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On the stability radii of continuous-time Markov jump linear systems

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Abstract— **This paper introduces the subject of stability radii for continuous-time infinite Markov jump linear systems (MJLS) with respect to complex unstructured perturbations. Among the results derived here we highlight: lower and upper bounds for the radius; a connection between the radius and a certain robust stability margin with respect to uncertainty on the transition rate matrix of the Markov process; and an explicit formula for the stability radius of two-mode scalar MJLS. Some examples are addressed in an attempt to illustrate the applicability of the main results, as well as some of their limitations.**

I. INTRODUCTION

Markov jump linear systems (MJLS) constitute an important class of hybrid systems, which have attracted considerable interest of the control research community over the last three decades or so. Its wide potential of applicability has already been illustrated by many applications in safety-critical and high-integrity systems (e.g., aircraft, chemical plants, nuclear power station, robotic manipulator systems, large scale flexible structures for space stations such as antenna, solar arrays, etc.), that is, systems which may experience abrupt changes in their structure (see, for instance, [1] and references therein).

Different studies of robust stability and stabilization problems for MJLS can be found from early references such as [2], [3] until more recent papers such as, e.g., [4]–[7]. Of particular interest here is the subject of *stability radii* of MJLS, which was previously considered, for example, in [8]–[13] (this last one being presently the only reference to have addressed the so-called *infinite* case, in which the Markov chain is assumed to take values in a countably infinite set). We remark that none of these papers, however, did consider the specific case where only unstructured perturbations may occur on the system model, which is our main concern in the present work.

This paper is devoted to the study of stability radii of continuous-time infinite MJLS in the *unstructured* case. Different from [13], in which the robust stabilization of infinite MJLS subjected to the more general structured uncertainties was tackled mainly with the aid of H_∞ control theory (the one from [14]), here we assume that only perturbations such as $A_i \rightsquigarrow A_i + \Delta_i$ (where A_i is the system matrix of the i-th subsystem) may affect the system. This, in a parallel to [15], [16], calls for an investigation of the stability criteria which apply to MJLS (such as, for instance, some results from the Lyapunov theory of [17] or the spectral approach presented in [18]).

Another problem of interest that we address here is that which come up when uncertainties in the transition rate matrix of the Markov process are allowed to occur. Although this issue has already been considered in the literature within a quite general framework (we refer to [5]), for the *finite* case, one novelty to be found here is a connection between the stability radius and a margin for robust stability with respect to uncertainty on the transition rate matrix of the Markov process. Moreover, the obtained results suggest that quite precise estimates could be attained if we were to take different definitions of stability radii into consideration, indicating a potentially fruitful topic for further research.

This paper is organized as follows. In section II we provide the bare essentials of notation to be employed in the rest of the paper. Section III introduces a fundamental model together with the notions of stability to be adopted, while in section IV some novel issues regarding the complex stability radii of infinite MJLS are presented. In section V attention is given to the problem of uncertainty on the transition rate matrix and its connection with the stability radius at hand. Finally, section VI presents specific results regarding two-mode jump systems, being followed by a few concluding remarks at section VII.

II. NOTATION

Let $\|\cdot\|$ denote the euclidean norm in the complex n-space $\mathbb{C}^n$. We write $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ as the Banach space of all matrices $M \in \mathbb{C}^{n\times m}$, equipped with the standard induced matrix norm, $\|\cdot\|$. Let also $\lambda(M)$ denote the spectrum of $M \in \mathbb{M}(\mathbb{C}^n, \mathbb{C}^n)$, with complex eigenvalues $\lambda_i(M)$, $i = 1, \dots, n$ and maximal real part $\mathbb{R}_e\{\lambda(M)\} := \max\{\mathbb{R}_e\{\lambda_i\} : \lambda_i \in \lambda(M)\}$. We define $\mathcal{S} := \{1, 2, \dots\}$ (unless otherwise stated) and, given any complex matrix N, denote by $M\otimes N$ and $M\oplus N$ the Kronecker product and sum, respectively (see, e.g., [19]). We also denote the complex conjugate, transpose, and conjugate transpose of such M by $\bar{M}$, M', and M^*, respectively, and let $\lambda_{\max}(M)$ stand for the greatest eigenvalue of $M = M^*$.

Let us introduce the infinite dimensional Banach space $\mathbb{H}^{m,n}_{\sup}$ of all matrices of the form $H = (H_1, H_2, \dots)$ with $H_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ for every $i \in \mathcal{S} := \{1, 2, \dots\}$, such that $\|H\|_{\sup} := \sup_{i\in\mathcal{S}} \|H_i\| < \infty$. We further write $\mathbb{H}^n_{\sup}$ in place of $\mathbb{H}^{n,n}_{\sup}$ and define $\tilde{\mathbb{H}}^{n+}_{\sup}$ as the set composed by all matrices $H = (H_1, H_2, \dots) \in \mathbb{H}^n_{\sup}$ such that $H_i^* = H_i \geq \varepsilon I_n$ for all $i \in \mathcal{S}$ and some $\varepsilon > 0$ independent of i (here I_n stands for the $n \times n$ identity matrix). Accordingly, we write that

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$L \in \tilde{\mathbb{H}}^{n-}_{\sup}$ whenever $-L \in \tilde{\mathbb{H}}^{n+}_{\sup}$. In addition, we introduce *scaling parameters* such as $\breve{\alpha} = (\alpha_1, \alpha_2, \ldots) \in \mathbb{H}^1_{\sup}$ such that $\alpha_i > 0$ for every $i \in \mathcal{S}$, or $\breve{\alpha} \succ 0$ for short, and for any $M \in \mathbb{H}^n_{\sup}$, we define $\breve{\alpha}M := (\alpha_1 M_1, \alpha_2 M_2, \ldots) \in \mathbb{H}^n_{\sup}$. Finally, we denote by $\mathrm{diag}(M_i)$ an infinite-sized matrix with block diagonal entries $M_1, M_2, \ldots$, and all the other entries equal to zero.

