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Robust Fault-tolerant Control Systems Design with Actuator Failures via Linear Fractional Transformations

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***Abstract*— This paper presents a linear matrix inequality (LMI) approach to solve the robust fault-tolerant control (FTC) problem with actuator failures. According to an equivalent transformation, fault effect factors can be put in linear fractional transformation (LFT) form. Then, based on the information from the fault detection and diagnosis (FDI) mechanism, the fault-tolerant control problem can be solved with a well-recognized design approach in robust control area called gain-scheduling control theory, and a fault-tolerant controller which provided with adaptive function can be developed for satisfactory performance. Also based on the LFT framework, the case of error estimation is considered in this paper with applying μ-theory for guaranteeing the closed-loop system's stability and performance. The proposed design technique is finally evaluated in the light of a simulation example.**

I. INTRODUCTION

In most practical control systems, a components' failure (including sensors, actuators and even the plant itself) may occur at uncertain time and the size of faults is also unknown. The fault may leads to performance deterioration or even instability of the system. Therefore, fault-tolerant control (FTC) system design, which can make the system operates in safety and with proper performance whenever components are healthy or faulted, has received significant attention over the past two decades. There broadly classified into two type approaches for design the fault-tolerant controller, namely passive approach [9, 10, 11, 12, 19, 22] and active approach [4, 5, 6, 7, 8, 13, 15, 20]. In the passive approach, using robust control techniques, a fixed controller is designed to guarantee system's stability and performance in fault case as well as in normal case. Recently, several approaches have been developed, such as ARE-based approach [9, 10, 12]; LMI-based approach [11, 19]; Pole region assignment technique [22], etc. The passive approach is relatively easy to design the controller for the presumed faults because they are not rely on on-line controller adjusting. However, it has also a very limited fault tolerant capability because as the number of possible failures and the degree of system redundancy increase, the controller design becomes more conservative and attainable control performances may not be satisfactory. On the other hand, a fault-tolerant control system based on active approach can compensate for faults either by selecting a pre-computed control law or by synthesizing a new control strategy on-line. There are primary two typical approaches for fault compensations in active fault-tolerant, such as fault detection and isolation (FDI) [4, 5, 20] and adaptive approach [7, 8, 13, 15]. Since the active FTC system offers the flexibility to select different controllers, the most suitable controller can be chosen for the situation and the better performance can be obtained than the passive FTC system.

As faults are unknown, many researchers have focused on the development of methodologies to estimate faults [5, 6, 7, 14, 20], so that measures could be taken to accommodate their effects in FTC systems. Although the fault effect factors are estimated as accurately as possible, there are always exist inaccuracies of estimation in most of applications. In this paper, all inaccuracies are considered as parametric uncertainties which is convenient to be described in LFT forms. And we assume the upper and lower bounds of inaccuracies are known according to the FDI mechanism's self-characteristic. Then, the estimation inaccuracies can be defined as a convex polytopic uncertain domain in terms of the upper and lower bounds of inaccuracies, and the FDI scheme is considered availably if the inaccuracy values within that domain.

In this paper, we consider the robust FTC problem for a linear discrete-time system, and put the state-space model in LFT form so that the system depends on the estimate of control effectiveness where comes from FDI mechanism. Then, the controller which also depends on the same values can be designed using the gain-scheduling technique, which is similar to design controller in linear parameter-varying (LPV) systems introduced in [1, 2, 18], to guarantee the closed-loop system safety and performance. Due to the adaptive nature, the fault-tolerant controller has adaptive robust characteristic. However, as a result of the controller designed based on the estimated values, the closed-loop system may have bad performances or even be unstable when some estimated values are absolutely wrong. For this case of FDI failures, μ-theory which can measure the minimum values of uncertainties which make the system unstable is utilized to guarantee the stability of closed-loop system. In other words, if we consider estimation errors as uncertainties, the maximum size of allowed uncertainties can be expressed by the structured singular value μ of the feedback system. Then we can adjust the bound of inaccuracies, reconstruct the controller and let the estimation error be in the minimum range, so that the new controller may tolerant the faults of FDI.

The rest of the paper is organized as follows. The FTC problem formulation and a system interpretation for this problem are described in Section 2. In Section 3, LMI synthesis conditions are presented, and the linear time-invariant control structure for FTC problem is also designed. Section 4 introduces μ-theory application for the case of FDI scheme be fail. Section 5 gives an example and simulation. Finally, conclusion is given in Section 6.

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II. PRELIMINARIES AND PROBLEM STATEMENT

We now introduce our notation and gather some elementary facts. R denotes the set of real numbers; C denotes the set of complex numbers. For real symmetric matrices M, the notation $M > 0(< 0)$ stands for positive (negative) definite and means that all the eigenvalues of M are positive (negative). For $M > 0$, $M^{1/2}$ denotes the unique positive definite square root. For an arbitrary matrix P, ker(P) stands for the null space of the linear operator associated with P. Given numbers $\rho_k, k = 1,\ldots,n$, the notation $\text{diag}_k[\rho_k]$ denotes the diagonal matrix with ρ_k along the diagonal.

