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Improved Robust $\mathscr{D}_u$-stability measures via S-procedure

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***Abstract*— In this paper we focus on the notion of robust matrix root-clustering analysis in a union of regions that are possibly disjoint and non symmetric. Indeed this work aims at computing a bound on the size of the uncertainty domain preserving matrix $\mathscr{D}_u$-stability. A Linear Fractional Transform (LFT) uncertainty is considered. To reduce conservatism, a new approach, based on some generalized S-procedure, is addressed. In the case where the studied matrices depend affinely on the uncertain parameters or when the studied matrices are subject to polytopic uncertainty, it is known that recently developed $\mathscr{LMI}$ conditions are effective to assess the robust performance in a less conservative fashion. This paper further extends the preceding results and propose a unified way to obtain new $\mathscr{LMI}$ conditions even in the case of rational parameter dependence. Some conservatism induced by some techniques encountered in the literature is here reduced .**

***Index Terms*— Robust matrix, $\mathscr{D}_u$-stability, S-procedure, $\mathscr{LMI}$.**

I. Introduction

Robust stability analysis of control systems with parameter uncertainties is one of the fundamental issues in system theory. Many important advances have been achieved, see [1]–[3] and the references therein.

The problem of robust matrix root-clustering in a region $\mathscr{D}$ of the complex plane, also referred to as matrix $\mathscr{D}$-stability problem, has been widely investigated in the last decades. Indeed, checking the robust $\mathscr{D}$-stability of the state matrix proves to be very useful to analyze the transient response of a linear model. It enables ones, for instance, to verify if some specified damping ratio and/or some settling time is reached.

Considering a complex uncertain matrix $\mathbf{A} = A(\theta, \Delta), (\theta \in \mathbb{R}^q, \theta \in [\theta_{inf}, \theta_{sup}]$ and Δ is an unknown matrix), our purpose is to find the largest bound on the domain in which Δ lies such that $\mathbf{A}$ remains $\mathscr{D}$-stable. Such bounds are called robustness bounds or robust $\mathscr{D}$-stability bounds. The way to estimate such a bound obviously depends on the structure of the uncertainty. The structured (parametric) case can be distinguished from the unstructured (non parametric) one as pointed out in one of the first contributions due to [4].

The maximal acceptable size of uncertainty was clearly defined, for LFT uncertainty, as the stability radius [5]. Such a stability radius was shown to equal the reciprocal of the H_∞-norm of a transfer in [6] and, thus, also appears to be the reciprocal of the maximal structured singular value μ [7] along frequency. Unfortunately, techniques related to μ analysis may sometimes induce heavy computations, this is even truer when some performance level is required.
The research of robustness bounds was later improved in [8]–[11] and extended to, for example, Ω-regions and $\mathscr{LMI}$-regions [12]–[14].

One of the first attempts to consider unions of regions is provided in [15]. The concept of $\mathscr{D}_u$-stability (root-clustering in some region $\mathscr{D}_u$ whose form encompasses many unions of possible disjoint and not symmetric subregions) enables more general results [16], [17]. However, these results remain quite conservative. The use of Lyapunov functions is certainly the main approach for this kind of analysis. Even if the use of a single quadratic Lyapunov function for the whole of the uncertainty domain [18] (quadratic stability), led to interesting results [19], it remains quite pessimistic. To reduce this conservatism, many authors proposed Parameter-Dependent Lyapunov Functions [20]–[23]. Moreover, the multiplier techniques have been exploited to develop less conservative robust stability criteria [24]–[27].

In this paper, we propose to compute a robust $\mathscr{D}_u$-stability bound in a different manner. To reduce conservatism in the derivation of the bound, the reasoning relies on a framework based upon explicitly rational parameter dependent Lyapunov functions [28]. Indeed, to achieve our goal, we take benefit of a quadratic $\mathscr{D}_u$-stability condition recently introduced in [17], we rewrite it as a parameter-dependent $\mathscr{LMI}$ conditions and transform it, using the S-procedure (see [24], [29] and the references therein), into a parameter-independent optimization problem that can be efficiently solved. Among the more recent papers on this subject, it is worth mentioning [25], [28], [30]. Recently the generalized S-procedure, introduced by [29], [31], has proved to be very useful for robustness analysis and synthesis of control systems. This procedure provides a non conservative way to convert inequality conditions on lossless sets into numerically verifiable conditions represented by linear matrix inequalities.
The paper is organized as follows: after this introduction, a large section is dedicated to preliminaries and problem statement including some root-clustering concepts; The formulation of $\mathscr{EEMI}$-regions is recalled, the modeling of the uncertain state-matrix is introduced and the basic tool in this work, which is the S-procedure, is emphasized. In the third section, previously existing results [17] are recalled. A new formulation of the problem in the presence of LFT uncertainty is then given. In the fifth section that constitutes the most original contribution, we explain how to derive a robust $\mathscr{D}_u$-stability bound via an extended version of the S-

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procedure [28], [29]. Section 6 provides a simple illustrative example before the paper is concluded in section 7.

