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Stabilizing Control of Symmetric Affine Systems by Direct Gradient Descent Control

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Abstract— **This paper is concerned with control of nonholonomic systems. As is well known, symmetric affine system is uncontrollable with continuous time-invariant differentiable state feedback. In this paper we apply Direct Gradient Descent Control (DGDC) for the symmetric affine system. The DGDC is such a method that we manipulate control inputs directly so as to decrease a performance function by the steepest descent method. Note that the DGDC is a dynamic controller that we can adjust not only its gain parameter but also its initial condition. Then, not only controllable part of symmetric affine system is asymptotically stabilized, but also uncontrollable part can be converged to the origin by choosing the initial condition appropriately. Applying the DGDC, we can control the symmetric affine system without transforming it into the "chained form". Simulation results for a four wheeled vehicle and a flying robot demonstrate the effectiveness of the proposed method.**

I. Introduction

Nonholonomic system is defined as a system having constraints which can not be described with only generalized positions and attitude angles. In recent years, mechanical systems with nonholonomic constraints were given attention from a viewpoint of nonlinear control theory, because their state equation is very hard to control.

The nonholonomic system, in most cases, is described as a nonlinear affine system. In particular, mechanical systems with nonholonomic speed constraints are represented with a symmetric affine system without a drift term.

It is well known that the symmetric affine system cannot be asymptotically stabilized by the continuous time-invariant state feedback, even if it is controllable (refer to Brockett's Theorem [1]). Accordingly, discontinuous switching feedback control [3], [4], [2] and/or time-varying feedback control[8], [10], [11], [12] have been proposed. However, most of them are restricted to the so-called "chained form". Khennouf et al. [3] utilized a structure of the chained form skillfully and designed a two-stage switching scheme using an invariant manifold. It is difficult to extend to a system besides the chained form, however.

Generally speaking, methods based on the transformation into the chained form is complex, individual and skillful. Further it yields a problem of singular point caused by the transformation. So it is expected to develop a control method without such transformation.

Mita et al.[2], [4] proposed Variable Constraint Control (VCC) which could be applied for a symmetric affine system without transforming into the chained form. (Note that it is also of two-stage scheme based on the use of invariant manifold.) But VCC adopts nonlinear non-interaction control which requires an inverse function matrix for its implementation, and hence a trouble of singular point arises.

Meanwhile, there exists Direct Gradient descent Control (DGDC),[5], [6], [7]) as an effective stabilizing control method for general nonlinear systems. The DGDC is a dynamic controller that manipulates control inputs directly so as to decrease a performance function by the steepest descent method. Its implementation is simple and so practical.

In Ref.[9] we proposed applying the DGDC to realize VCC, where each stage of VCC was executed by the DGDC for an individual performance function in each stage. Our method was then executed with the two-stage type DGDC.

In this paper, we propose to apply the DGDC for an original symmetric affine system without making any coordinate transformation, and report that asymptotical stabilization can be achieved by setting an initial condition of control input optimally. Furthermore, the best setting of the initial condition of input is not necessarily made one time. We can perform asymptotical stabilization by repeating it on all such occasions as the DGDC is executed at several stages.

Lastly, we report simulation results for a flying robot and a four-wheeled vehicle.

II. Direct Gradient Descent Control

First, we summarize Direct Gradient Descent Control. We consider here control problem for general nonlinear systems. The aim is to manipulate the control input in order to decrease the performance function $F(\boldsymbol{x}(t))$. Hence the problem is formulated as follows:

$$\underset{\boldsymbol{u}(t)}{\text{decrease}} \quad F(\boldsymbol{x}(t)) + P(\boldsymbol{u}(t)) \qquad (1a)$$

$$\text{subject to} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \qquad (1b)$$

where $\boldsymbol{x}(t) \in R^n$ is the state vector, $\boldsymbol{u}(t) \in R^r$ is the input vector, and $P(\boldsymbol{u}(t))$ denotes the input cost or the penalty function for the input. For any continuous $\boldsymbol{u} : \boldsymbol{u}(t),\ t \geq 0$, the system $(1b)$ has a unique continuous solution $\boldsymbol{x} : \boldsymbol{x}(t),\ t \geq 0$ under adequate assumptions. The value of the trajectory $\boldsymbol{x}$ associated with a given $\boldsymbol{u}$ at t, that is the state $\boldsymbol{x}(t)$, is denoted by $\boldsymbol{x}(t; \boldsymbol{u})$.

