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Convexification of the Range-Only Station Keeping Problem

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Abstract— **Using concepts from switched adaptive control theory plus a special parameterization of the class of 2×2 nonsingular matrices, a tractable and provably correct solution is given to the three landmark station keeping problem in the plane in which range measurements are the only sensed signals upon which station keeping is to be based. The performance of the overall system degrades gracefully in the face of increasing measurement and miss-alignment errors, provided the measurement errors are not too large.**

I. Introduction

In this paper as in [1], we take station keeping to mean the practice of keeping a mobile autonomous agent in a position in the plane which is determined by prescribed distances from two or more landmarks. We are particularly interested in solutions to the station keeping problem in which the only signals available to the agent whose position is to be maintained, are noisy range measurements from its neighbors[1].

Work on the range-only station keeping problem already exists [2], [3], [4]. Our approach to station keeping builds on the work initiated in [1] where we treated station keeping as a problem in switched adaptive control. We continue with the same approach in this paper but now deal directly with an important computational issue which was not addressed in [1]. In particular, the control system considered in [1] requires an algorithm capable of minimizing with respect to the four entries in a 2×2 nonsingular matrix P, a cost function of the form $M(X,P) = \mathrm{trace}\{\begin{bmatrix} I & P \end{bmatrix} X \begin{bmatrix} I & P \end{bmatrix}'\}$ where X is a 4×4 positive semi-definite matrix. What makes the problem difficult is the constraint that P must be non-singular, since this leads to a non-convex optimization problem. The main contribution of this paper is to explain how to avoid this difficulty by utilizing the fact that any 2×2 non-singular matrix B can be written as $B = U(I + L)S$ where U is a specially structured matrix from a finite set, L is strictly lower triangular and S is symmetric and positive definite [5]. This fact enables us to modify the optimization problem just described, so that instead of having non-convex problem to solve, one has a a finite set of convex problems. Not only does the modification lead to convex programming problems, but also programming problems which can each be solved efficiently using semi-definite programming methods [6].

In Section II we formulate the station keeping problem of interest. Error models appropriate to the solution to the problem are presented in Section III. In Section IV we present a switched adaptive control system which solves the three neighbor station keeping problem for a point modelled agent. In Section V we state our main results. In Section VI and VII we explain how to implement the proposed control system by re-formulating a non-convex optimization problem, specific to the problem at hand, as a semi-definite programming problem utilizing a matrix decomposition technique.

II. Formulation

Let $n > 1$ be an integer. The system of interest consists of $n + 1$ points in the plane, labelled $0, 1, 2, \ldots, n$, which will be referred to as agents. Let $x_0, x_1, \ldots, x_n$ denote the coordinate vector of current positions of agents $0, 1, 2, \ldots n$ respectively with respect to a common frame of reference. Assume

$$\dot{x}_i = 0, \quad i \in \{1, 2, 3, \ldots, n\} \tag{1}$$

We further assume that the nominal model for how agent 0 moves is a kinematic point model of the form

$$\dot{x}_0 = u \tag{2}$$

where u is an open loop control taking values in $\mathbb{R}^2$.

Suppose that agent 0 can sense its distances $y_1, y_2, y_3, \ldots, y_n$ from neighboring agents $1, 2, 3, \ldots, n$ with uniformly bounded, additive errors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ respectively. Thus

$$y_i = ||x_i - x_0|| + \epsilon_i, \quad i \in \{1, 2, \ldots, n\} \tag{3}$$

where $|| \cdot ||$ denotes the Euclidian 2-norm. Suppose in addition that agent 0 is given a set of non-negative numbers $d_1, d_2, \ldots, d_n$, where d_i represents a desired distance from agent 0 to agent i. The problem is to devise a control law depending on the d_i and the y_i which, were the ϵ_i all zero, would cause agent 0 to move to a position in the formation which, for $i \in \{1, 2, \ldots, n\}$, is d_i units from agent i. We call this the n *neighbor station keeping problem.*

Let x^* denote the target position to which agent 0 would have to move were the station keeping problem solvable. Then x^* would have to satisfy

$$d_i = ||x_i - x^*||, \quad i \in \{1, 2, \ldots, n\} \tag{4}$$

To account for the more realistic situation when points are out of alignment, we will assume instead of (4), that there is a value of x^* for which

$$d_i = ||x^* - x_i|| + \bar{\epsilon}_i, \quad i \in \{1, 2, \ldots, n\} \tag{5}$$

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[1]We are indebted to B.D.O. Anderson for making us aware of this problem.

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where each $\bar{\epsilon}_i$ is a small miss-alignment error.

Our specific control objective can now be stated. Devise a feedback control for agent 0, using the d_i and measurements y_i, which bounds the induced $\mathcal{L}^2$ gains from each ϵ_i and each $\bar{\epsilon}_i$ to each of the errors

$$e_i = y_i^2 - d_i^2, \quad i \in \{1, 2, 3, \ldots, n\} \tag{6}$$

We will address this problem using well known concepts and constructions from adaptive control.