Notations :

M' denotes the transpose conjugate of matrix M so $s' \in \mathbb{C}$ is the conjugate of complex number s. M^H equals matrix $M+M'$. *HPD* stands for Hermitian positive definite. The 2-norm of M (maximal singular value) is denoted by $||M||_2$. $[M_k]_{\rightarrow} = [M_1 \cdots M_{\bar{k}}]$, $[M_k]_{\downarrow} = [M_1' \cdots M_{\bar{k}}']'$. I and 0 are the identity matrix and the null matrix of appropriate dimensions respectively. In matrix inequalities >0 (resp. <0) means positive (resp. negative) definite. ≥ 0 (resp. ≤ 0) means positive (resp. negative) semi definite. Symbol $\otimes$ denotes the matrix Kronecker product. A Linear Fractional Transformation (LFT) defined, with $(I-A\Delta)$ invertible, by:
$\mathscr{F}_u\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right], \Delta\right) = A + B\Delta(I-D\Delta)^{-1}C.$

II. Problem statement

In this section, we introduce the structure of the uncertainty that is taken into account. The precise purpose is then formulated, the notion of $\mathscr{D}_u$-stability is recalled, and the S-procedure, that is the basic tool in this paper, is presented.

A. The uncertainty structure

Consider the uncertain system

$$\dot{x} = \mathbf{A}(\theta,\Delta)x. \tag{1}$$

where the parameter vector θ belongs to the set $\theta = \left\{(\theta_1,...,\theta_k) : \theta_j \in [\theta_{j_{min}}, \theta_{j_{max}}], j=1,..,k\right\}$, and Δ is unknown matrix.

In this paper we focus on rational parameter-dependence that allows to represent $\mathbf{A}(\theta,\Delta)$ as a linear fractional transformation (LFT)

$$\mathbf{A}(\theta,\Delta) = A(\theta) + B(\theta)\Delta(I-D(\theta)\Delta)^{-1}C(\theta), \tag{2}$$

where Δ is constant, unknown and belongs to $\mathscr{B}(\rho)$, the ball of all matrices $\Delta \in \mathbb{C}^{q\times r}$ satisfying $||\Delta||_2 \leq \rho$. Matrix $(I-D(\theta)\Delta)^{-1}$ exists (well posedness). $A(\theta) \in \mathbb{C}^{n\times n}$, $B(\theta) \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{r\times n}$ and $D(\theta) \in \mathbb{C}^{r\times q}$ are rational matrices on θ.

B. Problem statement

Let $\mathbf{A}(\theta,\Delta)$ be an uncertain matrix as defined in (2), $\mathscr{D}$ a clustering region of the complex plane that might be the union of several possibly disjoint and non symmetric subregions (a possible formulation will be given in the next subsection). This contribution aims at computing the complex $\mathscr{D}$-stability radius. More precisely, assume that $A(\theta)$ is $\mathscr{D}$-stable i.e that the whole of its spectrum lies in $\mathscr{D}$. Define $r_{\mathscr{D}}$ as the largest value of ρ, the radius of $\mathscr{B}(\rho)$, such that $\mathbf{A}(\theta,\Delta)$ remains $\mathscr{D}$-stable for any $\Delta \in \mathscr{B}(\rho)$. Such a value is the so-called complex $\mathscr{D}$-stability radius. A lower bound ρ^* of $r_{\mathscr{D}}$, as tight as possible, is to be computed.

C. Matrix $\mathscr{D}_u$ stability

In this subsection, the notion of $\mathscr{EEMI}$-regions ($\mathscr{EEMI}$ for Extended Ellipsoidal Matrix Inequality) is recalled [17]. Then, the concept of $\mathscr{D}_u$-stability is reminded to the reader.