[**Assumption 1**] $\boldsymbol{f}$ is continuously differentiable on $(\boldsymbol{x}, \boldsymbol{u})$

[**Assumption 2**] $\boldsymbol{f_u}$, $F_{\boldsymbol{x}}$ are Lipschitz's continuous.

As a means of on-line control for problem (1), we apply the steepest descent method on $\boldsymbol{u}(t)$. For that, we derive the

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gradient function of the functional

$$\phi[\boldsymbol{u}] \stackrel{\triangle}{=} F(\boldsymbol{x}(t;\boldsymbol{u})) \tag{2}$$

for the fixed t. Then, the following theorem holds.
[**Theorem 1**] Under assumptions 1 and 2, the functional $\phi : U_{[0,t]} \to R$, defined, by (2), is Gâteaux differentiable, and its gradient $\nabla\phi[\boldsymbol{u}] \in U_{[0,t]}$ on $\boldsymbol{u} \in U_{[0,t]}$, at time t, is given by:

$$\nabla\phi[\boldsymbol{u}](t) = \boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}), \boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}))^T \tag{3}$$

(proof) See Ref. [5]. ∎

Using this gradient function, we execute the steepest descent method in the continuous form for problem (1):

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}\left\{\nabla\phi[\boldsymbol{u}](t) + \nabla P(\boldsymbol{u}(t))\right\}, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{4}$$

where $\mathcal{L} > 0$ is a proportional coefficient. Substitute (3) into (4) to obtain

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}\left\{\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}), \boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}))^T + P_{\boldsymbol{u}}(\boldsymbol{u}(t))^T\right\} \tag{5}$$

As the penalty function, we adopte the quadric form

$$P(\boldsymbol{u}(t)) = \frac{1}{2}(\boldsymbol{u}_d - \boldsymbol{u}(t))^T R(\boldsymbol{u}_d - \boldsymbol{u}(t)) \tag{6}$$

where $\boldsymbol{u}_d$ is a value of control input corresponding to the desired equilibrium $\boldsymbol{x}_d$, and R is a positive definite matrix.

Substituting (6) in (5) we have the steepest descent method

$$\begin{aligned}\dot{\boldsymbol{u}}(t) = -\mathcal{L}\{&\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}), \boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}))^T \\ &-R(\boldsymbol{u}_d - \boldsymbol{u}(t))\}\end{aligned} \tag{7}$$

Note that the proportional coefficient is taken as $\mathcal{L} \stackrel{\triangle}{=} \mathrm{diag}(\alpha_1, \cdots, \alpha_r), \alpha_i > 0$.

The dynamic control law (7) is called "Direct Gradient Descent Control (DGDC)". The second term of (7) implies the negative feedback that contributes to stability.

In the regulator problem whose desired value is the origin without loss of generality, let us consider the following problem with a positive definite quadric performance:

$$\underset{\boldsymbol{u}(t)}{\text{decrease}}\ F(\boldsymbol{x}(t)) = \frac{1}{2}\boldsymbol{x}(t)^T Q\boldsymbol{x}(t) + \frac{1}{2}\boldsymbol{u}(t)^T R\boldsymbol{u}(t) \tag{8a}$$

$$\text{subject to } \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{8b}$$

From (7), the DGDC then becomes as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{9}$$

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}\left\{\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t), \boldsymbol{u}(t))^T Q\boldsymbol{x}(t) + R\boldsymbol{u}(t)\right\}, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{10}$$

The reader may refer to Refs. [5], [6], [7] about the stability of DGDC or how to decide the coefficient $\mathcal{L}$.

III. Control scheme for the nonholonomic systems

Nonholonomic system is defined as a system having constraints which cannot be described with only generalized positions and attitude angles (e.g. speed constraint and acceleration one). Typical examples of nonholonomic system are a space robot, an underactuated manipulator, etc.