Linear fractional transformation (LFT) is used extensively in control domain. For appropriately dimensioned matrices K and

$$E = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$$

assuming the inverses exist, the lower and upper LFT are defined as

$$\begin{aligned} F_l(E,K) &= E_1 + E_2K(I-E_4K)^{-1}E_3 \\ F_u(E,K) &= E_4 + E_3K(I-E_1K)^{-1}E_2, \end{aligned}$$

respectively. For a stable real-rational transfer function matrix G, the H_∞ norm is defined in the usual way for discrete time systems:

$$\|G(z)\|_\infty = \sup_{\theta\in[0,2\pi]} \bar{\sigma}(G(e^{j\theta}))$$

where $\bar{\sigma}(G)$ stands for the largest singular value of a matrix G.

Consider a linear time-invariant discrete-time model described by

$$\begin{aligned} x(k+1) &= Ax(k) + B_1w(k) + B_2u(k) \\ z(k) &= C_1x(k) + D_{12}u(k) \\ y(k) &= C_2x(k) + D_{21}w(k) \end{aligned} \tag{1}$$

where $x(k) \in R^n$ is the state, $u(k) \in R^m$ is the control input, and $y(k) \in R^p$ is the measured output, $z(k) \in R^q$ is the regulated output and $w(k) \in R^s$ is an exogenous disturbance in $L_2[0,\infty]$, respectively. All system matrices are known constant matrices of appropriate dimensions.

In this paper, we consider a general actuator fault model. Let $u_{ih}^F(k)$ represent the signal from the ith actuator that has failed in the hth faulty mode. Then we denote the fault model

$$u_{ih}^F(k) = (1-\sigma_i^h)u_i(k), \quad 0 \le \underline{\sigma}_i^h \le \sigma_i^h \le \bar{\sigma}_i^h \le 1$$

where $i = 1\ldots m$, $h = 1\ldots L$, σ_i^h is an unknown actuator efficiency factor, the index h denotes the hth faulty mode and L is the total faulty modes. let $\underline{\sigma}_i^h$ and $\bar{\sigma}_i^h$ represent the lower and upper bounds of σ_i^h, respectively. Note the practical case, when $\bar{\sigma}_i^h = \underline{\sigma}_i^h = 0$, there is no fault for the ith actuator u_i. when $\bar{\sigma}_i^h = \underline{\sigma}_i^h = 1$, the ith actuator u_i is outage. when $0 < \underline{\sigma}_i^h \le \bar{\sigma}_i^h < 1$, that means the type of actuator faults is loss of effectiveness.

Denote

$$u_h^F(k) = [u_{1h}^F(k), u_{2h}^F(k), \cdots, u_{mh}^F(k)]^T = (I-\sigma^h)u(k)$$

where $\sigma^h = \text{diag}[\sigma_1^h, \sigma_2^h, \cdots, \sigma_m^h]$, $\sigma_i^h \in [\underline{\sigma}_i^h, \bar{\sigma}_i^h]$. Then a set of operator with above structure is defined by

$$\Delta_{\sigma^h} = \{\sigma^h : \sigma^h = \text{diag}_i[\sigma_i^h], \sigma_i^h \in [\underline{\sigma}_i^h, \bar{\sigma}_i^h], i = 1,2,\ldots,m\}. \tag{2}$$

Letting $\hat{\sigma}_i$ stand for the estimate of σ_i, we define a set as

$$\Delta_{\hat{\sigma}} = \left\{\hat{\sigma} : \hat{\sigma} = \text{diag}_i[\hat{\sigma}_i], \hat{\sigma}_i \in [\min_h\{\underline{\sigma}_i^h\}, \max_h\{\bar{\sigma}_i^h\}]\right\}. \tag{3}$$

Let δ_i^+ and δ_i^- as the upper and lower bounds of the estimate inaccuracies δ_i, respectively, then the relationship of $\hat{\sigma}$ and σ^h can be described as:

$$\sigma^h = \hat{\sigma} - \delta \tag{4}$$

where $\delta = \text{diag}[\delta_1, \delta_2, \cdots, \delta_m]$, $\delta_i \in [\delta_i^-, \delta_i^+]$, and a set is defined by

$$\Delta_\delta = \{\delta : \delta = \text{diag}_i[\delta_i], \delta_i \in [\delta_i^-, \delta_i^+], i = 1,2,\ldots,m\}. \tag{5}$$

Here, we consider the estimate inaccuracies δ as a polytope of $m\times m$ diagonal matrices N_j. That is,

$$\Delta_\delta := Co(N_j, j = 1,2,\ldots,2^m) = \{\sum_{j=1}^{2^m} \alpha_j N_j : \alpha_j \ge 0, \sum_{j=1}^{2^m} \alpha_j = 1\}$$

where $Co(\cdot)$ denotes the convex hull, and every element of N_j is given by the corresponding extreme vertices of δ_i, i.e δ_i^+ or δ_i^-.

For the convenience of description in the following sections, for all possible faulty modes L, the following uniform actuator fault model is exploited:

$$u_i^F(t) = (I-\sigma_i)u_i(t), \quad \sigma_i \in \{\sigma_i^1 \cdots \sigma_i^L\}. \tag{6}$$

Hence, from the above description, dynamics with actuator fault (1) is described by

$$\begin{aligned} x(k+1) &= Ax(k) + B_2(I-\hat{\sigma}+\delta)u(k) + B_1w(k) \\ z(k) &= C_1x(k) + D_{12}(I-\hat{\sigma}+\delta)u(k) \\ y(k) &= C_2x(k) + D_{21}w(k). \end{aligned} \tag{7}$$

For the sake of fault effect factors can be of linear fractional transformation (LFT) parameter dependence. A transformation be introduced as:

$$I - \hat{\sigma} = F_u\left(\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}, \hat{\sigma}\right) \tag{8}$$

where $E_i = \text{diag}[E_{i1}, E_{i2}, \cdots, E_{im}]$, $i = 1,2,3,4$ are defined by the equation, and one of suitable choices is $E_1 = 0_m, E_2 = I_m, E_3 = -I_m$, $E_4 = I_m$, which satisfies (8).