Definition 1: Let $\mathscr{R}$ be a set of m Hermitian matrices R_k defined by

$$R_k = R_k' = \left[\begin{array}{cc} R_{k_{00}} & R_{k_{01}} \\ R_{k_{01}}' & R_{k_{11}} \end{array}\right] \in \mathbb{C}^{2d\times 2d}, \quad \forall k \in 1,...,m, \tag{3}$$

The set of points $\mathscr{D}_u$ defined by

$$\begin{array}{rcl} \mathscr{D}_u & = & \left\{ z \in \mathbb{C} \,|\, \exists w = \left[\begin{array}{c} w_1 \\ \vdots \\ w_m \end{array}\right] \in \left\{\mathbb{R}^{+*}\right\}^m \,| \right. \\ f_{\mathscr{D}_u(w,z)} & = & \displaystyle\sum_{k=1}^{m} \left(w_k \left[\begin{array}{cc} I_{dk} & z' I_{dk} \end{array}\right] R_k \left[\begin{array}{c} I_{dk} \\ zI_{dk} \end{array}\right] \right) \prec 0 \left.\right\} \end{array} \tag{4}$$

is called an open $\mathscr{EEMI}$-region of degree d.

$\mathscr{D}_u$ is the generic name for an $\mathscr{EEMI}$-region. When $m=1$, this description reduces to $\mathscr{EMI}$-formulation ($\mathscr{EMI}$ for Ellipsoidal Matrix Inequality) proposed in [23] which is rather equivalent to the $\mathscr{LMI}$ formulation [32].

Theorem 1: [17] Let $\mathscr{D}_u$ be an $\mathscr{EEMI}$-region as introduced in definition 1. A matrix A $\in \mathbb{C}^{n\times n}$ is $\mathscr{D}_u$-stable if and only if there exists a set $\mathscr{P}$ of m HPD matrices $P_k \in \mathbb{C}^{n\times n}$, $k=1,...,m$, such that

$$\left[\begin{array}{c|c} I_{dn} & I_d \otimes A' \end{array}\right] \mathscr{U}(\mathscr{P}) \left[\begin{array}{c} I_{dn} \\ \hline I_d \otimes A \end{array}\right] < 0 \tag{5}$$

with

$$\mathscr{U}(\mathscr{P}) = \sum_{k=1}^{m} (R_k \otimes P_k) \tag{6}$$

Remark 1: Note that this condition consists in finding m "Lyapunov" matrices P_k (as much as subregions in practice). Besides, the result of [23] is a special case of Theorem 1 for which $m=1$ and $\mathscr{R}$ reduces to only one matrix $R \in \mathbb{R}^{d\times d}$ (*i.e.* for one single symmetric $\mathscr{EMI}$-region).

D. S-procedure

In this section, we will first present a generalized version of the S-procedure that converts a constrained inequality to an unconstrained inequality with multiplier(s) [33]. The S-procedure is frequently used as a tool in system theory to derive stability and performance results for nonlinear and uncertain systems [33]. Various properties of linear systems can be characterized by inequality conditions in the frequency domain [19], [34]. It has been shown that frequency domain inequalities (frequency dependence) can be reformulated as a parameters independent $\mathscr{LMI}$ conditions by applying generalized S-procedure. It follows that we can verify various properties of linear systems without introducing any conservatism (under some conditions), by solving $\mathscr{LMI}$ resulting from the generalized S-procedure.

Theorem 2: Let a matrix F, a Hermitian matrix Θ and a subset $\mathscr{S}$ of Hermitian matrices be given. Suppose $\mathscr{S}$ is rank one separable (*i.e.* if and only if a certain rank-one property holds for an associated separating hyperplane [35]).Then the following statements are equivalent.

$$(i)\ \xi'\Theta\xi < 0, \forall \xi \in \mathscr{G} = \left\{\xi \in \mathbb{C}^n : \xi \neq 0,\ \xi' S \xi \leq 0 \right. \\ \left. \forall S \in \mathscr{S}\right\}. \tag{7}$$

$$(ii)\ \exists S \in \mathscr{S} \mid \Theta + F' S F < 0. \tag{8}$$

The procedure to replace the condition (7) by (8) is called the generalized S-procedure [29], [31]. When the rank one separability condition is not verified, this replacement introduces conservatism : The condition (8) is only sufficient for (7).

Let us now consider the state-space realization

$$\dot{x}(t) = A x(t) + B u(t). \tag{9}$$
$$y(t) = C x(t) + B u(t). \tag{10}$$

For the particular choice of ξ and F in (7) and (8)

$$\xi = \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}, F = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}.$$

And for the following choice of S

$S = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$ where P is a hermitian matrix. The S-procedure is then particularized as follows

$$(i)\ \Phi'(jw)\ \Theta\ \Phi(jw) < 0\ \forall w \in \mathbb{R} \cup \infty \Leftrightarrow$$
$$(ii)\ \Theta + \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}' \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} < 0.$$

It is the KYP lemma for continuous systems.