Nonholonomic systems, in most cases, are expressecd by the following nonlinear state equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + G(\boldsymbol{x}(t))\boldsymbol{u}(t) \tag{11}$$

where $\boldsymbol{x}(t) \in R^n$, $\boldsymbol{u}(t) \in R^r$ are the state vector and the input vector, respectively. (11) is called the nonlinear affine system, and the term $\boldsymbol{f}(\boldsymbol{x}(t))$ is called the drift term because it is composed of no input. In particular the nonlinear affine system without drift

$$\dot{\boldsymbol{x}}(t) = G(\boldsymbol{x}(t))\boldsymbol{u}(t) \tag{12}$$

is called the symmetric affine system.

In general, underactuated mechanical system with accelerative constraints is represented with the nonlinear affine system having a drift term, and one with speed constraints is represented by a driftless symmetric affine system.

Brockett[1] established a necessary condition in order that nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ has a smooth state feedback control law that asymptotically stabilizes an equilibrium point in the neighborhood of it. Since the symmetric affine system does not satisfy the condition of Brockett's theorem, one cannot asymptotically stabilize it by the continuous state feedback control law $\boldsymbol{u} = \boldsymbol{\alpha}(\boldsymbol{x})$. So several ideas have been proposed to overcome the severe Brockett's necessary condition. Among the proposed methods for nonholonomic systems, there are three types of control scheme as follows:

- Stabilization by time-varing state feedback[10], [12]
- Stabilization by discontinuous state feedback[3]
- Control scheme based on Time-State Control Form[11]

IV. DGDC for Symmetric Affine Systems

In this section, we investigate DGDC for the symmetric affine system (12) being nonholonomic one.

Since our purpose is to manipulate control input $\boldsymbol{u}(t)$ so as to decrease the performance function $F(\boldsymbol{x}(t))$, the problem is represented as follows:

$$\underset{\boldsymbol{u}(t)}{\text{decrease}}\ F(\boldsymbol{x}(t)) = \frac{1}{2}\boldsymbol{x}(t)^T Q\boldsymbol{x}(t) + \frac{1}{2}\boldsymbol{u}(t)^T R\boldsymbol{u}(t) \tag{13a}$$

$$\text{subject to } \dot{\boldsymbol{x}}(t) = G(\boldsymbol{x}(t))\boldsymbol{u}(t), \ \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{13b}$$

where $\boldsymbol{x}(t) \in R^n$ is the state vector and $\boldsymbol{u}(t) \in R^r$ is the input vector, and $Q > 0$, $R > 0$.

Then, the DGDC given in Section 2 is expressed with the following differential equations:

$$\dot{\boldsymbol{x}}(t) = G(\boldsymbol{x}(t))\boldsymbol{u}(t), \ \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{14}$$

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}\left\{G(\boldsymbol{x}(t))^T Q\boldsymbol{x}(t) + R\boldsymbol{u}(t)\right\}, \boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{15}$$

where $\mathcal{L} = \mathrm{diag}\ (\alpha_1, \cdots, \alpha_r),\ Q = \mathrm{diag}\ (q_1, \cdots, q_n),\ \ R = \mathrm{diag}\ (r_1, \cdots, r_r)$.

The linearlized system of symmetric affine system with DGDC (14),(15) becomes

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{u}}(t) \end{bmatrix} = \begin{bmatrix} O & G(\boldsymbol{0}) \\ -\mathcal{L}G(0)^T Q & -\mathcal{L}R \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \end{bmatrix} \tag{16}$$

In case that stability of this linearlized matrix is affirmed, the original nonlinear system is asymptotically stable in the neighborhood of the origin. Since this linearlized matrix has always 0 eigenvalues, however, it is impossible to check whether the system can be asymptotically stabilized from the linearlized system. When we must control the nonlinear system whose linearlized system has eigenvalues on an imaginary axis, we usually apply the analysis based on the center manifold theory[16], [13] or Lyapunov's theorem.

The symmetric affine system cannot be asymptotically stabilized by the time invariant state feedback because of Brockett's theorem. Namely, we cannot make all state variables converge to the origin by the state feedback, and can stabilize only to an equilibrium point satisfying $G(\boldsymbol{x}(t))\boldsymbol{u}(t) = \boldsymbol{0}$. Accordingly, it is necessary to use a switching state feedback law or a time-varying state feedback one.