Concerning the random objects, we fix a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ carrying a right-continuous filtration $\mathfrak{F}_t \subset \mathfrak{F}$ on $t \in \mathbb{R}_+ := [0, \infty)$. Additionally, we denote by $E(\cdot)$ the usual mathematical expectation and define L^n_2 as the space of all second order random variables $(\Omega, \mathfrak{F}) \mapsto \mathbb{C}^n$.

III. Model Setting and Preliminaries

Consider in $(\Omega, \mathfrak{F}, \mathbb{P})$ a homogeneous Markov process $\theta = \{(\theta_t, \mathfrak{F}_t), t \in \mathbb{R}_+\}$, with right continuous sample paths and state space $\mathcal{S} \subseteq \mathbb{N}$, such that

$$\mathbb{P}(\theta_{t+dt} = j | \theta_t = i) = \begin{cases} \lambda_{ij} dt + o(dt), & i \neq j, \\ 1 + \lambda_{ii} dt + o(dt), & i = j, \end{cases} \quad (1)$$

where $0 \leq \lambda_{ij}$ for $i \neq j$, and $0 \geq \lambda_{ii} = -\sum_{j \in \mathcal{S} \setminus \{i\}} \lambda_{ij}$ for all $i \in \mathcal{S}$. Most of the time we will be dealing with the countably infinite case $\mathcal{S} = \{1, 2, \ldots\}$, and put the additional *conservativeness* hypothesis that there is $\nu < \infty$ such that $|\lambda_{ii}| < \nu$ for each $i \in \mathcal{S}$. The initial condition $\theta_0 : \Omega \to \mathcal{S}$ is assumed to be a random variable with fixed distribution π_0.

With θ defined in this way consider the following system:

$$\{A, \Lambda\} : \qquad \dot{x}(t) = A_{\theta_t} x(t), \quad t \in \mathbb{R}_+, \quad (2)$$

with initial condition $x(0) = x_0 \in L^n_2$. We refer to this MJLS by the pair $\{A, \Lambda\}$, where $A = (A_1, A_2, \ldots) \in \mathbb{H}^n_{\sup}$ is the *state matrix* associated with (2) and $\Lambda = [\lambda_{ij}]$ is the *transition rate matrix* of the jump process. For the sake of brevity we shall denote this latter fact by $\theta \sim \Lambda$.

Before proceeding to the introduction of stability radii for $\{A, \Lambda\}$, in the next section, it is necessary to first present some basic facts and results concerning this system. To begin with, let us introduce the following notion of *nominal* L_2-*stability*, on the basis of a widely adopted terminology in the literature for MJLS.

Definition 1: System $\{A, \Lambda\}$ is said to be stochastically stable (SS, or just *stable*) if, for any initial condition $x_0 \in L^n_2$ and initial distribution π_0, we have that

$$\int_0^\infty E[\|x(t)\|^2] dt < \infty. \quad (3)$$

$\triangledown\triangledown$

The following lemma summarizes two important results borrowed from [18], regarding the stability of system $\{A, \Lambda\}$. Namely, it states that SS holds if and only if the entire spectrum of an augmented infinite dimensional matrix lies in the open left half plane or, equivalently, if and only if an infinite set of interconnected Lyapunov inequalities is feasible.

Lemma 1: Let $\boldsymbol{\mathcal{A}}(A, \Lambda) := \Lambda' \otimes I_{n^2} + \mathrm{diag}(\bar{A}_i \oplus A_i)$. Then system $\{A, \Lambda\}$ is SS if and only if one of the following equivalent conditions hold:

(i) $\mathbb{R}_e\{\lambda(\mathcal{A})\} := \sup\{\mathbb{R}_e\{\lambda_i\} : \lambda_i \in \lambda(\mathcal{A})\} < 0$, where

$$\mathcal{A} := \boldsymbol{\mathcal{A}}(A, \Lambda). \quad (4)$$

(ii) There is $P = (P_1, P_2, \ldots) \in \tilde{\mathbb{H}}^{n-}_{\sup}$ such that $\mathcal{T}(P) = (\mathcal{T}_1(P), \mathcal{T}_2(P), \ldots) \in \tilde{\mathbb{H}}^{n+}_{\sup}$, where

$$\mathcal{T}_i(P) := A_i^* P_i + P_i A_i + \sum_{j \in \mathcal{S}} \lambda_{ij} P_j, \quad i \in \mathcal{S} \quad (5)$$

$\triangledown\triangledown$

Notice that both Definition 1 and Lemma 1 deal only with the *nominal* stability of system $\{A, \Lambda\}$. In the next sections we shall be concerned with the main subject of this paper, which is the robust stability of system $\{A, \Lambda\}$ in face of specific classes of perturbations.