In the next, we will present a version of the S-procedure in the uncertain case.

We can use the LFT formulae to establish the relationship between transfer matrices and their state-space realizations. A system with a state-space realization (9) has a transfer matrix of

$$\Phi(s) = \mathscr{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{I}{s}\right). \tag{11}$$

Now take $\theta = \frac{1}{s}$, the transfer matrix can be written as

$$\Phi(s) = \mathscr{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \theta\right). \tag{12}$$

In the general case, for an uncertain state-space realization $(A(\theta), B(\theta), C(\theta), D(\theta))$, whose matrices rationally depend on θ, one can find an LFT representation of the system *i.e.*

$$\Phi(\theta) = \mathscr{F}_u\left(\begin{bmatrix} A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{bmatrix}, \theta\right). \tag{13}$$

We now present the basic theorem, which will be used to prove our main result.

Theorem 3: [28] Let $\Phi(\theta)$ be a rational matrix function of $\theta \in \mathbb{R}^q$, defined by its LFT realization as in (13), let Θ be a Hermitian matrix. Then the condition

$$\forall \theta \in \left[\theta_{inf}, \theta_{sup}\right]\ \Phi^T(\theta)\ \Theta\ \Phi(\theta) < 0 \tag{14}$$

holds if (and only if when $q = 1$) the following condition holds

$$\begin{bmatrix} C'_\Phi \\ D'_\Phi \end{bmatrix} \Theta \begin{bmatrix} C'_\Phi \\ D'_\Phi \end{bmatrix}' + \begin{bmatrix} I & 0 \\ A_\Phi & B_\Phi \end{bmatrix}' S \begin{bmatrix} I & 0 \\ A_\Phi & B_\Phi \end{bmatrix} < 0. \tag{15}$$

with $S \in \mathscr{S}$ is a lossless subset [31].

In the next sections, we adopt the following subset $\mathscr{S}$ of hermitian matrices, that is proven in [28] to be lossless (or rank-one separable).

$$\mathscr{S} = \left\{S = S^T \mid \exists \mathscr{Q}, G \in \mathbb{R}^{n_{A_\Phi} \times n_{A_\Phi}}, \mathscr{Q} < 0,\ G = -G^T \right. \\ \left. S = \begin{bmatrix} -2\mathscr{Q} & (\theta_{inf}+\theta_{sup})\mathscr{Q} + G \\ (\theta_{inf}+\theta_{sup})\mathscr{Q} - G & -2\theta_{inf}\theta_{sup}\mathscr{Q} \end{bmatrix}\right\}. \tag{16}$$

III. Starting point

In this section, some result on quadratic $\mathscr{D}_u$-stability analysis given in [17] is recalled, and then we will try to extend it in the robust case for more complicated uncertainties (rational parameter dependence and not structured uncertainties).

First, we consider that the uncertain matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathscr{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta\right). \tag{17}$$

in which $\Delta \in \mathbb{C}^{q \times r}$is unknown matrix and belongs to $\mathscr{B}(\rho)$, the ball of $(q \times r)$ complex matrices checking $||\Delta|| \leq \rho$, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is constant and the parameters variation is not take into account.

Theorem 4: [17] Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be a matrix as defined in the equation (17) with M constant and $\mathscr{D}_u$ be an $\mathscr{E}\mathscr{E}\mathscr{M}\mathscr{I}$-region as formulated in definition 1. Matrix $\mathbf{A}$ is quadratically $\mathscr{D}_u$-stable with respect to $\mathscr{B}(\rho)$ if (and only if when $d = 1$) there exists a set $\mathscr{P}$ of m HPD matrices P_k, $k = 1, \ldots, m$ such that

$$Q_u(M, \mathscr{P}) = \left[\begin{array}{c|c|c|c} I_{dn} & I_d \otimes A' & 0 & 0 \\ \hline 0 & I_d \otimes B' & I_{dq} & 0 \\ \hline 0 & 0 & 0 & I_{dr} \end{array}\right] \times \\ \left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{\mathscr{U}(\mathscr{P})} & 0 & \begin{array}{c} I_d \otimes C' \\ \hline 0 \end{array} \\ \hline \multicolumn{2}{c|}{0} & -I_{dq} & I_d \otimes D' \\ \hline I_d \otimes C & 0 & I_d \otimes D & -\gamma I_{dr} \end{array}\right] \left[\begin{array}{c|c|c} I_{dn} & 0 & 0 \\ \hline I_d \otimes A & I_d \otimes B & 0 \\ \hline 0 & I_{dq} & 0 \\ \hline 0 & 0 & I_{dr} \end{array}\right] < 0 \\ \forall i \in 1, \ldots, N \tag{18}$$

where $\gamma = \rho^{-2}$ and $\mathscr{U}(\mathscr{P})$ is given by equation (6)

Remark 2: When $m = d = 1$, the maximum value of ρ, easily obtained by convex programming, equals the complex $\mathscr{D}_u$-stability radius.