III. Error Model

The controllers which we propose to study will all be based on the following error model that has been developed in [1] for the case where $n = 3$. Let $e = \begin{bmatrix} e_1 - e_3 & e_2 - e_3 \end{bmatrix}'$ and define $q = B\bar{x}_0$, where $B = 2\begin{bmatrix} x_3 - x_1 & x_3 - x_2 \end{bmatrix}'$ and $\bar{x}_0 = x_0 - x^*$. The error model is then

$$e = q + \epsilon ||B^{-1}q|| + \eta \tag{7}$$
$$\dot{q} = Bu \tag{8}$$

where $\epsilon = 2\begin{bmatrix} \epsilon_1 - \epsilon_3 & \epsilon_2 - \epsilon_3 \end{bmatrix}'$, $\eta = \begin{bmatrix} \eta_1 - \eta_3 & \eta_2 - \eta_3 \end{bmatrix}'$ and $\eta_i = 2\epsilon_i||x_i - x_0|| + \epsilon_i^2 - 2\bar{\epsilon}_i||x_i - x^*|| - \bar{\epsilon}_i^2 - 2\epsilon_i||\bar{x}_0||$.

Our assumption that the x_i are not co-linear implies that B is non-singular. Note that since B is nonsingular, $x_0 = x^*$ whenever $q = 0$. This in turn will be the case when $e = 0$ provided $\epsilon = 0$ and $\eta = 0$. The term $||B^{-1}q||\epsilon$ can be regarded as a perturbation and can be dealt with using standard small gain arguments.

IV. Station Keeping Supervisory Controller

In the sequel we will assume that $||\epsilon|| \leq \epsilon^*, \;\; t \geq 0$ where ϵ^* is a positive constant which satisfies the constraint

$$\epsilon^* < \frac{1}{||B^{-1}||} \tag{9}$$

The type of control system we intend to develop assumes that B is unknown, but requires one to define at the outset a closed bounded subset of 2×2 non-singular matrices $\mathcal{P} \subset \mathbb{R}^{2\times 2}$ which is big enough so that it can be assumed that $B \in \mathcal{P}$. It is clear that because of the non-singularity requirement, just about any reasonably defined parameter space $\mathcal{P}$ which satisfies these conditions would not be convex, or even the union of a finite number of convex sets.

The supervisory control system to be considered consists of a "multi-estimator" $\mathbb{E}$, a "multi-controller" $\mathbb{C}$, a "monitor" $\mathbb{M}$ and a "dwell-time switching logic" $\mathbb{S}$. These terms and definitions have been discussed before in [7] and elsewhere. The numbered equations which follow, are the equations which define the supervisory controller we will consider.

A. *Multi-Estimator* $\mathbb{E}$

For the problem of interest, the multi-estimator $\mathbb{E}$ is defined by the two equations

$$\dot{z}_1 = -\lambda z_1 + \lambda e \tag{10}$$
$$\dot{z}_2 = -\lambda z_2 + u \tag{11}$$

where λ is a design constant which must be positive but is otherwise unconstrained.

Note that the signal $\rho = z_1 + Bz_2 - q$ satisfies

$$\dot{\rho} = -\lambda\rho + \lambda(\epsilon||B^{-1}q|| + \eta)$$

For $P \in \mathcal{P}$, let $\bar{e}_P$ denote the Pth *output estimation error*

$$\bar{e}_P = z_1 + Pz_2 - e$$

The relevant relationships between these signals when $P = B$ can be conveniently described by the block diagram in Figure 1. The diagram describes a nonlinear dynamical system with inputs η and $z_1 + Bz_2$ and outputs $\bar{e}_B$.

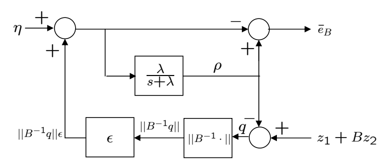

Fig. 1. Subsystem

B. *Multi-Controller* $\mathbb{C}$

The multi-controller $\mathbb{C}$ we propose to study is simply

$$u = -\lambda\widehat{B}^{-1}e \tag{12}$$

where $\widehat{B}$ is a suitably defined piecewise constant switching signal taking values in $\mathcal{P}$. The definition of u has been crafted so that the "closed-loop parameterized system" matrix $-\lambda PP^{-1}$ is stable with "stability margin" λ for all $P \in \mathcal{P}$. The consequence of this definition of u is predicted by the certainty equivalence stabilization theorem [8] and is as follows. Let $\bar{e}_{\widehat{B}} = z_1 + \widehat{B}z_2 - e$ and define the so-called *injected sub-system* to be the system with input $\bar{e}_{\widehat{B}}$ and output $z_1 + Bz_2$ which results when $z_1 + Bz_2 - \bar{e}_{\widehat{B}}$ is substituted for e in the closed loop system determined by (10), (11) and (12). Thus

$$\dot{z}_1 = \lambda\widehat{B}z_2 - \lambda\bar{e}_{\widehat{B}}$$
$$\dot{z}_2 = -\lambda\widehat{B}^{-1}z_1 - 2\lambda z_2 + \lambda\widehat{B}^{-1}\bar{e}_{\widehat{B}}$$