In terms of (8), we denote the following transform equations as:

$$\begin{aligned} (I-\hat{\sigma})u(k) &= E_3w_\sigma(k) + E_4u(k) \\ z_\sigma(k) &= E_1w_\sigma(k) + E_2u(k) \end{aligned} \tag{9}$$

where the internal signals $w_\sigma(k) \in R^m$ and $z_\sigma(k) \in R^m$.

Then (7) can be rewritten in the LFT form as follows:

$$\begin{bmatrix} x(k+1) \\ z_\sigma(k) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_1E_3 & B_2 & \overline{B_1E_4} \\ 0 & E_1 & 0 & E_2 \\ C_1 & D_{12}E_3 & 0 & \overline{D_{12}E_4} \\ C_2 & 0 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w_\sigma(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{10}$$

where $\overline{B_1E_4} = B_1E_4 + B_1\delta$, $\overline{D_{12}E_4} = D_{12}E_4 + D_{12}\delta$, and

$$w_\sigma(k) = \hat{\sigma}z_\sigma(k). \tag{11}$$

Therefore, we get the plant's LFT form. Due to the controller for this LFT plant will also be parameter dependent, and will be have a similar structure with the plant, the state-space model for the controller can be describe as:

$$\begin{bmatrix} x^K(k+1) \\ u(k) \\ z_\sigma^K(k) \end{bmatrix} = \begin{bmatrix} A^K & B_1^K & B_2^K \\ C_1^K & D_{11}^K & D_{12}^K \\ C_2^K & D_{21}^K & D_{22}^K \end{bmatrix} \begin{bmatrix} x^K(k) \\ y(k) \\ w_\sigma^K(k) \end{bmatrix} \tag{12}$$

where

$$w_\sigma^K(k) = \hat{\sigma}z_\sigma^K(k). \tag{13}$$

Based on the above description, the FTC closed-loop system can be drawn as in Fig.1. Note that the closed-loop system depend on parameter $\hat{\sigma}$, hence define an operator:

$$\begin{aligned} \Delta_{\sigma\oplus\sigma} = \{&\text{diag}(\hat{\sigma}, \hat{\sigma}) : \\ &\hat{\sigma} = \text{diag}_i[\hat{\sigma}_i], \hat{\sigma}_i \in [\min_h\{\underline{\sigma}_i^h\}, \max_h\{\bar{\sigma}_i^h\}] \subseteq [0,1]\} \end{aligned} \tag{14}$$

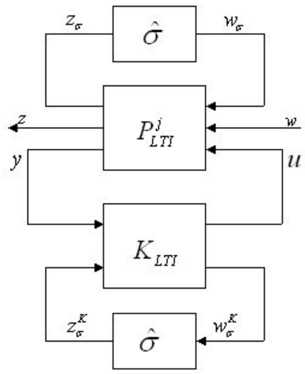

Fig. 1. LFT plant and controller.

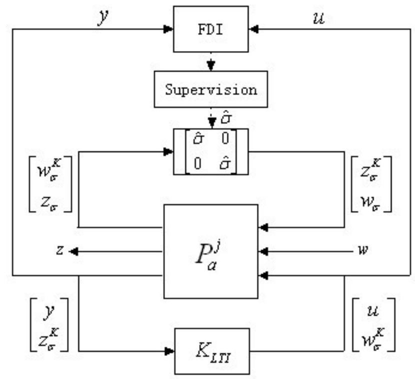

Fig. 2. Augmented LTI plant for fault-tolerant control design.

where $i = 1,2,\ldots,m$, $h = 1,2,\ldots,L$.

We modify block diagram for control design shows in Fig.2. The block P^j_a $(j = 1,2,\ldots,2^m)$ represents the augmented plant used for control design. The twice-repeated block $\hat{\sigma}$ represents parameter time variations which will be viewed as the estimate of actuator fault effect factors.

Note that we have collected all of the estimation parameters (both from the plant and controller) together, and the P^j_{LTI} and K_{LTI} represent the linear time-invariant portions of the system, and the LFT relationships are

$$P^j = F_u(P^j_{LTI},\hat{\sigma}), \quad K = F_l(K_{LTI},\hat{\sigma}), \quad j = 1,2,\ldots,2^m.$$

Now for the fault-tolerant control problem, our problem describe as follows:

Construct an LFT control structure K_{LTI} such that the closed-loop system T_{zw} is internally stable and the H_∞ norm of closed-loop transfer function matrix satisfies

$$\| T_{zw}(P^j_a,K_{LTI},\hat{\sigma}) \|_\infty \leq 1 \tag{15}$$

for all fault effect factors $\| \hat{\sigma} \|_\infty \leq 1$.