This situation does not change even if a dynamic controller is used. In case of using a dynamic controller, however, it is possible to asymptotically stabilize by manipulating its initial value without switching the control law. To do that, we propose a new method how to search the initial value of the dynamic controller below.

The DGDC for the symmetric affine system has been given by (14),(15). In case of no foresight information, we generally choose zero as $\boldsymbol{u}(0)$. But, it makes the system converge to the undesired equilibrium point $\boldsymbol{x}^s$ not being the origin. [1]. In this case, variables corresponding to the mode that the linearlized system is asymptotically stable can be asymptotically stabilized, but the original nonlinear system moves in accordance with the dynamics of uncontrollable variables corresponding to the mode that the linearlized system is not asymptotically stable, and reaches the undesired equilibrium $\boldsymbol{x}^s$. Nevertheless, utilizing initial value of DGDC, we can find $\boldsymbol{u}_0$ such that $\boldsymbol{x}(t) \to \boldsymbol{0}$ as $t \to \infty$.

Let us investigate a method to find such $\boldsymbol{u}_0$ which achieves asymptotical stabilization. For that purpose, we assume: [**Assumption 3**] System (14) is reachable and adjusting $\boldsymbol{u}_0$ in (15) has enough freedom such that DGDC (15) can make $\boldsymbol{x}(t)$ come close to the origin as $t \to \infty$.

To do that, we have to search such $\boldsymbol{u}_0$ by minimizing the norm of equilibrium point $\boldsymbol{x}(\infty)$ when executing (14),(15). Without letting $t \to \infty$ actually, however, we can find it by solving the following problem with sufficiently large t_1:

$$\min_{\boldsymbol{u}_0} \ \sum_{j=1}^{n} w_j x_j^2(t_1), \quad w_j > 0 \tag{17a}$$

$$\text{subj. to } \dot{\boldsymbol{x}}(t) = G(\boldsymbol{x}(t))\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{17b}$$

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}G(\boldsymbol{x}(t))^T Q\boldsymbol{x}(t) - \mathcal{L}R\boldsymbol{u}(t), \boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{17c}$$

where $w_j,\ j = 1, 2, \cdots, n$ are weight coefficients. Basically the optimum $\boldsymbol{u}_0$ is found one time. Optimum $\boldsymbol{u}_0$ is not necessarily found at one stage, however. When $x_j(\infty),\ j \in J$ has converged to the undesired equilibrium $x_j^s,\ j \in J$ by the DGDC with $\boldsymbol{u}_0$ being not optimal, we need search the initial value $\boldsymbol{u}_0$ at the second stage under setting

$$x_j(t_1) \approx \begin{cases} x_j^s, & j \in J \\ 0, & j \notin J \end{cases}$$

as the initial state. Note that J denotes the set of subscripts of the state variables which converged to the undesired equilibrium. Accordingly, we solve the following :

$$\min_{\boldsymbol{u}_0} \quad \sum_{j \in J} w_j x_j^2(t_1'), \quad w_j > 0 \tag{18a}$$

$$\text{subj. to } \quad \dot{\boldsymbol{x}}(t) = G(\boldsymbol{x}(t))\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}(t_1) \tag{18b}$$

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}G(\boldsymbol{x}(t))^T Q\boldsymbol{x}(t) - \mathcal{L}R\boldsymbol{u}(t),\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{18c}$$

where t_1' is sufficiently large.

Even when optimization of $\boldsymbol{u}_0$ has not been done exactly , it is possible to let all state variables converge to the orign gradually by repeating such a process several times. We can let them converge to a point in the neighborhood of the origin within permissible accuracy in practice. (We confirmed this fact by simulation for several plants, not proven mathematically though.)

Such an optimal design or updating $\boldsymbol{u}(0) = \boldsymbol{u}_0$ gives the same effect as the switching state feedback law.

[Nelder-Mead's Method] We apply Nelder-Mead's method [14], [15] to solve the optimization problem (17) or (18) on-line. This is an improved algorithm of Simplex method, which is a kind of optimization technique without using gradients. A man of business finds it useful because of the simplicity. It is very effective for problems with a relatively small number of decision variables. Since it brings an approximate solution within permissible accuracy in the finite number of searching steps, it may be said very convenience when one cannot calculate gradients of an objective function.