Certainty equivalence implies that this system, viewed as a dynamical system with input $\bar{e}_{\widehat{B}}$, is also stable with stability margin λ for each fixed $\widehat{B} \in \mathcal{P}$. In this special case one can deduce this directly using the state transformation $\{z_1, z_2\} \longmapsto \{z_1, z_1 + \widehat{B}z_2\}$. For this system to have stability margin λ means that for any positive number $\lambda_0 < \lambda$ the matrix $\lambda_0 I + A(\widehat{B})$ is exponentially stable for all constant $\widehat{B} \in \mathcal{P}$. Here

$$A(\widehat{B}) = \begin{bmatrix} 0 & \lambda\widehat{B} \\ -\lambda\widehat{B}^{-1} & -2\lambda I \end{bmatrix}$$

which is the state coefficient matrix of the injected system.

In the sequel, we fix λ_0 at any positive value such that $\lambda_0 < \lambda(1 - \epsilon^*)||B||^{-1}$. This number turns out to be a lower bound on the convergence rate for the entire closed-loop control system.

We need to pick one more positive design parameter, called a *dwell time* τ_D. This number has to be chosen large enough so that the injected linear system defined above is exponentially stable with stability margin λ for every "admissible" piecewise constant switching signal $\widehat{B} : [0, \infty) \to \mathcal{P}$, where by *admissible* we mean a piecewise constant signal whose switching instants are separated by at least τ_D time units. This is easily accomplished because each $\lambda_0 I + A(P),\ \ P \in \mathcal{P}$ is a stability matrix. All that's required then is to pick τ_D large enough so that the induced norm {any matrix norm} of each matrix $e^{\{\lambda_0 I + A(P)\}t},\ P \in \mathcal{P}$, is less than 1.

C. Monitor $\mathbb{M}$

The state dynamic of monitor $\mathbb{M}$ is defined by the equation

$$\dot{W} = -2\lambda_0 W + \begin{bmatrix} z_1 - e \\ z_2 \end{bmatrix} \begin{bmatrix} z_1 - e \\ z_2 \end{bmatrix}' \qquad (13)$$

where W is a "weighting matrix" which takes values in the linear space $\mathcal{X}$ of 4×4 symmetric matrices; although not crucial, for simplicity we will require $\mathbb{M}$ to be initialized at zero. This clearly implies that $W(t)$ is positive semi-definite for all $t \geq 0$. Note that it takes only 10 differential equations rather than 16 to generate W because of symmetry.

1) The output of $\mathbb{M}$ *- first pass:* The output of $\mathbb{M}$ is a parameter dependent "monitoring signal" which for the moment we define to be $\mu_P = M(W, P)$ where $M : \mathcal{X} \times \mathcal{P} \to \mathbb{R}$ is the scalar-valued function

$$M(X, P) = \text{trace}\{\begin{bmatrix} I & P \end{bmatrix} X \begin{bmatrix} I & P \end{bmatrix}'\}$$

The μ_P are helpful in motivating the definition of $\mathbb{M}$ and the switching logic $\mathbb{S}$ which follows; however, they are actually not used anywhere in the implemented system.

Note that for any $P \in \mathcal{P}$,

$$\dot{\mu}_P = -2\lambda_0 \mu_P + \text{trace}(\{z_1 - e + Pz_2\}\{z_1 - e + Pz_2\}')$$

so

$$\dot{\mu}_P = -2\lambda_0 \mu_P + ||z_1 - e + Pz_2||^2$$

But $\bar{e}_P = z_1 - e + Pz_2$. Therefore

$$\dot{\mu}_P = -2\lambda_0 \mu_P + ||\bar{e}_P||^2$$

Thus

$$M(W, P) = \int_0^t e^{-2\lambda_0 (t-s)} ||\bar{e}_P||^2 ds$$

Thus if we introduce the exponentially weighted 2-norm

$$||\omega||_t = \sqrt{\int_0^t \{e^{\lambda_0 s} ||\omega(s)||\}^2 ds}$$

where ω is a piecewise continuous signal, then

$$M(W(t), P) = e^{-2\lambda_0 t} ||\bar{e}_P||_t^2, \quad t \geq 0$$

Minimizing $M(W(t), P)$ with respect to P and setting $\widehat{B}(t)$ to the resulting minimizing value, would then yield an inequality of the form $||\bar{e}_{\widehat{B}}||_t \leq ||e_B||_t$. Were it possible to accomplish this at every instant of time and were $\widehat{B}$ changing slowly enough, then one could conclude that for ϵ^* sufficiently small, the resulting overall system with input η and output e would be stable with respect to the exponentially weighted norm we've been discussing. It is of course not possible to carry out these steps instantly. Were we to continue with this definition of μ_P, we would nonetheless, want to minimize $M(W(t), P)$ from time to time and in doing so would end up with an input-output stable system. In fact the implementation of dwell time switching proposed in [1] requires such minimizations to be carried out. But were we to proceed with this approach, we'd run head on into an important practical problem which we want to address.