Remark 1: The case of $\hat{\sigma}_i = 1$, $i = 1,2,\ldots,m$ is accepted in our design, since that just caused by the wrong estimation with FDI scheme which will be discussed in section 4. Moreover, if all actuators are outage, there are no input values to FDI mechanism, and the mechanism has also no output values. Here, if $\hat{\sigma}_i > \max_h\{\bar{\sigma}^h_i\}$, then we let $\hat{\sigma}_i = \max_h\{\bar{\sigma}^h_i\}$. Similarly, If $\hat{\sigma}_i < \min_h\{\underline{\sigma}^h_i\}$, we let $\hat{\sigma}_i = \min_h\{\underline{\sigma}^h_i\}$.

III. FAULT-TOLERANT CONTROL SYSTEM DESIGN

Since the plant and controller has same block structure and depend on the same parameters, we consider the set of positive definite similarity scalings associated with the structure $\hat{\sigma}$ in (3)

$$J_\sigma := \{J > 0 : J\hat{\sigma} = \hat{\sigma}J, \forall\hat{\sigma} \in \Delta_{\hat{\sigma}}\} \subset R^{m\times m}.$$

Using the small-gain theorem, we can easily have the following lemma for system's robust performance in the face of the fault factors $\sigma \bigoplus \sigma$, or equivalently for the existence of fault-tolerant controllers to satisfy our objective. Define

$$J_{\sigma\oplus\sigma} := \left\{ \begin{pmatrix} J_{11} & J_{12} \\ J^T_{12} & J_{22} \end{pmatrix} \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \begin{pmatrix} J_{11} & J_{12} \\ J^T_{12} & J_{22} \end{pmatrix} \right\} \tag{16}$$

where $J_{if} = \mathrm{diag}[J_{if1}, J_{if2}, \cdots, J_{ifm}] > 0, \quad i,f = \{1,2\}$.

Lemma 1: Consider an uncertainty structure $\hat{\sigma}$ and the associated set of similarity scalings $J_{\sigma\oplus\sigma}$. If there exists a scaling matrix $J \in J_{\sigma\oplus\sigma}$ and an LTI control structure K_{LTI} such that the nominal closed-loop system $F_l(P^j_a,K_{LTI})$, $j = 1,2,\ldots,2^m$ is internally stable and satisfies

$$\left\| \begin{bmatrix} J & 0 \\ 0 & I_q \end{bmatrix} F_l(P^j_a,K_{LTI}) \begin{bmatrix} J^{-1} & 0 \\ 0 & I_s \end{bmatrix} \right\|_\infty \leq 1 \tag{17}$$

then $F_l(K_{LTI},\hat{\sigma})$ is a standard H_∞ controller.

Proof: The proof is a straightforward application of the small gain theorem. See [16], [23] for more details.

Consider the linear discrete time system (10) and controller (12), The closed-loop system with state-space description is simply given by:

$$\begin{bmatrix} x(k+1) \\ z_\sigma(k) \\ x^K(k+1) \\ z^K_\sigma(k) \\ z(k) \end{bmatrix} := M_j \begin{bmatrix} x(k) \\ w_\sigma(k) \\ x^K(k) \\ w^K_\sigma(k) \\ w(k) \end{bmatrix}. \tag{18}$$

Design $\bar{J}_{\sigma\oplus\sigma}$ as the set of invertible matrices with block diagonal structure given as

$$\bar{J}_{\sigma\oplus\sigma} : = \left\{ \begin{bmatrix} J_{11o} & 0 & J_{12o} & 0 \\ 0 & J_{11} & 0 & J_{12} \\ J_{21o} & 0 & J_{22o} & 0 \\ 0 & J_{21} & 0 & J_{22} \end{bmatrix} : J_{ifo} \in R^{n\times n}, J_{if} \in R^{m\times m} \right\}. \tag{19}$$

Lemma 2[1]: The linear discrete-time system described in (10) is internally exponentially stable, and there exists a

$$J \in J_{\sigma\oplus\sigma}$$

such that

$$\left\| \begin{bmatrix} J & 0 \\ 0 & I_q \end{bmatrix} G_j \begin{bmatrix} J^{-1} & 0 \\ 0 & I_s \end{bmatrix} \right\|_\infty < 1$$

if and only if there exists a matrix

$$\bar{J} \in \bar{J}_{\sigma\oplus\sigma}$$

such that

$$\bar{\sigma}\left(\begin{bmatrix} \bar{J} & 0 \\ 0 & I_q \end{bmatrix} M_j \begin{bmatrix} \bar{J}^{-1} & 0 \\ 0 & I_s \end{bmatrix} \right) < 1$$

where G is the transfer function from $[w_\sigma(k)\ w^K_\sigma(k)\ w(k)]^T$ to $[z_\sigma(k)\ z^K_\sigma(k)\ z(k)]^T$.