[A Study on Adjustment of Initial Value of dynamic controller] Lyapunov's stability theory is discussed based on only a differential equation and the time derivative of Lyapunov's function. But we should consider about not only the differential equation (state equation) but also its initial value dependence of the solution for the asymptotical stabilization of nonholonomic system.

Namely it is necessary to choose an adequate initial value in order to make the solution reach the desired point (the origin) because a solution to the differential equation is a function of the initial value. This indicates that the initial value can be utilized as a parameter to make the trajectory reach the desired point. Such initial value $\boldsymbol{u}_0$ is not always unique, and may innumerably exist.

[1] It is a difficult point for symmetric affine system that the state converges to $\boldsymbol{x}^s \neq \boldsymbol{0}$ satisfying $G(\boldsymbol{x}^s)\boldsymbol{u}^s = \boldsymbol{0}$ under the inappropriate control.

The DGDC for the symmetric affine system is given by (14),(15), and its linearlized system is given by (16). In case that (16) has eigenvalues on an imaginary axis, (16) is transformed into the following diagonal block form:

$$\begin{bmatrix} \dot{\overline{\boldsymbol{x}}}^S \\ \dot{\overline{\boldsymbol{x}}}^C \end{bmatrix} = \begin{bmatrix} A^S & O \\ O & A^C \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{x}}^S \\ \overline{\boldsymbol{x}}^C \end{bmatrix} \tag{19}$$

by the transformation

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = T \begin{bmatrix} \overline{\boldsymbol{x}}^S \\ \overline{\boldsymbol{x}}^C \end{bmatrix}, \tag{20}$$

where $\begin{bmatrix} A^S & O \\ O & A^C \end{bmatrix} = T^{-1} \begin{bmatrix} O & G(\mathbf{0}) \\ -\mathcal{L}G(\mathbf{0})^T Q & -\mathcal{L}R \end{bmatrix} T$

Here A^S is Hurwitz and all eigenvalues of A^C is supposed to exist on the imaginary axis.

Meanwhile, when representing DGDC (14),(15) as

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{u}} \end{bmatrix} = F\left(\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} \right), \begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{u}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_0 \end{bmatrix} \tag{21}$$

, this equation can be transformed into

$$\begin{bmatrix} \dot{\overline{\boldsymbol{x}}}^S \\ \dot{\overline{\boldsymbol{x}}}^C \end{bmatrix} = T^{-1} F\left(T \begin{bmatrix} \overline{\boldsymbol{x}}^S \\ \overline{\boldsymbol{x}}^C \end{bmatrix} \right), \begin{bmatrix} \overline{\boldsymbol{x}}^S(0) \\ \overline{\boldsymbol{x}}^C(0) \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{x}}_0^S \\ \overline{\boldsymbol{x}}_0^C \end{bmatrix} \tag{22}$$

by the transformation (20).

In case where the linearlized system possesses 0 eigenvalues, its stability is generally discussed by using center manifold theory. The following is well known.

"All trajectories starting from a neighborhood of the orgin are attracted exponentially to the center manifold $S^c(0)$, and stability of an equilibrium point can be checked only from the dynamics on the center manifold. Namely, if an equilibrium point is stable, asymptotically stable, unstable on $S^c(0)$, then the equilibrium point is stable, asymptotically stable, unstable in the whole region, respectively.

Put the center manifold mapping of system (22) as $\overline{\boldsymbol{x}}^S = \boldsymbol{\pi}(\overline{\boldsymbol{x}}^C)$. Then the dynamics on the center manifold is expressed as

$$\dot{\boldsymbol{x}}^C = F^C(\boldsymbol{\pi}(\overline{\boldsymbol{x}}^C), \overline{\boldsymbol{x}}^C) \tag{23}$$

where F^S and F^C are a part of $T^{-1}F$ corresponding to $\overline{\boldsymbol{x}}^S$ and $\overline{\boldsymbol{x}}^C$, respectively. Since the dynamics of $\overline{\boldsymbol{x}}^S$ is exponentially stable, from the mentioned above, the stability of (22) is guaranteed if (23) is stable [15], [16].