IV. The Unstructured Stability Radius

Consider the following perturbed version of system (2),

$$\dot{x}(t) = (A_{\theta_t} + \Delta_{\theta_t}) x(t), \quad t \in \mathbb{R}_+ \quad (6)$$

with $x(0) = x_0 \in L^n_2$, where $\Delta = (\Delta_1, \Delta_2, \ldots) \in \mathbb{H}^n_{\sup}$. We denote such model for uncertainty by the map $\{A, \Lambda\} \rightsquigarrow \{A + \Delta, \Lambda\}$ and assume, from now on, that the nominal system $\{A, \Lambda\}$ is SS.

In this paper we consider the following definition for the stability radius of system $\{A, \Lambda\}$.

Definition 2: The (complex) stability radius of the SS system $\{A, \Lambda\}$ in face of unstructured perturbations such as $\{A, \Lambda\} \rightsquigarrow \{A + \Delta, \Lambda\}$ is defined as

$$\mathbf{r}(\{A, \Lambda\}) = \inf_{\Delta \in \mathbb{H}^n_{\sup}} \{\|\Delta\|_{\sup} : \text{system (6) is not SS}\}.$$

$\triangledown\triangledown$

The stability radius corresponds to the size of the smallest destabilizing perturbation, in an appropriate sense. It is a direct measure of robustness for system $\{A, \Lambda\}$ in that, the larger the radius is, the more robust is the (stochastic) stability of that system with respect to the class of perturbations taken into consideration.

The following proposition states that the stability radius of system $\{A, \Lambda\}$ is homogeneous with respect to multiplication by a scalar.

Proposition 1: The stability radius of system $\{A, \Lambda\}$ is such that, for every constant $\tau > 0$,

$$\tau \, \mathbf{r}(\{A, \Lambda\}) = \mathbf{r}(\{\tau A, \tau \Lambda\}). \quad (7)$$

Proof: See the appendix. ■

Remark 1: A consequence of the above result is that a scaling of the radius corresponds to a change on the jump process's dynamics – in the sense that, in general (we refer to (29)), one has $\tau \, \mathbf{r}(\{A, \Lambda\}) \neq \mathbf{r}(\{\tau A, \Lambda\})$. This illustrates the fact that, as an extension of a well-known feature of MJLS, the *robustness* of SS of system $\{A, \Lambda\}$ depends, among other things, on the switching behavior of the jump process, θ. $\triangledown\triangledown$

The following lemma shows how the maximal real part of the spectrum of $\mathcal{A}$ may be used to obtain an upper bound for the stability radius of (2).

Lemma 2: The stability radius of system $\{A, \Lambda\}$ satisfies

$$\mathbf{r}(\{A, \Lambda\}) \leq -\frac{1}{2} \mathbb{R}_e\{\lambda(\mathcal{A})\} \quad (8)$$

Proof: Denote $\xi := -\mathbb{R}_e\{\lambda(\mathcal{A})\} > 0$. Then, since for all $i \in \mathcal{S}$ we have $\lambda_i(\mathcal{A}+\xi I) = \lambda_i(\mathcal{A}) + \xi$, it follows that $\mathbb{R}_e\{\lambda(\mathcal{A}+\xi I)\} = 0$. But since $\xi \in \mathbb{R}$, we have

$$\xi I_{n^2} = \tfrac{1}{2}\bar{\xi} I_{n^2} + \tfrac{1}{2}\xi I_{n^2} = \tfrac{1}{2}\{(\bar{\xi} I_n) \oplus (\xi I_n)\}, \qquad (9)$$

and hence $\mathcal{A} + \xi I = \boldsymbol{\mathcal{A}}(A + \frac{1}{2}\xi I_n, \Lambda)$. This means, the multiple-of-identity perturbation $\{A, \Lambda\} \rightsquigarrow \{A + \frac{1}{2}\xi I_n, \Lambda\}$ draws the system into instability, from which the result follows. ∎

An important result from H_∞ control theory is the following extension of [8, Theorem 4] to the infinite setting (see also [20]).

Lemma 3: System (6) is SS for any $\Delta = (\Delta_1, \Delta_2, \ldots) \in \mathbb{H}^n_{\sup}$ such that $\|\Delta\|_{\sup} < \rho$ whenever there exist $P = (P_1, P_2, \ldots) \in \tilde{\mathbb{H}}^{n-}_{\sup}$ and $\breve{\alpha} = (\alpha_1, \alpha_2, \ldots) \succ 0$ such that

$$\begin{bmatrix} \mathcal{T}(P) - \rho^2 \breve{\alpha} I_n & P \\ P & \breve{\alpha} I_n \end{bmatrix} \in \tilde{\mathbb{H}}^{2n+}_{\sup}, \qquad (10)$$

with $\mathcal{T}(P) = (\mathcal{T}_1(P), \mathcal{T}_2(P), \ldots)$ given by (5). Moreover, $\rho = \rho(\breve{\alpha})$.

Proof: This is a direct consequence of [13, Theorem 3.3]. The proof relies on an appropriate re-scaling of P and $\breve{\alpha}$, together with the identity $\gamma = \rho^{-1}$. ∎

It is important to notice that, in the preceding lemma, we have $\rho = \rho(\breve{\alpha}) \le \sup_{\breve{\gamma} \succ 0} \rho(\breve{\gamma})$. A consequence of this fact is the following unifying theorem, which states what bounds on the stability radius can we obtain by means of appropriate optimization of the scaling parameters.

