx z_\varepsilon + \frac{\gamma_1}{6}(z_\varepsilon^2 - 1) \quad (17)$$

Another approach is to approximate x_ε by the ε-quantile of a standardized distribution having the same skewness as ξ. The use of

$$P(a \pm \exp[\mu + \sigma\eta] < u_\varepsilon) = \varepsilon \quad (18)$$

with η a standard normal random variable where the plus sign is used if $\gamma_1 > 0$ and the minus sign is used for $\gamma_1 < 0$, often gives estimates that are close to the NP-2 approximation. The left hand side of the inequality in the probability in (18) is a standardized random variable with the skewness γ_1 if the following are satisfied

$$a \pm \sqrt{q}\exp(\mu) = 0 \quad (19)$$
$$q(q-1)\exp(2\mu) = 1 \quad (20)$$
$$(q-1)(q+2)^2 = \gamma_1^2 = \frac{(\mathrm{Thr}(U))^2}{(\mathrm{Var}(U))^3} \quad (21)$$

where

$$q = \exp(\sigma^2) .$$

If γ_1 is given then q is obtained from (21) by the Cardano formula.

It follows from (16) that

$$u_\varepsilon \approx EU + \sqrt{\mathrm{Var}(U)}\left(z_\varepsilon + \frac{\gamma_1}{6}(z_\varepsilon^2 - 1)\right) . \qquad (22)$$

Since the inequality

$$P(U < 0) \leq \varepsilon$$

is equivalent to the inequality

$$u_\varepsilon \geq 0 . \qquad (23)$$

(22) changes (23) to

$$EU + z_\varepsilon \sqrt{\mathrm{Var}(U)} + \frac{\mathrm{Thr}(U)}{6\mathrm{Var}(U)}(z_\varepsilon^2 - 1) \geq 0 . \qquad (24)$$

Similarly the log normal approximation implies that

$$u_\varepsilon \approx EU \pm \frac{\sqrt{\mathrm{Var}(U)}}{\sqrt{q-1}}(\exp(\pm z_\varepsilon \sqrt{\log q})/\sqrt{q} - 1) . \qquad (25)$$

VIII. Control of the Risk Reserve

The contributions to the reserve contain mostly a risk loading. In the course of time this would cause an excessive growth of the risk reserve if no funds were withdrawn from it. This problem, more precisely the reduction in the contributions, is the main topic in [4]. In this section, regular withdrawals are investigated using the models described above.

A linear control of the risk reserve is used for its reduction at the beginning of each period by the amount

$$j_t(U_{t-1} - k_t) .$$

Thus the control is given by the sequence of pairs $(j_t, k_t; t \in \mathbb{N})$ of gain factors j_t and levels k_t. Equation (9) is modified as follows

$$U_t = (1+i)(1-j_t)U_{t-1} + (1+i)j_t k_t + b_t - \eta_t . \qquad (26)$$

For simplicity the time homogeneous case is studied, that is,

$$\begin{aligned} b_t &= b \\ E\eta_t &= E\eta \\ \mathrm{Var}(\eta_t) &= \mathrm{Var}(\eta) \\ \mathrm{Thr}(\eta_t) &= \mathrm{Thr}(\eta) \\ j_t &= j \\ k_t &= k . \end{aligned}$$

Let f and g be given by

$$\begin{aligned} f &= b - E\eta \\ g &= (1+i)(1-j) . \end{aligned}$$

If $|g| < 1$ then the stationary values of the three moments of $(U_t, t \in \mathbb{N})$ are given by

$$\begin{aligned} EU &= \frac{(1+i)jk + f}{1-g} \\ \mathrm{Var}(U) &= \frac{\mathrm{Var}(\eta)}{1-g^2} \\ \mathrm{Thr}(U) &= \frac{\mathrm{Thr}(\eta)}{1-g^3} . \end{aligned}$$

Since the negative values of U_t should have small probability, for a given factor j it is important to determine the level k such that

$$P(U < 0) = \varepsilon .$$

Using (22) this level k is estimated by the NP-2 approximation as

$$k \approx \left[\frac{\left((1-g)\left(\sqrt{\mathrm{Var}(U)}z_{1-\varepsilon} - \frac{\mathrm{Thr}(U)}{6\mathrm{Var}(U)}(z_{1-\varepsilon}^2 - 1)\right) - f\right)}{j(1+i)}\right] \qquad (27)$$

where

$$\Phi(z_{1-\varepsilon}) = 1 - \varepsilon .$$

If the log normal approximation is used then

$$k \approx \frac{\left[\frac{\sqrt{\mathrm{Var}(U)}}{\sqrt{q-1}}(\exp(z_{1-\varepsilon}\sqrt{\ln q})/\sqrt{q} - 1)(1-g) - f\right]}{j(1+i)} . \qquad (28)$$