Now, taking the Lyapunov function candidate of the symmetric affine system with DGDC (14), (15) as $V(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}^T Q \boldsymbol{x} + \boldsymbol{u}^T R \boldsymbol{u}$, we can show that its time derivative $\dot{V}(\boldsymbol{x}, \boldsymbol{u})$ is negative semi-definite in the neighborhood of the equilibrium $(\boldsymbol{x}, \boldsymbol{u}) = (\mathbf{0}, \mathbf{0})$ under a certain condition on $\mathcal{L}, Q, R$. Then the origin is locally stable in the sense of Lyapunov.

Accordingly, since the solution $\begin{bmatrix} \overline{\boldsymbol{x}}^S(t) \\ \overline{\boldsymbol{x}}^C(t) \end{bmatrix}$ of (22) is a function of $\overline{\boldsymbol{x}}(0) = \begin{bmatrix} \overline{\boldsymbol{x}}_0^S \\ \overline{\boldsymbol{x}}_0^C \end{bmatrix}$, we have to choose $\overline{\boldsymbol{x}}(0) = \begin{bmatrix} \overline{\boldsymbol{x}}_0^S \\ \overline{\boldsymbol{x}}_0^C \end{bmatrix}$ adequately in order to make $\overline{\boldsymbol{x}}^C(t)$ converge to $\mathbf{0}$ as $\overline{\boldsymbol{x}}^S(t)$ converges to $\mathbf{0}$ from any $\overline{\boldsymbol{x}}(0)$.

Consider the above mentioned in regard to the original system before transformation. Then, under the Assumption 3, we have to choose $\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{u}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_0 \end{bmatrix}$ appropriately to let both the controllable variables being aymptotically stable and uncontrollable variables but being Lyapunov stable converge to $\mathbf{0}$. That can be achieved by choosing an appropriate $\boldsymbol{u}(0) = \boldsymbol{u}_0$ as the function of $\boldsymbol{x}_0$, i.e. $\boldsymbol{u}_0(\boldsymbol{x}_0)$ for arbitrarily given $\boldsymbol{x}_0$.

V. CONTROL OF FLYING ROBOT

We consider a flying robot with extending and contracting lower limbs [4] shown in **Fig.1**. As such a robot, a hopping robot during jump or a space robot without a thruster exists.

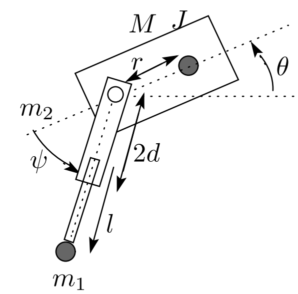


Fig.1. Flying robot with extending and contracting lower limbs

This robot is composed of three parts, i.e. the lower limb, the upper limb and the body. The lower limb with variable length l has mass m_1 which is concentrated on the top point, the upper limb with length $2d$ has mass m_2 concentrated on the center point and the body with mass M has moment of inertia J. It possesses three generalized coordinates, i.e. absolute angle of body θ, relative angle from the body to the upper limb ψ and variable length of the lower limb l. Control inputs are stretching speed of the lower limb $u_1 = \dot{l}$ and relative angular velocity $u_2 = \dot{\psi}$.

This flying robot can be represented as the follows.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\dfrac{b(\boldsymbol{x}(t))}{a(\boldsymbol{x}(t))} & -\dfrac{c(\boldsymbol{x}(t))}{a(\boldsymbol{x}(t))} \end{bmatrix} \boldsymbol{u}(t) \tag{24}$$