Theorem 1: The stability radius of system $\{A, \Lambda\}$ satisfies

$$\hat{\rho} \le \mathbf{r}(\{A, \Lambda\}) \le -\frac{1}{2}\mathbb{R}_e\{\lambda(\mathcal{A})\} \qquad (11)$$

where $\hat{\rho} := \sup_{\breve{\alpha} \succ 0} \rho(\breve{\alpha})$. ▽▽

One last result is as follows. It states that, in some cases, an alternative estimate for the stability radius of system $\{A, \Lambda\}$ may be obtained in terms of the radius of a certain real scalar system $\{\tilde{a}, \Lambda\}$.

Lemma 4: Suppose system $\{\tilde{a}, \Lambda\}$ is SS, where $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots)$ with $\tilde{a}_i = \lambda_{\max}(A_i + A_i^*)/2$, $i \in \mathcal{S}$. Then

$$\mathbf{r}(\{\tilde{a}, \Lambda\}) \le \mathbf{r}(\{A, \Lambda\}) \qquad (12)$$

Proof: The proof goes in the same manner as that of [18, Corollary 4.20]. ∎

Remark 2: Notice that the estimate of Lemma 4 may only be obtained in the particular case where the jump system $\dot{\tilde{x}}(t) = \tilde{a}_{\theta_t}\tilde{x}(t)$ is SS. Furthermore, we remark that in the case $\mathcal{S} = \{1, 2\}$ an exact formula for the radius of $\{\tilde{a}, \Lambda\}$ may be employed at this point (see section VI-B). ▽▽

V. Uncertainty in the transition rate matrix

In this section we suppose that, for fixed $A \in \mathbb{H}^n_{\sup}$, system (2) is subjected to some uncertainty on the transition rate matrix, Λ. We assume that the rate of leaving the ℓ-th mode, for a given $\ell \in \mathcal{S}$, is not precisely known. More specifically, we try to figure out how robust is the stochastic stability of system (2) with respect to parametric perturbations such as

$$\lambda_{\ell j} \rightsquigarrow \kappa \lambda_{\ell j}, \qquad (13)$$

for a given ℓ and every j in $\mathcal{S}$, (with λ_{ij} unchanged for $i \neq \ell$), where $\kappa > 0$ is the uncertain parameter. The perturbed system, from now on represented by the map $\{A, \Lambda\} \rightsquigarrow \{A, \Lambda^\ell(\kappa)\}$, is the following:

$$\dot{x}(t) = A_{\tilde{\theta}_t} x(t), \quad t \in \mathbb{R}_+, \qquad (14)$$

with initial condition $x(0) = x_0 \in L_2^n$ and uncertain jump process $\tilde{\theta} \sim \Lambda^\ell(\kappa)$. To some extent, our aim is to characterize how much uncertainty on the Markov switching properties can the nominal model tolerate without becoming unstable.

Remark 3: Although this uncertainty structure may not be so general as to include each vicinity of Λ around its nominal value, we underline here the important class of finite Markov chains described by

$$\Lambda = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \ldots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_N & 0 & 0 & \ldots & -\lambda_N \end{bmatrix} \qquad (15)$$

with $\lambda_i \ge 0$ for each $i \in \{1, \ldots, N\}$. This includes the 2-mode example treated in section VI and is capable of describing, for instance, an "ordered" Markov chain (maybe a tandem of N queues like the one arising in token ring networks). An important issue here is that a perturbation such as $\Lambda \rightsquigarrow \Lambda^\ell(\kappa)$ does not destroy qualitative properties of the Markov chain, such as its natural ordering or the presence/absence of absorbing modes. ▽▽

Our first result here goes as follows. Roughly, it states that all the uncertainty related to the jump process may be somewhat transferred to the state matrix, $A \in \mathbb{H}^n_{\sup}$.

Lemma 5: System $\{A, \Lambda^\ell(\kappa)\}$ is SS if and only if system $\{A^\ell(\kappa), \Lambda\}$, governed by

$$\dot{x}(t) = A^\ell_{\theta_t}(\kappa) x(t), \quad t \in \mathbb{R}_+ \qquad (16)$$

with Markov switching $\theta \sim \Lambda$ is also SS, where

$$A_i^\ell(\kappa) = \begin{cases} A_i, & i \neq \ell \\ \kappa^{-1} A_i, & i = \ell \end{cases}$$

Proof: From Lemma 1 we have that system $\{A, \Lambda^\ell(\kappa)\}$ is SS if and only if there is $P = (P_1, P_2, \ldots) \in \tilde{\mathbb{H}}^{n-}_{\sup}$ such that $\mathcal{T}^{\kappa,\ell}(P) = (\mathcal{T}_1^{\kappa,\ell}(P), \mathcal{T}_2^{\kappa,\ell}(P), \ldots) \in \tilde{\mathbb{H}}^{n+}_{\sup}$, where $\mathcal{T}_i^{\kappa,\ell}(P) = A_i^* P_i + P_i A_i + \sum_{j \in \mathcal{S}} (\kappa \lambda_{ij}) P_j$ whenever $i = \ell$, and $\mathcal{T}_i^{\kappa,\ell}(P) = \mathcal{T}_i(P)$ otherwise. To prove the result, simply notice that this is equivalent to $\tilde{\mathcal{T}}^{\kappa,\ell}(P) = (\tilde{\mathcal{T}}_1^{\kappa,\ell}(P), \tilde{\mathcal{T}}_2^{\kappa,\ell}(P), \ldots) \in \tilde{\mathbb{H}}^{n+}_{\sup}$, where

$$\tilde{\mathcal{T}}_i^{\kappa,\ell}(P) = [A_i^\ell(\kappa)]^* P_i + P_i A_i^\ell(\kappa) + \sum_{j \in \mathcal{S}} \lambda_{ij} P_j, \quad i \in \mathcal{S}.$$