We express the state-space description of the dynamic system as an LFT of the two constant matrices, S_P^j and S_K:

$$\begin{bmatrix} x(k+1) \\ z_\sigma(k) \\ x^K(k+1) \\ z_\sigma^K(k) \\ z(k) \end{bmatrix} := F_l(S_p^j, S_K) \begin{bmatrix} x(k) \\ w_\sigma(k) \\ x^K(k) \\ w_\sigma^K(k) \\ w(k) \end{bmatrix}$$

Hence, the following statements are equivalent:

(1) there exists a matrix $J \in J_{\sigma\oplus\sigma}$, and a time-invariant controller K_{LTI} such that

$$\left\| \begin{bmatrix} J & 0 \\ 0 & I_q \end{bmatrix} F_l(P_a^j, K_{LTI}) \begin{bmatrix} J^{-1} & 0 \\ 0 & I_s \end{bmatrix} \right\|_\infty < 1.$$

(2) There exists a matrix $Y \in \bar{J}_{\sigma\oplus\sigma}$ and a constant matrix S_K such that

$$\bar{\sigma}\left(\begin{bmatrix} Y^{1/2} & 0 \\ 0 & I_q \end{bmatrix} F_l(S_p^j, S_K) \begin{bmatrix} Y^{-1/2} & 0 \\ 0 & I_s \end{bmatrix} \right) < 1. \tag{20}$$

Therefore, if there exist a matrix K_{LTI} and $Y \in \bar{J}_{\sigma\oplus\sigma}, Y = Y^T > 0$, satisfy (20), we can achieve the H_∞ controller that provided with fault-tolerant function.

For the sake of solve the problem, we reformulate it into a finite-dimensional convex feasibility program using following Lemma.

Lemma 3: Let $R_j \in F^{l\times l}$, $U_j \in F^{l\times m}$, and $V_j \in F^{p\times l}$, $j = 1,2,\ldots,2^m$, Suppose the columns of $U_\perp^j \in F^{l\times(l-m)}$ and $V_\perp^j \in F^{(l-p)\times l}$ be bases for the null space of U_j^T and V_j^T such that $[U_j \;\; U_\perp^j]$, $[V_j^T \;\; V_\perp^{jT}]^T$ are both invertible, and that $U_j^T U_\perp^j = 0$, $V_j V_\perp^{jT} = 0$. Let $N_Z \subset F^{l\times l}$ be a given set of positive matrices. Then there exists a Q such that

$$\bar{\sigma}[Z^{1/2}(R_j + U_j Q V_j) Z^{-1/2}] < 1$$

if and only if there is a $Z \in N_Z$ such that

$$\begin{aligned} &V_\perp^j (R_j^T Z R_j - Z) V_\perp^{jT} < 0 \\ &U_\perp^{jT} (R_j Z R_j^T - Z^{-1}) U_\perp^j < 0. \end{aligned} \tag{21}$$

The above lemma can be proved using standard matrix dilations arguments [1] or Schur complement arguments [21] when Z is positive definite.

Using the above lemma, we can turn fault-tolerant problem into solve the Affine matrix inequality (AMIs) (21). Divide S_P^j into

$$S_P^j = \begin{bmatrix} R_j & U_j \\ V_j & 0 \end{bmatrix}$$

with

$$R_j = \begin{bmatrix} M_{11}^j & 0 & M_{12}^j \\ 0 & 0 & 0 \\ M_{21}^j & 0 & M_{22}^j \end{bmatrix}, U_j = \begin{bmatrix} M_{13}^j & 0 \\ 0 & I \\ M_{23}^j & 0 \end{bmatrix},$$

$$V_j = \begin{bmatrix} M_{31}^j & 0 & M_{32}^j \\ 0 & I & 0 \end{bmatrix}.$$

Note that by Lemma 3, and the properties of $\bar{J}_{\sigma\oplus\sigma}$, the LFT synthesis problem becomes whether the following form:

$$\bar{\sigma}[Z^{1/2}(R_j + U_j Q V_j) Z^{-1/2}] < 1$$

has a feasible solution of Q with the above R_j, U_j, V_j?

As we can see, the conditions in (21) are matrix inequalities, one in Z, and the other in Z^{-1}. Due to the structure of P_a, however, only portions of Z and Z^{-1} appear in the matrix inequalities. As is demonstrated in [l], this allows one to express the two matrix inequalities in terms of variables X and Y , where both X and Y are conformal with $\hat{\sigma}$ and are in set Z, and additional coupling conditions between the subblocks of X and Y. In particular, for each i, there must exist X_{2i} and $X_{3i} = X_{3i}^T$ such that

$$\begin{bmatrix} X_i & X_{2i} \\ X_{2i}^T & X_{3i} \end{bmatrix}^{-1} = \begin{bmatrix} Y_i & * \\ * & * \end{bmatrix} \tag{22}$$

and in addition, the following must be satisfied as well:

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} \geq 0.$$

Lemma 4: There always exist X_{2i} and $X_{3i} = X_{3i}^T$ such that the condition in (22) satisfied.

Proof: The proof may be found in [1].

This yields the following LMI synthesis condition:

Theorem 1: There exists a Q and Z such that the inequalities in (21) are satisfied if and only if there exists X and Y in N_Z, where X and Y are conformal to σ, such that

$$\begin{aligned} &\tilde{V}_\perp^{jT}\left(\tilde{R}_j^T Y \tilde{R}_j - Y\right)\tilde{V}_\perp^j < 0 \\ &\tilde{U}_\perp^{jT}\left(\tilde{R}_j X \tilde{R}_j^T - X\right)\tilde{U}_\perp^j < 0 \\ &\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} \geq 0 \end{aligned} \tag{23}$$

where $i = 1,2,\ldots,m,\; j = 1,2,\ldots,2^m$,

$$\tilde{R}_j = \begin{bmatrix} M_{11}^j & M_{12}^j \\ M_{21}^j & M_{22}^j \end{bmatrix}$$

and the columns of $\tilde{U}_\perp^j$ and $\tilde{V}_\perp^j$ form bases for the null space of $[M_{13}^{jT} \;\; M_{23}^{jT}]^T$ and $[M_{31}^j \;\; M_{32}^j]^T$.