For the determination of j there are two concurrent criteria. The mean percentage withdrawn from the reserve

$$j(EU - k)/[EU - j(EU - k)] \qquad (29)$$

should be large. On the other hand, the probability of having to add funds to the reserve if

$$U_t < k$$

should be small. The log normal approximation implies that

$$P(U_t < k) \approx 1 - \Phi\left(\ln\left[\frac{EU - k}{\sqrt{\mathrm{Var}(U)}} \cdot \sqrt{q}(\sqrt{q-1} + 1)/\sqrt{\ln q}\right]\right) . \qquad (30)$$

An application of this approach is given in the following.

IX. Example

This example uses the data of one scenario from an investigation of the Motor Third Party Liability Insurance in the Czech Republic [3]. The risk volume is one million vehicles and the premiums are paid for a calendar year and are due by the end of February. The amounts are given in millions of Czech crowns that are rounded off to the first decimal place.

The time homogeneous case is considered. The yearly contributions (i.e., the total premiums after the deductions for administration and operating expenses) are

$$B = 822.5 .$$

No contributions reserve is made.

The moments of Y_t for the given risk volume are

$$EY_t = 319.9$$
$$\mathrm{Var}(Y_t) = 2603.7$$
$$\mathrm{Thr}(Y_t) = 116484.1\ .$$

For modelling the loss development as in (9) the following values are chosen

$$d_1 = .80$$
$$d_2 = .30$$
$$d_3 = .15$$
$$d_4 = .10$$
$$d_5 = .05$$
$$d_6 = d_7 = \cdots = 0\ .$$

Furthermore,

$$\mathrm{Var}(e(t,j)) = 4\mathrm{Var}(Y_t)$$
$$\mathrm{Thr}(e(t,j)) = 0$$

and the rate of return is

$$i = .05\ .$$

From these assumptions it follows that

$$E\eta_t = 749.0$$
$$\mathrm{Var}(\eta_t) = 22640.1$$
$$\mathrm{Thr}(\eta_t) = -1495304.7\ .$$

The following table contains the values of (27), (28), (29), and (30) together with the expected value, the standard deviation and the skewness of U for the values of the gain $j = 0.4$, 0.3, 0.25, 0.1972. To illustrate the performance of the stationary controls in a finite time horizon, the initial reserve that is needed to guarantee $P(U < 0) = \varepsilon$ with $\varepsilon = 0.01$ during ten years was computed for the controls in the second and the fourth column of Table 1. The initial reserves and the expected values of the risk reserve resulting from the NP-2 approximation and the log normal approximation are given in Table 2. In these cases there is a remarkable agreement of the results that are obtained by the two methods.

In recent years, a development of control theory applications in insurance has occurred and it is hoped that this paper will stimulate further developments of control theory having applications in the risk management of an insurance company.

Table 1

j	0.4	0.3	0.25	0.1972
(27)	175.0	123.5	78.6	-0.1
(28)	175.3	123.7	78.8	0.1
(29)	34.6%	30.4%	27.8%	24.6%
(30)	5.9%	3.2%	2.2%	1%
EU	490.0	553.45	604.2	685.8
$\sqrt{\mathrm{Var}(U)}$	193.8	221.9	244.1	279.7
γ_1	−0.2741	−0.2270	−0.2008	−0.1704

Table 2

$j = .3$	0	1	2	3	4	5
$k = 123.5$	413.3	541.7	544.7	547.0	548.6	549.8
$k = 123.7$	414.0	542.4	545.3	547.5	549.0	550.2
$j = .3$	6	7	8	9	10	
$k = 123.5$	550.7	551.4	551.8	552.2	552.5	
$k = 123.7$	551.1	551.7	552.1	552.5	552.7	
$j = .1972$	0	1	2	3	4	5
$k = 0$	497.7	630.3	639.0	646.3	652.5	657.7
$k = 0$	498.8	631.4	639.9	647.1	653.2	658.3
$j = .1972$	6	7	8	9	10	
$k = 0$	662.1	665.8	668.9	671.6	673.8	
$k = 0$	662.6	666.2	669.3	671.8	674.0	

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