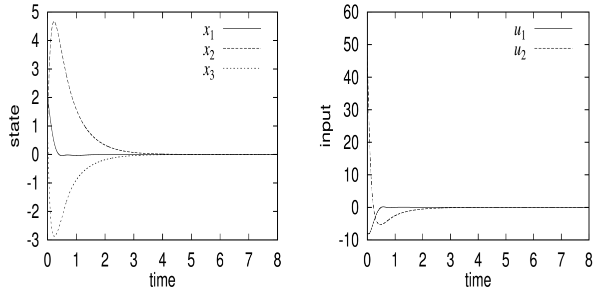
where

$$\begin{aligned} a(x_1, x_2) &= J + \frac{c_1(x_1) + 2c_2(x_1)\cos(x_2) + c_3}{m_0} \\ b(x_2) &= \frac{m_1 M r \sin x_2}{m_0} \\ c(x_1, x_2) &= \frac{c_1(x_1) + c_2(x_1)\cos(x_2)}{m_0} \\ c_1(x_1) &= m_1(m_2 + M)x_1^2 + 2m_1(m_2 + 2M)x_1 d \\ &\quad + (m_1 m_2 + 4m_1 M + m_2 M)d^2 \\ c_2(x_1) &= rM\{m_1(x_1 + 2d) + m_2 d\} \\ c_3 &= M(m_1 + m_2)r^2, \quad m_0 = m_1 + m_2 + M \end{aligned}$$

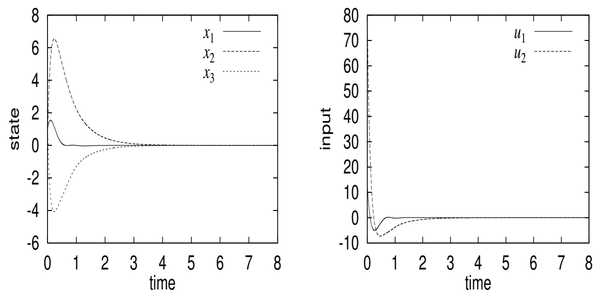
and $\boldsymbol{x} = [l,\psi,\theta]^T$. This system is controllable and $rank(G(\boldsymbol{x})) = 2$ around the goal $\boldsymbol{x}_d = [x_{1d}\ x_{2d}\ x_{3d}]^T$[4].

And the following parameters are used:

$$M = 2,\ J = \frac{2}{3},\ m_1 = 0.5,\ m_2 = 0.1,\ d = 1,\ r = 1$$



(a) $\boldsymbol{x}(0) = [2,\ \pi/4,\ \pi/4]^T$, $\boldsymbol{u}_0 = [-7.614,\ 50.31]^T$



(b) $\boldsymbol{x}(0) = [1,\ \pi/6,\ \pi/3]^T$, $\boldsymbol{u}_0 = [12.45,\ 76.38]^T$

Fig.2. The simulation results of flying robot

The DGDC (15) for (24) is obtained as follows:

$$\begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} = \begin{bmatrix} -\alpha_1(q_1 x_2(t) - q_3 x_3(t)\frac{b(\boldsymbol{x}(t))}{a(\boldsymbol{x}(t))} + r_1 u_1(t)) \\ -\alpha_2(q_2 x_2(t) - q_3 x_3(t)\frac{c(\boldsymbol{x}(t))}{a(\boldsymbol{x}(t))} + r_2 u_2(t)) \end{bmatrix}$$
$$\boldsymbol{u}(0) = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} \tag{25}$$

We choose the parameters $Q, \mathcal{L}, R$ as $Q = \mathrm{diag}\{5, 1, 1\}$, $\mathcal{L} = \mathrm{diag}\{10, 10\}, R = \mathrm{diag}\{1, 1\}$. This flying robot yields an off set when $\boldsymbol{u}(0) = \boldsymbol{0}$ is used. So we try to search initial value $\boldsymbol{u}_0$ such that all variables converge to the origin.

Here, we consider two cases of initial state, (a) $\boldsymbol{x}(0) = [2,\ \pi/4,\ \pi/4]^T$ and (b) $\boldsymbol{x}(0) = [1,\ \pi/6,\ \pi/3]^T$.

Set $t_1 = 8$ and solve problem (18) by using Nelder-Mead's method to obtain optimal initial values, respectively. As the results, in case (a), $\boldsymbol{u}_0 = \begin{bmatrix} -7.614 \\ 50.31 \end{bmatrix}$ was obtained with less than 20 iterations, and in case (b), $\boldsymbol{u}_0 = \begin{bmatrix} 12.45 \\ 76.38 \end{bmatrix}$ was obtained with less than 20 iterations.

Note that in Nelder-Mead's method, we set an initial simplex for both cases (a) and (b) as

$$\boldsymbol{u}_0^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \boldsymbol{u}_0^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \boldsymbol{u}_0^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

, weights in performance function (18a) as $w_1 = 1, w_2 = 1,\ w_3 = 1$, a reflection coefficient α, an expansion coefficient γ, a contraction coefficient β as $\alpha = 1.0,\ \gamma = 1.5,\ \beta = 0.5$, respectively, and the iteration was terminated as the performance function became around zero.