∎

Remark 4: Notice that we may write $A^\ell(\kappa) = A + \Delta^\ell(\kappa)$, with $\Delta^\ell(\kappa) := (0, \ldots, 0, (\kappa^{-1} - 1)A_\ell, 0, \ldots) \in \mathbb{H}^n_{\sup}$. This fact is employed in the sequel. ▽▽

The main result of this section is the following. It provides us with a margin for the robust SS of system $\{A, \Lambda\}$ with respect to perturbations such as $\Lambda \rightsquigarrow \Lambda^\ell(\kappa)$.

Theorem 2: Let $\mu^{(\ell)} := \mathbf{r}(\{A,\Lambda\})/\|A_\ell\|$. Then, whenever $\mu^{(\ell)} < 1$, we have the uncertain system $\{A, \Lambda^\ell(\kappa)\}$ stable for every $\kappa \in \mathbb{R}$ such that

$$\frac{1}{1+\mu^{(\ell)}} \leq \frac{\|A_\ell\|}{\|A_\ell\|+\hat{\rho}} < \kappa < \frac{\|A_\ell\|}{\|A_\ell\|-\hat{\rho}} \leq \frac{1}{1-\mu^{(\ell)}} \quad (17)$$

On the other hand, whenever $\mu^{(\ell)} \geq 1$ we have the stability of system $\{A, \Lambda^\ell(\kappa)\}$ guaranteed for any κ such that

$$\frac{1}{1+\mu^{(\ell)}} \leq \frac{\|A_\ell\|}{\|A_\ell\|+\hat{\rho}} < \kappa < \infty. \quad (18)$$

Proof: Suppose first that (17) holds true together with $\mu^{(\ell)} < 1$. In this case, it may be easily verified that

$$\mu^{(\ell)} \geq \tfrac{\hat{\rho}}{\|A_\ell\|} > \left(\tfrac{1}{\kappa} - 1\right) > -\tfrac{\hat{\rho}}{\|A_\ell\|} \geq -\mu^{(\ell)}, \quad (19)$$

from which we immediately have that (see Remark 4 above)

$$\|\Delta^\ell(\kappa)\|_{\sup} := |\tfrac{1}{\kappa} - 1|\|A_\ell\| < \hat{\rho} \leq \mathbf{r}(\{A,\Lambda\}),$$

in such a way that the perturbed system is SS. Finally, the infinity-bound in (18) is a consequence of the fact that $\left|\frac{1}{\kappa} - 1\right| < 1$ for every $\kappa \geq 1$, which, together with $\mu^{(\ell)} \geq 1$, implies that $\|\Delta^\ell(\kappa)\|_{\sup} < \|A_\ell\| \leq \mathbf{r}(\{A,\Lambda\})$ for the perturbation introduced in Remark 4. ■

Remark 5: One natural way to obtain better estimates for the acceptable uncertainty on $\Lambda \rightsquigarrow \Lambda^\ell(\kappa)$ would be to further explore the structure of the perturbation $A_\ell \rightsquigarrow A_\ell + (\kappa^{-1} - 1)A_\ell$, defined in Remark 4. It seems that the more direct – and innovative – approach to do so would be to seek for an estimate for the values of κ such that the system $\{(A_1, \ldots, \kappa^{-1}A_\ell, \ldots), \Lambda\}$ is SS. But another approach, perhaps more appealing in view of recent results from [13], is to describe such perturbation as a *structured* one. This will be considered in a future work. ▽▽

VI. Case studies

In this section we show how the preceding results can be used to study the robust stability of two-mode jump systems. Suppose that $\mathcal{S} = \{1,2\}$, so that the transition rate matrix of θ is, without loss of generality, given by

$$\Lambda = [\lambda_{ij}] = \begin{bmatrix} -\beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{bmatrix}, \qquad \beta_1, \beta_2 > 0, \quad (20)$$

what we denote by $\theta \sim (\beta_1, \beta_2)$. Notice that we explicitly rule out the case in which $\beta_1\beta_2 = 0$, i.e., we assume that none of the modes is absorbing. In fact, the existence of an absorbing mode would lead the system to an LTI regime after a finite number of jumps had occurred (at most one, in this case), so that the problem would be easily tackled by well-known results from, e.g., [21, Section 5.3].

A. Two-mode jump systems (general case)

With respect to the jump process (20) defined above, let us consider the following MJLS:

$$\dot{x}(t) = A_{\theta_t} x(t), \quad \theta \sim (\beta_1, \beta_2), \quad t \in \mathbb{R}_+, \quad (21)$$

with $A = (A_1, A_2) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$. We also introduce $\Xi_i = \bar{A}_i \oplus A_i$, $i \in \{1,2\}$, and rewrite $\boldsymbol{\mathcal{A}}(A,\Lambda)$ in (4) as

$$\boldsymbol{\mathcal{A}}(A_1, A_2, \beta_1, \beta_2) := \begin{bmatrix} \Xi_1 - \beta_1 I_{n^2} & \beta_2 I_{n^2} \\ \beta_1 I_{n^2} & \Xi_2 - \beta_2 I_{n^2} \end{bmatrix} \quad (22)$$

The following proposition states that the problem of optimizing the lower bound in (11) can be reduced to a one-dimensional problem, in the two-mode case.