2) A Non-Convex Parameter Space: Note that even though $M(X, P)$ is a quadratic positive semi-definite function of the elements of P, the problem of minimizing $M(X, P)$ over $\mathcal{P}$ is still very complex because $\mathcal{P}$ is not typically convex or even a finite union of convex sets. Thus if we were to use such a parameter space and proceed as we've just outlined, we'd be faced with an intractable non-convex optimization problem. The root of the problem stems from the requirement that the algebraic curve

$$\mathcal{C} = \{P : p_{11}p_{22} - p_{12}p_{21} = 0\}$$

in $\mathbb{R}^{2\times 2}$ on which P is singular cannot intersect $\mathcal{P}$. The key idea in our approach to avoid the tractability problem is to use a different parameterization which we describe next.

3) Re-parameterization: Let $\mathcal{U}$ denote the set of all 2×2 matrices U, where each U is a matrix of 0's, 1's and -1's having exactly one nonzero entry in each row and column; there are exactly eight such matrices. It is known [5] that any 2×2 nonsingular matrix M can be written as $M = U(I + L)S$ for some $U \in \mathcal{U}$, some strictly lower triangular matrix L and some symmetric positive definite matrix S. This suggests that we consider a parameter space

$$\mathcal{P} = \{U(I + L)S : \{U, L, S\} \in \mathcal{U} \times \mathcal{L} \times \mathcal{S}\}$$

where $\mathcal{L}$ is a compact, convex subset of the linear space of strictly lower triangular 2×2 matrices and $\mathcal{S}$ a compact, convex subset of the convex set of all 2×2 positive definite matrices. Notice that this definition of $\mathcal{P}$ satisfies both the compactness requirement and the requirement that its elements are all non-singular matrices. Of course one needs to also make sure that $\mathcal{L}$ and $\mathcal{S}$ are large enough so that $B \in \mathcal{P}$. For the present we will assume that $B \in \mathcal{P}$ and thus that there are matrices $U_B \in \mathcal{U},\ L_B \in \mathcal{L}$ and $S_B \in \mathcal{S}$ such that $B = U_B(I + L_B)S_B$.

In the sequel we will show that it is possible to meaningfully redefine the type of optimization referred to above as the problem of minimizing a function $J(U, L, S)$ over the set $\mathcal{U} \times \mathcal{L} \times \mathcal{S}$. While this set is not convex, $\mathcal{L} \times \mathcal{S}$ is. Moreover, as we shall see, for each fixed $U \in \mathcal{U},\ J(U, L, S)$ is a convex, quadratic function of the entries in L and S. Because of this, the minimization of $J(U, L, S)$ over $\mathcal{U} \times \mathcal{L} \times \mathcal{S}$ boils down to solving eight convex programming problems, one for each $U \in \mathcal{U}$.

4) The output of $\mathbb{M}$ *- second pass:* In the light of the preceding discussion we now re-define $\mathbb{M}$'s output to be $\mu_{\{U,L,S\}} = M(W, U, L, S)$ where now $M : \mathcal{X} \times \mathcal{U} \times \mathcal{L} \times \mathcal{S} \to$

$\mathbb{R}$ is

$$M(X,U,L,S) = \text{trace}\{\left[(I-L)U' \quad S\right] X \left[(I-L)U' \quad S\right]'\} \quad (14)$$

In this case it is easy to see that

$$M(W(t),U,L,S) = e^{-2\lambda_0 t}||(I-L)U'\bar{e}_P||_t^2, \quad t \geq 0$$

where $P = U(I+L)S$. In deriving this expression for M, we have made use of the easily verified formulas $U' = U^{-1},\ U \in \mathcal{U}$ and $(I+L)^{-1} = I-L,\ L \in \mathcal{L}$.

The matrix $\widehat{B}$ used in the definition of u in (12) is now defined by the formula

$$\widehat{B} = \widehat{U}(I+\widehat{L})\widehat{S} \quad (15)$$

where $\{\widehat{U}, \widehat{L}, \widehat{S}\}$ is a piecewise constant switching signal taking values in $\mathcal{U}\times\mathcal{L}\times\mathcal{S}$. This signal will be generated by a "dwell-time switching logic".

D. Dwell-time Switching Logic $\mathbb{S}$

For our purposes a *dwell-time switching logic* $\mathbb{S}$, is a hybrid dynamical system whose input and output are W and $\widehat{B}$ respectively, and whose state is the ordered triple $\{X, \tau, \{\widehat{U}, \widehat{L}, \widehat{S}\}\}$. Here X is a discrete-time matrix which takes on sampled values of W, and τ is a continuous-time variable called a *timing signal*. τ takes values in the closed interval $[0, \tau_D]$. Also assumed pre-specified is a *computation time* $\tau_C \leq \tau_D$ which bounds from above for any $X \in \mathcal{W}$, the time it would take to compute a value $\{U, L, S\} \in \mathcal{U}\times\mathcal{L}\times\mathcal{S}$ which minimizes $M(X,U,L,S)$. Between "event times," τ is generated by a reset integrator according to the rule $\dot{\tau} = 1$. Event times occur when the value of τ reaches either $\tau_D - \tau_C$ or τ_D; at such times τ is reset to either 0 or $\tau_D - \tau_C$ depending on the value of $\mathbb{S}$'s state. $\mathbb{S}$'s internal logic is defined by the flow diagram shown in Figure 2 where $\{U_X, L_X, S_X\}$ denotes a value of $\{U, L, S\} \in \mathcal{U}\times\mathcal{L}\times\mathcal{S}$ which minimizes $M(X,U,L,S)$.