The extension of this theorem for the robust case is presented in the next section.

IV. New formulation of the problem

In this section, the problem presented by Theorem 4 is extended for both structured and non structured uncertainties. The uncertain matrix $\mathbf{A}$ is given by the equation (2) *i.e.* it is affected by both LFT for Δ (unknown matrix) and LFT for θ (parameters variation).

Theorem 5: Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be a matrix as defined in the equation (2), $\mathscr{D}_u$ be an $\mathscr{E}\mathscr{E}\mathscr{M}\mathscr{I}$-region as formulated in the definition (1). Let $M = \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix}$ is a matrix of rational functions on θ. Matrix $\mathbf{A}$ is robustly $\mathscr{D}_u$-stable with respect to $\mathscr{B}(\rho)$ if (and only if when $d=1$) there exists a set $\mathscr{P}(\theta)$ of m HDP matrices $P_k(\theta)$, $k=1,...,m$ such that

$$\left[\begin{array}{c|c|c|c} I_{dn} & I_d\otimes A'(\theta) & 0 & 0 \\ \hline 0 & I_d\otimes B'(\theta) & I_{dq} & 0 \\ \hline 0 & 0 & 0 & I_{dr} \end{array}\right]\times$$
$$\left[\begin{array}{c|c|c} \mathscr{U}(\mathscr{P}(\theta)) & 0 & \begin{array}{c} I_d\otimes C'(\theta) \\ \hline 0 \end{array} \\ \hline 0 & -I_{dq} & I_d\otimes D'(\theta) \\ \hline I_d\otimes C(\theta) \;|\; 0 & I_d\otimes D(\theta) & -\gamma I_{dr} \end{array}\right]\times \tag{19}$$
$$\left[\begin{array}{c|c|c} I_{dn} & 0 & 0 \\ \hline I_d\otimes A(\theta) & I_d\otimes B(\theta) & 0 \\ \hline 0 & I_{dq} & 0 \\ \hline 0 & 0 & I_{dr} \end{array}\right]<0$$

where $\gamma=\rho^{-2}$ and $\mathscr{U}(\mathscr{P}(\theta))$ is given by

$$\mathscr{U}(\mathscr{P}(\theta))=\sum_{k=1}^{m}(R_k\otimes P_k(\theta)) \tag{20}$$

The proof is obvious from Theorem 1.

This optimization problem involving constraint (19) is convex in the decision variables $P_k(\theta)$. Unfortunately, it is infinite dimensional and the decision variables are in an infinite dimensional space. In this form, this prevents an efficient computation of the solution [28]. We can therefore impose a structure on $P_k(\theta)$ in order to arrive at inequalities on unknowns matrices [24].

When M varies in a polytopic manner, the condition (19) can be treated in a simple fashion, according to [9], as mentioned in the next theorem. Matrix $M\in\mathbb{C}^{(n+r)(n+q)}$ is assumed to belong to a polytope $\mathscr{M}$:

$$\mathscr{M}=\left\{M=M(\delta)|M=\sum_{i=1}^{h}\delta_i M_i=\sum_{i=1}^{h}\left(\delta_i\left[\begin{array}{cc} A_i & B_i \\ C_i & D_i\end{array}\right]\right);\right.$$
$$\left.\delta\in\underline{\delta}=\left\{\delta=[\delta_1,\ldots,\delta_h]'\in\{\mathbb{R}^+\}^h\,|\sum_{i=1}^{h}\delta_i=1\right\}\right\}.$$

where $M_i, i=1,\ldots,h$ are the vertices of $\mathscr{M}$ and $\underline{\delta}$ is the set of coordinates.