Proposition 2: The stability radius of the two-mode system (21) is such that $\hat{\rho} \leq \mathbf{r}(\{A,\Lambda\})$, where

$$\hat{\rho} = \sup_{\alpha_1>0} \sup_{\alpha_2>0} \rho(\alpha_1, \alpha_2) = \sup_{\alpha_1>0} \rho(\alpha_1, \alpha_2) \quad (23)$$

for any given $\alpha_2 > 0$ or, alternatively,

$$\hat{\rho} = \sup_{\alpha_2>0} \sup_{\alpha_1>0} \rho(\alpha_1, \alpha_2) = \sup_{\alpha_2>0} \rho(\alpha_1, \alpha_2) \quad (24)$$

for any $\alpha_1 > 0$.

Proof: Since $\hat{\rho} = \sup_{\alpha_1>0}\sup_{\alpha_2>0}\rho(\alpha_1,\alpha_2)$ and (10) depends affinely on P and $\breve{\alpha}$ we have

$$\hat{\rho} = \sup_{\alpha_1>0} \sup_{\alpha_2>0} \rho(\alpha_1/\alpha_2, 1) = \sup_{\omega>0} \rho(\omega, 1),$$

where $\omega := \alpha_1/\alpha_2$. Hence, defining $\tilde{\omega} = \alpha_2\omega$ it follows that $\hat{\rho} = \sup_{\tilde{\omega}>0} \rho(\tilde{\omega}, \alpha_2)$, which is just (23). By a similar argument, we have that

$$\hat{\rho}_0 := \sup_{\alpha_2>0} \sup_{\alpha_1>0} \rho(\alpha_1, \alpha_2) = \sup_{\hat{\omega}>0} \rho(1, \hat{\omega}) = \sup_{\alpha_2>0} \rho(\alpha_1, \alpha_2)$$

for $\hat{\omega} := \alpha_2/\alpha_1$, and so it only remains to show that $\hat{\rho} = \hat{\rho}_0$. This is proven by writing down

$$\hat{\rho} = \sup_{\omega>0} \rho(\omega, 1) = \sup_{\omega>0} \rho(1, 1/\omega) = \sup_{\hat{\omega}>0} \rho(1, \hat{\omega}) = \hat{\rho}_0.$$

■

Remark 6: A direct consequence of the above proposition is that, in order to optimize the scaling parameters in the two-mode case, one only has to determine the optimal *ratio* between α_1 and α_2. The result is illustrated by the following example. ▽▽

Example 1: Consider system (21) with A_1, A_2 given by

$$A_1 = \begin{bmatrix} 0.5 & -1 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -1 \\ 0 & 0.5 \end{bmatrix}$$

and $(\beta_1, \beta_2) = (3, 4)$. The relationship between the maximal ρ of Lemma 3 and different choices of scaling parameters (α_1, α_2) is depicted in Figure 1 below. As it suggests, the optimal lower bound $\hat{\rho}$ in (11) can be attained by the ratio $\alpha_1/\alpha_2 = 1$ (shown as a bold line). Perhaps this can be better observed in Figure 2, where solutions to the fixed-α_2 maximization problem $\sup \rho(\cdot, \alpha_2)$ are compared for three diferent values of α_2.

Another important feature to be observed in Figure 2 is that, according to what we stated in Proposition 2, the maximum of $\sup \rho(\cdot, \alpha_2)$ does not in fact depend on α_2. The two bounds on the stability radius which we stated in (11) are, in this case, given by

$$0.1579 \leq \mathbf{r}(\{A,\Lambda\}) \leq 0.2087, \quad (25)$$

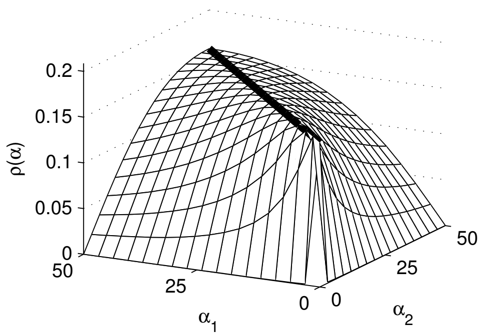

Fig. 1. Dependence of $\rho(\alpha)$ on the scaling parameters.

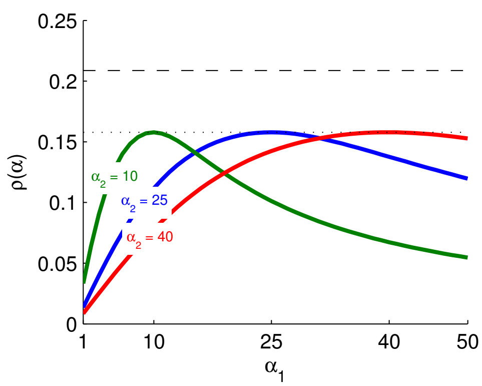

Fig. 2. Estimates for $\mathbf{r}(\{A, \Lambda\})$ in terms of (α_1, α_2).

corresponding to dashed and dotted lines in Figure 2. Also, notice that a bound such as (12) may not be constructed in this case, since both $\tilde{a}_1 := \lambda_{\max}(A_1 + A_1^*)/2$ and $\tilde{a}_2 := \lambda_{\max}(A_2 + A_2^*)/2$ are positive (so that the corresponding scalar system of Lemma 4 is unstable; see Remark 2).