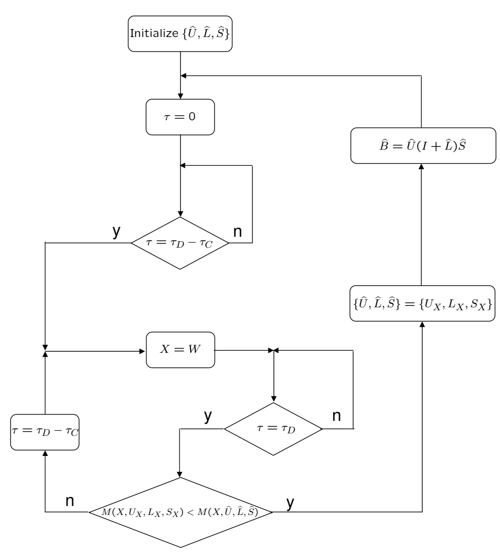

Fig. 2. Dwell-Time Switching Logic $\mathbb{S}$

The definition of $\mathbb{S}$ clearly implies that its output $\widehat{B}$ is an admissible switching signal. This means that switching cannot occur infinitely fast and thus that existence and uniqueness of solutions to the differential equations involved is not an issue.

Note that implementation of the switching logic just described requires an algorithm capable of minimizing $\text{trace}\{M(X,U,L,S)\}$ over $\mathcal{U}\times\mathcal{L}\times\mathcal{S}$ for various values of $X \in \mathcal{X}$. As we've already explained, for each fixed $U \in \mathcal{U}$, and $X \in \mathcal{X}$, minimization of $\text{trace}\{M(X,U,L,S)\}$ reduces to a convex programming problem. Thus for each $X \in \mathcal{X}$, it is enough to solve eight convex programming problems, one for each value of $U \in \mathcal{U}$; the results of these eight computations can then be compared to find the values of U, L and S which attain a global minimum of $\text{trace}\{M(X,U,L,S)\}$ over $\mathcal{U}\times\mathcal{L}\times\mathcal{S}$. In other words, by making use of the parameterization we've been discussing, we've been able to reformulate the overall adaptive algorithm in such a way that at each event time all that is necessary is to solve eight, independent quadratic programming problems, one for each $U \in \mathcal{U}$. Of course each of these eight problems may still be challenging. In Section VII we will explain how each can be reformulated as a semi-definite programming problem.

V. Results

The results which follow rely heavily on the following proposition which characterizes the effect of the monitor-dwell time switching logic subsystem.

Proposition 1: Suppose that $W(0) = 0$, that $\widehat{B} = \widehat{U}(I+\widehat{L})\widehat{S}$ is the response of the monitor-switching logic subsystem $\{\mathbb{M}, \mathbb{S}\}$ to any continuous input signals e, z_1, and z_2 taking values in $\mathbb{R}^2$, and that for $\{U,L,S\} \in \mathcal{U}\times\mathcal{L}\times\mathcal{S}$, $\bar{e}_P = (z_1 - e) + Pz_2$ where $P = U(I+L)S$. For each real number $\gamma > 0$ and each fixed time $T > 0$, there exists piecewise-constant signals $H : [0,\infty) \to \mathbb{R}^{2\times 4}$ and $\psi : [0,\infty) \to \{0,1\}$ such that

$$|H(t)| \leq \gamma, \quad t \geq 0 \quad (16)$$

$$\int_0^\infty \psi(t)dt \leq 4(\tau_D + \tau_C) \quad (17)$$

and

$$||(1-\psi)(\bar{e}_{\widehat{B}} - Hz) + \psi\bar{e}_B||_T \leq \delta||\bar{e}_B||_T \quad (18)$$

where $z = \left[z_1' \quad z_2'\right]'$, $\delta = 1 + 8\alpha^2\left(\frac{1+\text{diameter}\{\mathcal{P}\}}{\gamma}\right)^4$, and $\alpha = \max_{L\in\mathcal{L}}||I+L||$.