Following the above theorem, we can get the linear time-invariant control structure K_{LTI} such that the nominal closed-loop system $F_l(P_a^j, K_{LTI})$ is internally stable and let (15) come into existence. Clearly, although K_{LTI} can be obtained in terms of system's structure, but fault factors with the initial condition $\hat{\sigma}(0) = 0$ are estimated on-line. So, the controller is reconstructed on-line when the value of $\hat{\sigma}$ is changed.

The H_∞ controller $F_l(K_{LTI}, \hat{\sigma})$ has fault-tolerant function when fault effect factors σ have been estimated accurately. But when some estimates are absolutely wrong, the wrong controller may lead to performance deterioration or even instability of the closed-loop system. The next section will introduce how to solve the case of FDI failures, and maintain the system's stability.

IV. μ-THEORY APPLICATION FOR ERROR ESTIMATION

In this section, we apply the μ-theory to avoid the system become unstable when the case of error estimation occurs, which results from the FDI mechanism failures.

Let Δ stand for the estimation error of fault effect factors, and $\Delta = \mathrm{diag}[\Delta_1, \Delta_2, \ldots, \Delta_m]$, where $\Delta_i \in [\Delta_i^-, \Delta_i^+]$, Δ_i^- and Δ_i^+ represent the low and upper bounds of Δ_i, respectively, and the set is denoted by $\Delta_\Delta = \{\Delta : \Delta = \mathrm{diag}_i[\Delta_i], \Delta_i \in [\Delta_i^-, \Delta_i^+], i = 1,2,\ldots,m\}$, then $\sigma^h = \hat{\sigma} - \Delta$.

For convenience of description as exploited (12), the closed-loop system can be described as:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1(I - \hat{\sigma} + \Delta)u(k) + B_2 w(k) \\ z(k) &= C_1 x(k) + D_{12}(I - \hat{\sigma} + \Delta)u(k) \\ y(k) &= C_2 x(k) + D_{21} w(k). \end{aligned} \tag{24}$$

In terms of section 3, if $\Delta_\Delta \subseteq \Delta_\delta$, the designed controller is suitable for the FTC problem. But for the other cases, $\sigma \in \Delta_\sigma^{'}$, where $\Delta_\sigma^{'} := \{\sigma : \sigma = \mathrm{diag}_i[\sigma_i], \sigma_i \in [\hat{\sigma}_i + \delta_i^+, \hat{\sigma}_i + \Delta_i^+]$ or $\sigma_i \in$

$[\hat{\sigma}_i+\Delta_i^-,\hat{\sigma}_i+\delta_i^-],i=1,2,\dots,m\}$, the wrong controller may lead to performance deterioration or even instability of the closed-loop system.

Define uncertainties set:

$$N_{\Delta_u}=\{\Delta_u:\Delta_u=\mathrm{diag}_i[\Delta_{ui}],\Delta_{ui}\in\{\underline{\Delta}_{ui},\bar{\Delta}_{ui}\},\\ \bar{\Delta}_{ui}\in[0,\Delta_i^+-\delta_i^+],\underline{\Delta}_{ui}\in[\Delta_i^--\delta_i^-,0]\}. \tag{25}$$

Consider the transform such as (8), we let $E_2=E_3=I_m$ and $E_1=E_4=0_m$, then the transform equation can be written as:

$$\Delta u(k)=E_3 w_\Delta(k),\quad z_\Delta(k)=E_2 u(k) \tag{26}$$

Hence, we can get the closed-loop system with the controller (12) in the LFT form:

$$\begin{bmatrix} x_{cl}(k+1)\\ z(k)\\ z_\Delta(k)\end{bmatrix}=\begin{bmatrix} A_{cl} & B_{cl}^1 & B_{cl}^2\\ C_{cl}^1 & D_{cl}^{11} & D_{cl}^{12}\\ C_{cl}^2 & D_{cl}^{21} & 0\end{bmatrix}\begin{bmatrix} x_{cl}(x)\\ w(k)\\ w_\Delta(k)\end{bmatrix} \tag{27}$$

where

$$w_\Delta(k)=\Delta_u z_\Delta(k) \tag{28}$$

and $x_{cl}:=(x,\,x^K)$.

Define two augmented block structures, N_{Δ_N}, N_{Δ_S} as

$$N_{\Delta_N}:=\left\{\mathrm{diag}[\tau I_m,\,\Delta_2]:\ \tau\in C,\,\Delta_2\in C^{q\times q}\right\}\\ N_{\Delta_S}:=\left\{\mathrm{diag}[\Delta_N,\,\Delta_u]:\ \Delta_N\in N_{\Delta_N},\,\Delta_u\in N_{\Delta_u}\right\}. \tag{29}$$

where τ stands for the $\mathscr{Z}$ transform variable in the discrete-time context, and Δ_2 viewed as a norm bounded perturbation from an allowable perturbation class.

we rewrite (27) as a simply form:

$$\begin{bmatrix} x_{\Delta_N}(k+1)\\ z_\Delta(k)\end{bmatrix}=\begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix} x_{\Delta_N}(k)\\ w_\Delta(k)\end{bmatrix}:=M\begin{bmatrix} x_{\Delta_N}(k)\\ w_\Delta(k)\end{bmatrix} \tag{30}$$

where $x_{\Delta_N}:=(x_{cl},\,z)$. Then the feedback connection can be describe as Fig.3.