Theorem 6: [17] Let $\mathbf{A}\in\mathbb{C}^{n\times n}$ be a matrix as defined in the equation (17) and $\mathscr{D}_u$ be an $\mathscr{EEMI}$-region as formulated in definition 1. $\mathbf{A}$ is robustly $\mathscr{D}_u$-stable with respect to $\mathscr{B}(\rho)$ and $\mathscr{M}$ if there exists N set $\mathscr{P}_i$, each one made up by m HPD matrices P_{ki}, $k=1,...,m$, and a matrix $G_U\in\mathbb{C}^{d(2n+q+r)\times dn}$ such that

$$\left[\begin{array}{c|c|c} \mathscr{U}(\mathscr{P}_i) & 0 & \begin{array}{c} I_d\otimes C_i' \\ \hline 0 \end{array} \\ \hline 0 & -I_{dq} & I_d\otimes D_i' \\ \hline I_d\otimes C_i & I_d\otimes D_i & -\gamma I_{dr} \end{array}\right]+$$
$$\{G_U\left[\begin{array}{c|c|c|c} I_d\otimes A_i & -I_{mdn} & I_d\otimes B_i & 0 \end{array}\right]\}^H<0. \tag{21}$$

$\forall i\in\{1,\ldots,h\}$ where $\gamma=\rho^{-2}$.

This theorem ensures the existence of polytopic Lyapunov matrices in the form $P_k(\delta)=\sum_h^{i=1}\delta_i P_i$. This kind of condition is inspired from [9].

Remark 3: Note that the $\mathscr{LMI}$ condition (19) can be written as follows

$$\forall\theta\in[\theta_{inf},\theta sup]\;\;\{F_1(\theta)(\Gamma(\theta)+C)F_2(\theta)\}^H<0 \tag{22}$$

$$F_1(\theta)=\left[\begin{array}{c|c|c} (I_d\otimes A'(\theta))(R'_{10k}\otimes I_n)_\rightarrow & (R'_{10k}\otimes B'(\theta))_\rightarrow & 0 \\ 0 & 0 & I_d\otimes C(\theta) \\ 0 & 0.5(R'_{11k}\otimes B'(\theta))_\rightarrow & 0 \\ 0 & 0 & -0.5I_{dr} \\ 0.5(R_{00k}\otimes I_n)_\rightarrow & 0 & 0 \\ 0 & \frac{-1}{2}I_{dq} & 0 \\ 0.5(I_d\otimes A'(\theta))(R_{11k}\otimes I_n)_\rightarrow & (R'_{11k}\otimes B'(\theta))_\rightarrow & 0 \\ 0 & 0 & I_d\otimes D(\theta) \end{array}\right]',$$

$$F_2(\theta)=\left[\begin{array}{c|c|c|c|c|c|c|c} I & I & 0 & 0 & I & 0 & I_d\otimes A'(\theta) & 0 \\ 0 & 0 & I_d\otimes B'(\theta) & 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & I_{dr} & 0 & 0 & 0 & 0 \end{array}\right]',$$

$$\Gamma(\theta)+C=\left[\begin{array}{c|c|c|c|c|c|c|c} [I_d\otimes P_k(\theta)]_\downarrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [I_d\otimes P_k(\theta)]_\downarrow & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma I_{dr} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [I_d\otimes P_k(\theta)]_\downarrow & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{dq} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & [I_d\otimes P_k(\theta)]_\downarrow & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Our goal is the transformation of this infinite dimensional optimization problem into a finite dimensional one. For this, we first try to search for a rational decision variables. We then introduce the following finite parameterization

$$P_k(\theta)=\frac{\sum_{i=0}^{N}\theta^i P_{ki}}{1+\sum_{i=0}^{N}\theta^i d_{ki}}\quad\forall k\in\{1,\ldots,m\}. \tag{23}$$

Each region is parameterized by $(N+1)$ matrices P_i and N scalars d_i. In order to obtain a finite number of constraints, the second step is based on the extended version of S-procedure given by theorem 3.

V. ROBUST $\mathscr{D}_u$-STABILITY ANALYSIS VIA S-PROCEDURE

The main result of this paper is summarized in the next theorem

Theorem 7: Given N, let $\mathbf{A}\in\mathbb{C}^{n\times n}$ be a matrix as defined in the equation (2), $M=\left[\begin{array}{cc} A(\theta) & B(\theta) \\ C(\theta) & D(\theta)\end{array}\right]$ is a matrix of rational functions on θ, and $\mathscr{D}_u$ be an $\mathscr{EEMI}$-region as formulated in definition 1. Matrix $\mathbf{A}$ is robustly $\mathscr{D}_u$-stable with respect to $\mathscr{B}(\rho)$ if there exists m set $\mathscr{P}(\theta)$, each one made up by N matrices P_k, $k=1,...,N$, parameterized by the equation (23) and well-posed on $[\theta_{inf},\theta_{sup}]$, N scalars $d_i, i=1,...,N$ such that the three following conditions hold. (i) There exist a symmetric positive definite matrix $\mathscr{Q}$ and a skew-symmetric matrix G such that

$$\left[\begin{array}{c} C'_\Phi \\ D'_\Phi \end{array}\right]\Theta\left[\begin{array}{c} C'_\Phi \\ D'_\Phi \end{array}\right]'+\left[\begin{array}{cc} I & 0 \\ A_\Phi & B_\Phi \end{array}\right]'S\left[\begin{array}{cc} I & 0 \\ A_\Phi & B_\Phi \end{array}\right]<0, \tag{24}$$

where S is given by equation (16), Θ is given by

$$\Theta=\left[\begin{array}{cc} 0 & \Psi \\ \Psi' & 0 \end{array}\right] \tag{25}$$

with $\Psi = \Gamma_N + d_N C \dots \Gamma_0 + d_0 C$.