This proposition is a minor modification of a similar proposition proved in [7]. The proposition summarizes the key consequences of dwell time switching which are needed to analyze the system under consideration. While the inequality in (18) is more involved than the inequality $||\bar{e}_{\widehat{B}}||_t \leq ||\bar{e}_B||_t$ mentioned earlier, the former is provably correct whereas the latter is not. Despite its complexity, (18) can be used to establish input-output stability with respect to the exponentially weighted norm $||\cdot||_t$. The idea is roughly as follows. Fix $T > 0$ and pick γ small enough so that

$\lambda_0 I + A(\widehat{B}) + (1-\psi)D(\widehat{B})H$ is exponentially stable where $A(\widehat{B})$ is the state evolution matrix of the injected system defined at the beginning of Section IV-B and $D(\widehat{B}) = \begin{bmatrix} -\lambda I' & \lambda(\widehat{B}^{-1})' \end{bmatrix}'$. The fact that ψ has a finite $\mathcal{L}^1$ norm {cf. (17)}, implies that $\lambda_0 I + A(\widehat{B}) + (1-\psi)D(\widehat{B})H + \psi \begin{bmatrix} 0 & \widehat{B} - B \end{bmatrix}$ is exponentially stable as well. Next define $\bar{e} = (1-\psi)(\bar{e}_{\widehat{B}} - Hz) + \psi \bar{e}_B$. Then

$$||\bar{e}||_T \leq \delta ||\bar{e}_B||_T \tag{19}$$

because of (18). The definition of $\bar{e}$ implies that

$$\bar{e}_{\widehat{B}} = \bar{e} + (1-\psi)Hz + \psi \begin{bmatrix} 0 & \widehat{B} - B \end{bmatrix} z$$

Substitution into the injected system defined earlier yields the exponentially stable system

$$\dot{z} = \{A(\widehat{B}) + (1-\psi)D(\widehat{B})H + \psi \begin{bmatrix} 0 & \widehat{B} - B \end{bmatrix}\} z + D(\widehat{B})\bar{e}$$

with input $\bar{e}$. Now add to Figure 1, two copies of the system just defined, one $\{\bar{\Sigma}_1\}$ with output $e = \begin{bmatrix} I & \widehat{B} \end{bmatrix} z - \{\bar{e} + (1-\psi)Hz + \psi \begin{bmatrix} 0 & \widehat{B} - B \end{bmatrix} z\}$ and the other $\{\bar{\Sigma}_2\}$ with output $z_1 + Bz_2 = \begin{bmatrix} I & B \end{bmatrix} z$. The multiple copies are valid because the matrix $A(\widehat{B}) + (1-\psi)D(\widehat{B})H + \psi \begin{bmatrix} 0 & \widehat{B} - B \end{bmatrix}$ is exponentially stable. The resulting overall system is shown in Figure 3.

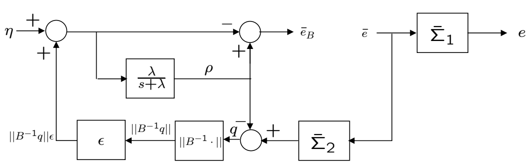

Fig. 3. Snapshot at time T of the Overall Subsystem for Analysis

In the light of (19), it is easy to see that if the bound ϵ^* on ϵ is sufficiently small, the induced gain of this system from η to e with respect to $||\cdot||_T$ is bounded by a finite constant g_T. It can be shown that g_T in turn, is bounded above by a constant g not depending on T [7]. Since this is true for all T, it must be true that g bounds the induced gain from η to e with respect to $||\cdot||_\infty$.

The following results are fairly straightforward consequences of these ideas. (a) If all measurement errors ϵ_i and all miss-alignment errors $\bar{e}_i$ are zero, then, no matter what its initial value, $x_0(t)$ tends to the unique solution x^* to (4) as fast as $e^{-\lambda_0 t}$. (b) If the measurement errors ϵ_i and the miss-alignment errors $\bar{e}_i$ are not all zero, and the ϵ_i sufficiently small, then no matter what its initial value, $x_0(t)$ tends to a value for which the norm of the error e is bounded by a constant times the sum of the norms of the ϵ_i and the $\bar{e}_i$.

VI. Definitions for $\mathcal{L}$ and $\mathcal{S}$

So far we've assumed that $\mathcal{L}$ is a compact, convex subset of the linear space of strictly lower triangular 2×2 matrices and that $\mathcal{S}$ is a compact, convex subset of the set of positive definite 2×2 matrices. The assumptions are sufficient to ensure that any matrix in $\mathcal{P} = \{U(I+L)S : (U, L, S) \in \mathcal{U} \times \mathcal{L} \times \mathcal{S}\}$ is invertible and also that the minimization of

$$M(X, U, L, S) = \text{trace}\{\begin{bmatrix} (I-L)U' & S \end{bmatrix} X \begin{bmatrix} (I-L)U' & S \end{bmatrix}'\} \tag{20}$$

over $\mathcal{L} \times \mathcal{S}$ for any fixed $U \in \mathcal{U}$ and any fixed positive semi-definite 2×2 matrix X, is a convex programming problem. But we've not yet explained how to explicitly define $\mathcal{L}$ and $\mathcal{S}$. To do this, it makes sense to first define bounds for B which are meaningful for the problem at hand. Towards this end, suppose that agent 0 has a limited sensing radius ρ. Since we've assumed that agent 0 can sense the distances to agents $1, 2$, and 3, it must be true that $||x_3 - x_1|| \leq 2\rho$ and $||x_3 - x_1|| \leq 2\rho$. But $B = 2\begin{bmatrix} x_3 - x_1 & x_3 - x_2 \end{bmatrix}'$. Prompted by this we will assume that $\sqrt{B'B} \leq \beta_2 I$ where $\beta_2 = 4\rho$.