Following the μ theory, if the estimation error Δ_u be in some range, which is the smallest structured Δ_u that causes instability of the feedback system, the closed-loop system will not became unstable. Therefore, we can enlarge the bound of inaccuracies δ_i^+ or δ_i^- for extending the stability region, and let the estimation error be in that range, then the controller has degined in section 3 can tolerant the FDI failures. We can define the following set:

$$N_{\Delta_u^j}=\{\Delta_u^j:\Delta_u^j=\mathrm{diag}_i[\Delta_{ui}],\Delta_{ui}\in\{\Delta_i^+-\delta_i^+,\Delta_i^--\delta_i^-\}\} \tag{31}$$

where $i=1\dots m,\ j=1\dots 2^m$. Then there are 2^m modes for the estimation errors.

Lemma 5 (Main loop theorem):

$$\mu_{\Delta_S}(M)<1\Leftrightarrow\begin{cases}\mu_{\Delta_u}(M_{22})<1\\ \max\limits_{\Delta_u\in N_{\Delta_u}}\mu_{\Delta_N}(F_l(M,\Delta_u))<1.\end{cases} \tag{32}$$

Proof: The proof can be found in [3].

Consider the following structure:

$$\begin{bmatrix} A^j & B^j\\ C^j & D^j\end{bmatrix}=\begin{bmatrix} A_{cl} & B_{cl}^1\\ C_{cl}^1 & D_{cl}^{11}\end{bmatrix}+\begin{bmatrix} B_{cl}^2\\ D_{cl}^{12}\end{bmatrix}\Delta_u^j\begin{bmatrix} C_{cl}^2 & D_{cl}^{21}\end{bmatrix}.$$

Hence, the uncertain system's output signal $z(k)$ is driven by the input disturbance $w(k)$, and the state equations are given as

$$\begin{bmatrix} x_{cl}^j(k+1)\\ z_j(k)\end{bmatrix}=\begin{bmatrix} A^j & B^j\\ C^j & D^j\end{bmatrix}\begin{bmatrix} x_{cl}^j(k)\\ w_j(k)\end{bmatrix}. \tag{33}$$

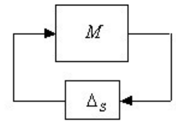

Fig. 3. Feedback Connection with M-Δ_S.

Theorem 2: The closed-loop system (27) is well-posed, stable, and strictly contractive if there exist symmetric matrix $X_j>0$, such that:

$$\begin{bmatrix} A^j & B^j\\ C^j & D^j\end{bmatrix}^T\begin{bmatrix} X_j & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} A^j & B^j\\ C^j & D^j\end{bmatrix}-\begin{bmatrix} X_j & 0\\ 0 & I\end{bmatrix}<0. \tag{34}$$

Proof: In terms of the μ-theory, if the estimation error Δ_u satisfy $\mu_{\Delta_S}(M)<1$, then the closed-loop system is internally stable. Following Lemma 5, the condition is equality with the right side of (32). Since $M_{22}\equiv 0$, then the first condition is always satisfied. For the second condition, the proof can be found in [3, 24]. ■

Remark 2: In virtue of $\sigma_i\in[\min\limits_h\{\underline{\sigma}_i^h\},\max\limits_h\{\bar{\sigma}_i^h\}]\subseteq[0,1]$, the estimation error satisfies $|\Delta_{ui}|\leq 1$ and also $\|\Delta_{ui}\|_\infty\leq 1$. Therefore, the worst case of FDI failure is the estimation equate to $\min\limits_h\{\underline{\sigma}_i^h\}$ or $\max\limits_h\{\bar{\sigma}_i^h\}$. In other words, if we design controller in the jth error mode for the case: $\Delta_u^j=\mathrm{diag}_i[\bar{\Delta}_{ui}]$, $\bar{\Delta}_{ui}\in\{\min\limits_h\{\underline{\sigma}_i^h\}-(1-\hat{\sigma}_i)-\delta_i^-,\max\limits_h\{\bar{\sigma}_i^h\}-(1-\hat{\sigma}_i)-\delta_i^+\}$, $i=1,\dots,m$. Then the closed-loop system will be stable for any estimation in that mode.

V. SIMULATION EXAMPLE

In this section, an example of fault-tolerant control system design is given to demonstrate the proposed method. We consider a randomly generated discrete-time system:

$$\begin{aligned} x(k+1)&=\begin{bmatrix} -0.01 & 0.57\\ -0.19 & -0.63\end{bmatrix}x(k)\\ &\quad+\begin{bmatrix} 0.12 & -1.19\\ 0.06 & -0.18\end{bmatrix}u(k)+\begin{bmatrix} 0.48 & 0.53\\ 0.52 & 0.19\end{bmatrix}w(k)\\ z(k)&=\begin{bmatrix} -0.18 & -0.71\\ 0.35 & -0.05\end{bmatrix}x(k)+\begin{bmatrix} -0.28 & 0.29\\ -0.73 & 0.01\end{bmatrix}u(k)\\ y(k)&=\begin{bmatrix} 0.22 & -0.11\\ 0.89 & 0.35\end{bmatrix}x(k)+\begin{bmatrix} -0.27 & -0.17\\ -0.43 & -0.61\end{bmatrix}w(k).\end{aligned} \tag{35}$$

There are two outputs signals from the controller using two actuators in the system. To verify the effectiveness of the proposed method, We consider the FDI estimation accuracy grade is ± 0.05 for every actuators, namely, $\delta_i\in[\delta_i^+,\delta_i^-]$, where $\delta_i^+=0.05$ and $\delta_i^-=-0.05$ for $i=1,2$.