$$\mathscr{F}_u\left(\left[\begin{array}{cc} A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{array}\right], \theta\right) = \left[\begin{array}{c} F_1^T(\theta) \\ F_3(\theta)F_2(\theta) \end{array}\right]. \quad (26)$$

where $F_1(\theta), F_2(\theta)$ are given in remark 3, and F_3 is given by

$$F_3(\theta) = \left[\begin{array}{c} \theta^N I \\ \theta^{N-1} I \\ \dots \\ I \end{array}\right]. \quad (27)$$

(ii) There exist a symmetric positive definite matrix $\mathscr{Q}_p$ and a skew-symmetric matrix G_p such that

$$\left[\begin{array}{c} C_p' \\ D_p' \end{array}\right] \Theta_p \left[\begin{array}{c} C_p' \\ D_p' \end{array}\right]' + \left[\begin{array}{cc} I & 0 \\ A_p & B_p \end{array}\right]' W_p \left[\begin{array}{cc} I & 0 \\ A_p & B_p \end{array}\right] < 0, \quad (28)$$

where W_p is given by equation (16) with G and $\mathscr{Q}$ replaced by G_p and $\mathscr{Q}_p$ respectively. Θ_p is given by

$$\Theta_p = \left[\begin{array}{cc} 0 & -(d_N \ d_{N-1} \ \dots 1) \\ -(d_N \ d_{N-1} \ \dots 1)' & 0 \end{array}\right]. \quad (29)$$

Morever

$$\mathscr{F}_u\left(\left[\begin{array}{cc} A_p & B_p \\ C_p & D_p \end{array}\right], \theta\right) = \left[\begin{array}{c} I \\ F_3(\theta) I \end{array}\right], \quad (30)$$

and F_3 is given by the equation (27).

(iii) There exist a symmetric positive definite matrix $\mathscr{Q}_b$ and a skew-symmetric matrix G_b such that

$$\left[\begin{array}{c} C_b' \\ D_b' \end{array}\right] \Theta_{bk} \left[\begin{array}{c} C_b' \\ D_b' \end{array}\right]' + \left[\begin{array}{cc} I & 0 \\ A_b & B_b \end{array}\right]' W_{bk} \left[\begin{array}{cc} I & 0 \\ A_b & B_b \end{array}\right] < 0, \quad (31)$$

$\forall k \in \{1, \dots, m\}$, where W_{bk} is also given by equation (16) with G and $\mathscr{Q}$ replaced by G_b, and $\mathscr{Q}_b$ respectively. Θ_{bk} is given by

$$\Theta_{bk} = \left[\begin{array}{cc} 0 & (P_N \ P_{N-1} \ \dots P_0) \\ (P_N \ P_{N-1} \ \dots P_0)' & 0 \end{array}\right]. \quad (32)$$

Morever

$$\mathscr{F}_u\left(\left[\begin{array}{cc} A_b & B_b \\ C_b & D_b \end{array}\right], \theta\right) = \left[\begin{array}{c} -I_n/2 \\ F_3(\theta) I_n \end{array}\right], \quad (33)$$

where F_3 is still given by the equation (27).

Proof: By applying the S-procedure in the form of Theorem 3, the $\mathscr{LMI}$ condition (24) is equivalent to (14) with

$$\Phi(\theta) = \mathscr{F}_u\left(\left[\begin{array}{cc} A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{array}\right], \theta\right) \quad (34)$$

and Θ is given by the equation (25). Then, it can be written as follows

$$\left[\begin{array}{c} F_1^T(\theta) \\ F_3(\theta)F_2(\theta) \end{array}\right]' \left[\begin{array}{cc} 0 & \Psi \\ \Psi' & 0 \end{array}\right] \left[\begin{array}{c} F_1^T(\theta) \\ F_3(\theta)F_2(\theta) \end{array}\right], \quad (35)$$

with $\Psi = \Gamma_N + d_N C \dots \Gamma_0 + d_0 C$. that is equivalent to (22) which is a simple formulation of $\mathscr{LMI}$ (19).