Finally, according to Theorem 2 a stability margin with respect to uncertainties on β_1 or β_2 may be constructed in terms of the lower bound in (25). If we perturb, say, β_1 onto $\kappa\beta_1$ then we have that the resulting uncertain system is stable for any perturbation such that

$$0.9344 < \kappa < 1.0755, \tag{26}$$

which is consistent with the actual margin $\kappa \in (2/3, 4)$ that one can obtain experimentally. Such conservatism also suggests that a destabilizing perturbation such as $A \rightsquigarrow A+\Delta$ with norm close to $\mathbf{r}(\{A, \Lambda\})$ should have a quite different structure from that of (A_1, A_2). $\triangledown\triangledown$

B. Two-mode scalar jump systems

In what follows we provide an explicit formula for the stability radii of system (21) in the scalar case, $n = 1$. To this end, let $\imath := \sqrt{-1}$ together with $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$ define the following system,

$$\dot{x}(t) = (a_{\theta_t} + \imath b_{\theta_t})\, x(t), \quad \theta \sim \tilde{\beta} := (\beta_1, \beta_2) \tag{27}$$

on $t \in \mathbb{R}_+$, which we denote by $\{a + \imath b, \tilde{\beta}\}$. The main result of this section is presented as follows.

Theorem 3: The stability radius of the two-mode scalar system (27) is given by

$$\begin{aligned}\mathbf{r}(\{a + \imath b, \tilde{\beta}\}) \;&=\; -\frac{a_1 + a_2}{2} + \frac{\beta_1 + \beta_2}{4} \\ &\quad -\frac{1}{2}\sqrt{\beta_1\beta_2 + \left(\tfrac{\beta_1 - \beta_2}{2} - a_1 + a_2\right)^2},\end{aligned} \tag{28}$$

in such a way that $\mathbf{r}(\{a + \imath b, \tilde{\beta}\}) = \mathbf{r}(\{a, \tilde{\beta}\})$. In particular,

$$\mathbf{r}(\{a + \imath b, \tilde{\beta}\}) = \frac{\beta - (a_1 + a_2) - \sqrt{\beta^2 + (a_1 - a_2)^2}}{2} \tag{29}$$

whenever $\tilde{\beta} = (\beta, \beta)$.

Proof: See the appendix. ■

Remark 7: From the proof of the above result we have that the minimal destabilizing disturbance is a *real* one. Hence, if we were to define a *real* stability radius here, it would equal the one from Definition 2, in this case. $\triangledown\triangledown$

In contrast to Example 1, in the sequel we present a particular situation in which exact values for stability radii and uncertain switching rates may be obtained. This also illustrates how the formulas of Theorem 3 may be employed for non-scalar systems, in the spirit of Lemma 4.

Example 2: Consider system (21) with A_1, A_2 given by

$$A_1 = \begin{bmatrix} 0.25 & -2 \\ 2 & 0.25 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 10 \\ -10 & -2 \end{bmatrix}$$

and $\tilde{\beta} = (\beta_1, \beta_2) = (2, 1)$. Then, the scalar system

$$\{\tilde{a}, \tilde{\beta}\}: \quad \dot{\tilde{x}}(t) = \tilde{a}_{\theta_t}\tilde{x}(t) \tag{30}$$

with $\tilde{a}_i = \lambda_{\max}(A_i + A_i^*)/2$, $i \in \{1, 2\}$ is such that, in view of Lemma 2, Lemma 4 and (28),

$$\tfrac{1}{2} = \mathbf{r}(\{\tilde{a}, \tilde{\beta}\}) \le \mathbf{r}(\{A, \tilde{\beta}\}) \le -\tfrac{1}{2}\mathbb{R}_e\{\lambda(\mathcal{A})\} = \tfrac{1}{2}, \tag{31}$$

where $\mathcal{A} = \boldsymbol{\mathcal{A}}(A_1, A_2, 2, 1)$, in this case. If we suppose now that the rate of leaving the first mode is uncertain, such as

$$\theta \sim \begin{bmatrix} -2\kappa & +2\kappa \\ +1 & -1 \end{bmatrix} \tag{32}$$

then, in view of Theorem 2 and the fact that

$$\mu^{(1)} := \mathbf{r}(\{\tilde{a}, \tilde{\beta}\})/|\tilde{a}_1| = 2 > 1, \tag{33}$$

we conclude that for every $\kappa \in (1/3, \infty)$, stability of the scalar system is preserved (which, from [18, Corollary 4.20], also guarantees stability of system $\{A, \tilde{\beta}\}$). Finally, it is not difficult to prove (either experimentally or with the aid of (29)) that such estimate coincides with the largest admissible uncertainty (32) that the system at hand can tolerate without becoming unstable. $\triangledown\triangledown$

VII. Conclusions

In this paper we have considered a novel stability radius for continuous-time infinite MJLS. Different from previous approaches in the literature (even if we restricted ourselves to the finite case) we focused here on the particular situation in which only unstructured additive disturbances are allowed to occur. By doing so, we were able to provide a new upper bound for the radius in terms of the maximal real part of the spectrum of an augmented infinite dimensional matrix. Also, two alternative lower bounds have been presented: one corresponding to the radius of a certain scalar MJLS, and other one in terms of an optimal choice of scaling parameters in an LMI problem. Besides, a new connection between stability radii and a robust stability margin with respect to uncertainty on the transition rate matrix of the Markov chain has been unveiled. Finally, particular attention was given to the so-called two-mode case. By this, not only we could show that the optimal scaling problem may be reduced by one parameter, but also it was possible to obtain an explicit formula for the radius of two-mode scalar jump systems. Two examples have then been studied, showing that even if restricted to the finite case, the obtained results can in fact provide new quantitative measures for the robust stability of MJLS.