Here, we consider the following three possible faulty modes:

Normal mode 1: Both of the two actuators are normal, that is, $\sigma_1^1=\sigma_2^1=0$.

Fault mode 2: The first actuator is outage or loss of effectiveness and the second actuator may be normal or loss of effectiveness, described by $a_1\leq\sigma_1^2\leq 1, 0\leq\sigma_2^2\leq a_2, a_1=0.8, a_2=0.3$ which denotes the maximum loss of effectiveness for the first actuator and the second actuator, respectively.

Fault mode 3: The second actuator is outage or loss of effectiveness and the first actuator may be normal or loss of effectiveness, that is, $0\leq\sigma_1^3\leq b_1, b_2\leq\sigma_2^3\leq 1, b_1=0.3, b_2=0.8$ which denotes the maximum loss of effectiveness for the first actuator and the second actuator, respectively.

Clearly, $\max\limits_h\{\bar{\sigma}_i^h\}=1$, $\min\limits_h\{\underline{\sigma}_i^h\}=0$ for $h=1,2,3$. We let $E_1=0_2$, $E_2=I_2$, $E_3=-I_2$, $E_4=I_2$, which satisfying (8).

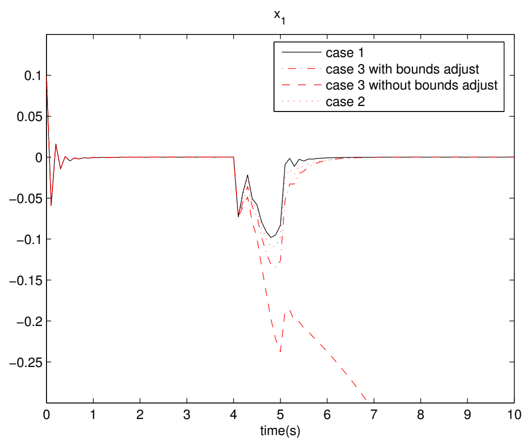

Fig. 4. Response curve of the first state x_1 in there cases with LFT controllers.

Using the process described in the third section, a control structure can be developed, and we consider the following three possible faulty cases:
Case 1: Both of the two actuators are normal, that is, $\sigma_1 = \sigma_2 = 0$.
Case 2: The first actuator loss 30% control effectiveness, the second actuator loss 80% control effectiveness, and the FDI scheme can estimate the faults accurately.
Case 3: With the same faults of case 2, the FDI scheme fail to estimate faults correctly and it gets the estimation $\hat{\sigma}_1 = 0$ and $\hat{\sigma}_2 = 0$, respectively.

Fig.4. shows the simulation result of the above fault cases. we assume that the closed-loop system is operate without faults within the first 4 seconds, and the disturbances $w(k) = [0.1,\ 0.3 \times sin(0.5t)]^T$ enter into the system in $4 \le t \le 5$(s). The solid curves of Fig.4. describes the first state x_1 in normal case with LFT output feedback controller depended on the fault factors $\sigma_1 = \sigma_2 = 0$. The dot curves corresponding to the case 2, where the LTI controller structure is designed via the polytopic method with vertices $\delta_1 \in \{0.05,\ -0.05\}$, $\delta_2 \in \{0.05,\ -0.05\}$, and $\hat{\sigma}_1 = 0.3$, $\hat{\sigma}_2 = 0.8$.

When the FDI scheme failed to estimate the faults, the closed-loop system may be unstable since using the wrong controller. The response curves (dash) is drawn with the case that the FDI get $\hat{\sigma}_1 = 0$ and $\hat{\sigma}_2 = 0$ rather than $\sigma_1 = 0.3$ and $\sigma_2 = 0.8$. If we adjust to $\delta_2^+ = 0.2$ and other bound remains, the response curve (dash-dot) illuminate that the new controller can tolerant that faults. In other words, the uncertainties $\Delta_u = \mathrm{diag}[0.25, 0.6]$ will not make the system unstable.

Through computing, we obtain that the LFT H_∞ performances of the closed-loop system are 0.0834, 0.1563, 0.3267 in the case 1, case 2 and case 3 with bounds adjust, respectively. It is easy to see that the proposed method is work for the above cases of FTC problem.

VI. CONCLUSIONS

In this paper, an LMI method for FTC system design via linear fractional transformations is proposed. Based on the information of control effectiveness from FDI mechanism, the plant and controller can depend on the same estimated values of fault factors to build the closed-loop systems. Then, the LFT controller can be designed using gain-scheduling control approach introduced in [1, 2, 18]. When the fault occurs, for both cases of enough accuracy and failure estimate of fault effect factors, the LFT controller reconstructed on-line to guarantee stability, robustness and satisfactory H_∞ performances for using the μ-theory in the case of fail FDI scheme. The simulation results of the example indicate that the performance has been achieved using this method.

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