We've also assumed that agents $1, 2$ and 3 are not positioned along a line; this is equivalent to B being nonsingular. One measure of B's nonsingularity, is its smallest singular value. Prompted by this, we will assume that there is a positive number β_1 such that $\sqrt{B'B} \geq \beta_1 I$; β_1 might be chosen empirically to reflect the degree to which the three leader agents are supposed non-colinear in a given formation. We shall assume that such a number has been chosen and moreover that $\beta_1 < \beta_2$. In summary we suppose that bounds β_1 and β_2 have been derived such that

$$\beta_1 I \leq \sqrt{B'B} \leq \beta_2 I \tag{21}$$

where β_1 and β_2 are distinct positive numbers. It is obvious that the set of matrices B satisfying these inequalities is not convex.

Our next objective is to define $\mathcal{L}$ and $\mathcal{S}$ so that any matrix B satisfying (21) is in $\mathcal{P}$. Let $\mathcal{L}$ be the set of all strictly lower triangular 2×2 matrices $L = \begin{bmatrix} l_{ij} \end{bmatrix}$ for which

$$|l_{21}| \leq 1 + \sqrt{2}\frac{\beta_2}{\beta_1} \tag{22}$$

In addition, let $\mathcal{S}$ be the set of all 2×2, symmetric matrices satisfying

$$\sigma_1 I \leq S \leq \sigma_2 I \tag{23}$$

where

$$\sigma_1 = \frac{1}{\left(2\sqrt{1 + \left(\frac{\beta_2}{\beta_1}\right)^2}\right)}\beta_1 \quad \sigma_2 = \left(2\sqrt{1 + \left(\frac{\beta_2}{\beta_1}\right)^2}\right)\beta_2 \tag{24}$$

It will now be shown that any matrix B satisfying (21) is in $\mathcal{P}$.

As a first step, let us note that b_{11} and b_{21} cannot both be zero because B is nonsingular. If $|b_{11}| \geq |b_{21}|$, let

$$\begin{aligned} U &= \begin{bmatrix} \text{sign}\{b_{11}\} & 0 \\ 0 & \text{sign}\{b_{11}d\} \end{bmatrix} \\ L &= \begin{bmatrix} 0 & 0 \\ \frac{u_{22}b_{21} - u_{11}b_{12}}{|b_{11}|} & 0 \end{bmatrix} \\ S &= \begin{bmatrix} |b_{11}| & u_{11}b_{12} \\ u_{11}b_{12} & \frac{b_{12}^2 + |d|}{|b_{11}|} \end{bmatrix} \end{aligned} \tag{25}$$

On the other hand, if $|b_{21}| > |b_{12}|$, let

$$U = \begin{bmatrix} 0 & -\text{sign}\{b_{21}d\} \\ \text{sign}\{b_{21}\} & 0 \end{bmatrix}$$
$$L = \begin{bmatrix} 0 & 0 \\ \frac{u_{12}b_{11}-u_{21}b_{22}}{|b_{21}|} & 0 \end{bmatrix}$$
$$S = \begin{bmatrix} |b_{21}| & u_{21}b_{22} \\ u_{21}b_{22} & \frac{b_{22}^2+|d|}{|b_{21}|} \end{bmatrix} \qquad (26)$$

In either case it is easy to verify that $B = U(I + L)S$. It is also clear that in either case $U \in \mathcal{U}$, that L is strictly lower triangular and that S is symmetric. Thus to prove that $B \in \mathcal{P}$, it is sufficient to show that in either of the two cases, L and S satisfy (22) and (23) respectively. We will do this only for the case $|b_{11}| \geq |b_{21}|$ as similar reasoning applies to the case $|b_{21}| < |b_{11}|$.

Let us note from (25) that $|l_{21}| \leq \left|\frac{b_{21}}{b_{11}}\right| + \left|\frac{b_{12}}{b_{11}}\right|$. By assumption $|b_{11}| \geq |b_{21}|$; this implies that $\left|\frac{b_{21}}{b_{11}}\right| \leq 1$, so $|l_{21}| \leq 1 + \left|\frac{b_{12}}{b_{11}}\right|$. Now from (21), $\beta_1 \leq \sqrt{b_{11}^2 + b_{21}^2}$, so $\beta_1 \leq \sqrt{2b_{11}^2} = \sqrt{2}|b_{11}|$; also from (21), $|b_{12}| \leq \beta_2$. Therefore $\left|\frac{b_{12}}{b_{11}}\right| \leq \sqrt{2}\frac{\beta_2}{\beta_1}$. It follows that l_{21} satisfies (22).