APPENDIX

Proof of Proposition 1. Bearing in mind Lemma 1, that the Kronecker product is homogeneous with respect to multiplication by a scalar [19] and that $\mathbb{R}_e\{\lambda(\tau^{-1}(\cdot))\} \equiv \tau^{-1}\mathbb{R}_e\{\lambda(\cdot)\}$, it follows that $\tau\boldsymbol{\mathcal{A}}(A,\Lambda) = \boldsymbol{\mathcal{A}}(\tau A, \tau\Lambda)$ and

$$\begin{aligned}\tau\mathbf{r}(\{A,\Lambda\}) &= \tau \inf\{\|\Delta\|_{\sup} : \mathbb{R}_e\lambda[\boldsymbol{\mathcal{A}}(A+\Delta,\Lambda)] \geq 0\} \\ &= \inf\{\|\tilde{\Delta}\|_{\sup} : \mathbb{R}_e\lambda[\boldsymbol{\mathcal{A}}(A+\tau^{-1}\tilde{\Delta},\Lambda)] \geq 0\} \\ &= \inf\{\|\tilde{\Delta}\|_{\sup} : \mathbb{R}_e\lambda[\boldsymbol{\mathcal{A}}(\tau A+\tilde{\Delta},\tau\Lambda)] \geq 0\},\end{aligned}$$

which corresponds to (7). ■

The following auxiliary remark should ease the proof of Theorem 3 below.

Remark 8: For any $z_1, z_2 \in \mathbb{C}$, it may be easily verified that $\boldsymbol{\mathcal{A}}(z_1, z_2, \cdot, \cdot) \equiv \boldsymbol{\mathcal{A}}(\mathbb{R}_e z_1, \mathbb{R}_e z_2, \cdot, \cdot)$. ▽▽

Proof of Theorem 3. First notice that, from Remark 8, we immediately have $\mathbf{r}(\{a+\imath b,\tilde{\beta}\}) = \mathbf{r}(\{a,\tilde{\beta}\})$. Moreover:

$$\begin{aligned}\mathbf{r}(\{a,\tilde{\beta}\}) &= \inf\{\|\underline{\delta}\|_{\max}; \mathbb{R}_e\lambda[\boldsymbol{\mathcal{A}}(a+\underline{\delta},\tilde{\beta})] \geq 0\} \\ &= \inf\{\|\underline{\delta}^R\|_{\max}; \mathbb{R}_e\lambda[\boldsymbol{\mathcal{A}}(a+\underline{\delta}^R,\tilde{\beta})] = 0\}\end{aligned}$$

from continuity of the spectrum together with Remark 8, in which $\underline{\delta}^R = (\delta_1^R, \delta_2^R)$ is the real part of $\underline{\delta} = (\delta_1, \delta_2)$ and $\|\cdot\|_{\max}$ is the maximum norm in $\mathbb{R}^2$. But since the SS of systems $\{a+\underline{\delta}^R,\tilde{\beta}\}$ and $\{(a+\underline{\delta}^R)/\tilde{\beta},\mathbb{1}\}$, in which $\mathbb{1} := (1,1)$, are equivalent (due to Lemma 5), it follows that

$$\mathbf{r}(\{a,\tilde{\beta}\}) = \inf\{\|\underline{\delta}^R\|_{\max}; \det\boldsymbol{\mathcal{A}}((a+\underline{\delta}^R)/\tilde{\beta},\mathbb{1}) = 0\},$$

because $\boldsymbol{\mathcal{A}}((a+\underline{\delta}^R)/\tilde{\beta},\mathbb{1})$ is self-adjoint. The solution of this problem satisfies $\mathbf{r}(\{a,\tilde{\beta}\}) = \inf_{\delta>0}\{\beta_1\delta; \det\mathcal{A} = 0\}$, in which

$$\mathcal{A} = \begin{bmatrix} 2(a_1+\beta_1\delta)/\beta_1 - 1 & 1 \\ 1 & 2(a_2+\beta_1\delta)/\beta_2 - 1 \end{bmatrix}.$$

After performing some calculations (namely, solving for δ the second-order algebraic equation $\det\mathcal{A} = 0$ at the above problem), we finally have that the only candidates for solution are given by

$$\begin{aligned}\delta_\pm \quad := \quad & \frac{1}{4} + \frac{1}{4\beta_1}\Big\{\beta_2 - 2(a_1+a_2) \\ & \pm\sqrt{4\beta_1\beta_2 + (\beta_1-\beta_2-2a_1+2a_2)^2}\Big\},\end{aligned}$$

so that $\mathbf{r}(\{a,\tilde{\beta}\}) = \beta_1\delta_-$ yields the desired results. ■

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