In the same manner, by applying the S-procedure (Theorem 3), the $\mathscr{LMI}$ condition (28) can be written as follows

$$\left[\begin{array}{c} I \\ F_3(\theta) I \end{array}\right]' \left[\begin{array}{cc} 0 & -(d_N \ d_{N-1} \ \dots 1) \\ -(d_N \ d_{N-1} \ \dots 1)' & 0 \end{array}\right] \left[\begin{array}{c} I \\ F_3(\theta) I \end{array}\right] < 0 \quad (36)$$

$$\Leftrightarrow \left\{1 + \sum_{i=1}^{N} d_i \theta_i\right\}^H > 0. \quad (37)$$

that is a necessary and sufficient condition for the well posedness of our decision variables $P_k(\theta)$ given by the equation (23).

By applying the S-procedure another time, the $\mathscr{LMI}$ condition (31) is equivalent to the condition

$$\left[\begin{array}{c} -I_n/2 \\ F_3(\theta) I_n \end{array}\right]' \left[\begin{array}{cc} 0 & (P_N \ P_{N-1} \ \dots P_0) \\ (P_N \ P_{N-1} \ \dots P_0)' & 0 \end{array}\right] \left[\begin{array}{c} -I_n/2 \\ F_3(\theta) I_n \end{array}\right] < 0, \quad (38)$$

where F_3 is given by the equation (27). This condition (38) is a simple formulation of the positivity of our decision variables, *i.e.* $P_k(\theta) > 0, \forall k \in \{1, \dots, m\}$ for $\theta \in [\theta_{inf}, \theta_{sup}]$. ■

Remark 4: To make calculation simpler and to avoid numerical problems, one can be satisfied, according to [36], by polynomial decision variables i.e. denominator of $P_k(\theta)$ for $k = 1, \dots, m$ equals 1.

VI. Short illustration

This simple example highlights the improvement of the present work compared to [17]. Consider equation (2) with

$$A(\theta) = \left[\begin{array}{cc} -15.1073 + (1+\theta) & -13.9317 + \frac{1}{(1+\theta)} \\ 8.5267 & 6.1073 + \frac{1}{(1+\theta)^2} \end{array}\right],$$

$B = [0.7150 \ 0.1215]'$, $C = [0.8989 \ 0.6582]$,

$D = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$, with $\theta \in [-0.047, 0.047]$.

The nominal spectrum is $\lambda(A) = \{-2; -5\}$. The clustering region $\mathscr{D}_u$ is chosen as the union of two discs centred around the eigenvalues of $A(\theta)$ and both of radius 1. The approach used in [17] introduces a conservatism and is simply found infeasible. Indeed, since $A(\theta)$ is affected by a rational uncertainty, [17, Theorem 6] cannot be applied directly. The solution is a change of variables ($\theta_1 = \theta$, $\theta_2 = \frac{1}{(1+\theta)}$ and $\theta_3 = \frac{1}{(1+\theta)^2}$), and then instead of considering a single uncertain parameter θ, two other parameters must be introduced, what implies 8 vertices of polytope, and the approach brings too much conservatism. Using our approach and by applying Theorem 7, we obtain the complex $\mathscr{D}_u$-stability radius $\rho^\star = 0.0577$. Although some conservatism sometimes induced by the technique of [17] is here circumvented, it has to be mentioned that the proposed approach could sometimes become numerically more demanding especially when several regions are considered.

Figure 1 shows the pole migration plotted for several values of Δ such that $||\Delta|| \leq \rho^\star$.

VII. Conclusion

We have proposed new $\mathscr{LMI}$-based tests for the robust analysis of matrix root-location when this matrix is subject

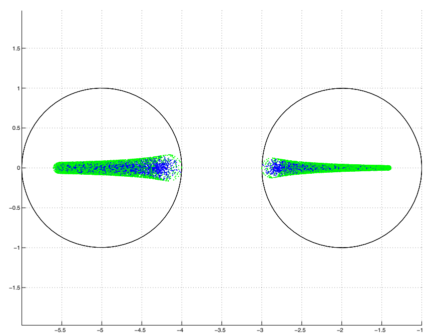

Fig. 1. pole migration for the uncertain matrix $A(\theta)$

to both LFT and rational parameter dependence uncertainties. These tests rely on an extended version of the S-procedure and involve explicitly parameter-dependent Lyapunov functions with a rational dependence on the parameter uncertainty. The simple example of the preceding section highlights the limits of the results provided in [17] and shows the relevance of the present contribution when [17, Theorem 6] fails or brings too much conservatism.

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