Next observe that $B'B = S(I+L)'U'U(I+L)S = S(I+L)'(I+L)S$. Now $(I+L)'(I+L) \leq (2+|l_{12}|^2)I$. Therefore $B'B \leq (2 + |l_{12}|^2)S^2$. From this and (21), it follows that $S^2 \geq \frac{\beta_1^2}{2+|l_{12}|^2}I$. From (22),

$$l_{21}^2 \leq 2\left(1 + 2\frac{\beta_2^2}{\beta_1^2}\right) \qquad (27)$$

Therefore $S^2 \geq \frac{\beta_1^4}{4(\beta_1^2+\beta_2^2)}I = \sigma_1^2 I$.

Finally observe that $S = (I - L)U'B$ and thus that $S^2 = B'U(I-L)'(I-L)U'B$. But $(I-L)'(I-L) \leq (2+|l_{12}|^2)I$. Therefore $S^2 \leq (2 + |l_{12}|^2)B'UU'B = (2 + |l_{12}|^2)B'B$. From this (21), and (27) it follows that $S^2 \leq 4(1 + \frac{\beta_2^2}{\beta_1^2})\beta_2^2 I$. Therefore S satisfies both inequalities in (23). This means that $B \in \mathcal{P}$.

VII. Semi-Definite Programming Formulation

Fix $U \in \mathcal{U}$, and let $X \in \mathcal{X}$ be a given positive semi-definite matrix. To implement the dwell time switching logic defined in Section IV-D, it is necessary to make use of an algorithm capable of minimizing over $\mathcal{L} \times \mathcal{S}$, a cost function of the form

$$N(L, S) = \text{trace}\{\begin{bmatrix}(I - L)U' & S\end{bmatrix} X \begin{bmatrix}(I - L)U' & S\end{bmatrix}'\} \qquad (28)$$

Our aim is to explain how to reformulate this convex optimization problem as a convex semi-definite programming problem over the space $\mathcal{Y} \times \mathcal{L} \times \mathcal{Y}$ where $\mathcal{Y}$ is the linear space of 2×2 symmetric matrices[2]. As a first step towards this end, we exploit two easily proved facts. First, if (L_1, S_1) minimizes $N(L, S)$ over $\mathcal{L} \times \mathcal{S}$, then $(\{\begin{bmatrix}(I - L_1)U_1' & S_1\end{bmatrix} X \begin{bmatrix}(I - L_1)U_1' & S_1\end{bmatrix}'\}, L_1, S_1)$ minimizes $\bar{N}(Y, L, S) = \text{trace}\{Y\}$ over $\mathcal{Y} \times \mathcal{L} \times \mathcal{S}$ subject to the constraint that $Y - [(I - L_1)U_1' \ S_1]X[(I - L_1)U_1' \ S_1]'$ is positive semi-definite. Second, if (Y_2, L_2, S_2) minimizes $\bar{N}(Y, L, S)$ over $\mathcal{Y} \times \mathcal{L} \times \mathcal{S}$ subject to the constraint that $Y - [(I - L_1)U_1' \ S_1]X[(I - L_1)U_1' \ S_1]'$ is positive semi-definite, then (L_2, S_2) minimizes $N(L, S)$ over $\mathcal{L} \times \mathcal{S}$. In other words, the optimization problem of interest is equivalent to minimizing the cost $\bar{N}(Y, L, S)$ over $\mathcal{Y}\times\mathcal{L}\times\mathcal{S}$ subject to the constraint

$$Y - \begin{bmatrix}(I - L)U' & S\end{bmatrix} X \begin{bmatrix}(I - L)U' & S\end{bmatrix}' \geq 0 \qquad (29)$$

To proceed, let us next observe that the matrix to the left in the above inequality, is the Schur complement of the matrix

$$Q = \begin{bmatrix} I & R'\begin{bmatrix}(I - L)U' & S\end{bmatrix}' \\ \begin{bmatrix}(I - L)U' & S\end{bmatrix} R & Y \end{bmatrix}$$

where R is any matrix such that $X = RR'$. Thus the matrix inequality in (29) is equivalent to the matrix inequality $Q \geq 0$ Moreover the constraint that $S \in \mathcal{S}$ is equivalent to $S \in \mathcal{Y}$ and the pair of linear matrix inequality constraints $\sigma_2 I - S \geq 0$ and $S - \sigma_1 I \geq 0$. These constraints can be combined with $Q \geq 0$ to give finally the constraint

$$\begin{bmatrix} Q & 0 & 0 \\ 0 & \sigma_2 I - S & 0 \\ 0 & 0 & S - \sigma_1 I \end{bmatrix} \geq 0 \qquad (30)$$

Thus we've reduced the optimization problem of interest to minimizing $\bar{N}(Y, L, S)$ over $\mathcal{Y}\times\mathcal{L}\times\mathcal{Y}$ subject to (30). Since (22) is equivalent to two linear inequality constraints, the problem to which we've been led is a conventional convex, semi-definite programming problem [6].

VIII. Concluding Remarks

In this paper we have used standard constructions from adaptive control to devise a tractable solution to the three neighbor station keeping problem. The solution is the same as that in [1] except that here a special parameterization is used to avoid the non-convex optimization problem which must be solved in order to implement the algorithm in [1].

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[2]We are indebted to Ali Jadbabai for making us aware of this simplification.