The simulation result is shown in Fig.2. It is observed that all state variables converge to the origin rapidly in both cases. Note that the optimal initial value $\boldsymbol{u}_0$ is not unique.

A flying robot with turning lower limb could be stabilized, also. But it is omitted because of page limitation.

VI. Control of a four-wheeled vehicle

We consider a four-wheeled vehicle[4] shown in **Fig.3**.

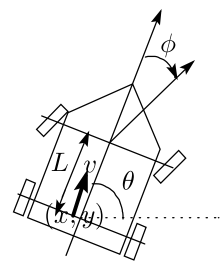


Fig.3. four-wheeled vehicle

This vehicle has a distance L from a middle point of a rear wheel shaft to that of a front wheel shaft.

Let four generalized coordinates be the plane position of body (x, y), the attitude angle of body θ and the steering angle of front wheels ϕ, and let control inputs be moving velocity $u_1 = v$ and steering angular velocity $u_2 = \dot{\phi}$. Then this system can be modeled as follows[4].

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ \frac{1}{L}\tan x_4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = G(\boldsymbol{x}(t))\boldsymbol{u}(t) \tag{26}$$

where $\boldsymbol{x} = \begin{bmatrix} x, & y, & \theta, & \phi \end{bmatrix}^T$.

The DGDC for this system becomes:

$$\begin{cases} \dot{u}_1 = -\alpha_1\Big\{q_1 x_1 \cos x_3 + q_2 x_2 \sin x_3 \\ \qquad\qquad + q_3 x_3 \frac{1}{L}\tan(x_4) + r_1 u_1\Big\} \\ \dot{u}_2 = -\alpha_2 \left\{q_4 x_4 + r_2 u_2\right\} \end{cases} \tag{27}$$

We choose the parameters $Q, \mathcal{L}, R$ as $Q = \mathrm{diag}\{5, 1, 1, 1\}$, $\mathcal{L} = \mathrm{diag}\{10, 10\}, R = \mathrm{diag}\{1, 1\}$. Since this vehicle yields an offset with initial value $\boldsymbol{u}(0) = \boldsymbol{0}$, we try to find an initial value $\boldsymbol{u}_0$ such that all state variables converge to the origin. Here, we consider two cases of initial state, (a) $\boldsymbol{x}(0) = [6,\ 5,\ \pi/4,\ 0]^T$ and (b) $\boldsymbol{x}(0) = [15,\ -10,\ \pi/6,\ \pi/4]^T$.

Set $t_1 = 10$ and solve problem (17) by use of Nelder-Mead's method to obtain optimal initial values $\boldsymbol{u}_0$, respectively. As the results, in case (a), $\boldsymbol{u}_0 = \begin{bmatrix} -62.31 \\ 2.023 \end{bmatrix}$ and in case (b), $\boldsymbol{u}_0 = \begin{bmatrix} -48.46 \\ -5.442 \end{bmatrix}$ were obtained with less than 70 iterations. Note that in Nelder-Mead's method, we set an initial simplex in both cases (a) and (b) as

$$\boldsymbol{u}_0^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \boldsymbol{u}_0^2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix},\ \boldsymbol{u}_0^3 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

, weights in performance function (18a) as $w_1 = 1, w_2 = 1,\ w_3 = 1$, a reflection coefficient α, an expansion coefficient γ, a contraction coefficient β as $\alpha = 1.0,\ \gamma = 1.5,\ \beta = 0.5$, respectively, and the iteration was terminated as the performance function became around zero.

The simulation results are shown in Fig.4 and 5. All state variables converged to the origin rapidly in both cases (a) and (b). It is noted that optimal initial value $\boldsymbol{u}_0$ is not unique.

VII. Conclusion

We showed that the DGDC worked effectively for the symmetric affine system. In particular the initial value of DGDC was used effectively for asymptotical stabilization.

Note that a dynamic controller can be effectively applied for the system which cannot be controlled by a static continuous state feedback without changing it into the chained form. In addition, it is possible to avoid a difficulty of singular point avoidance at the time of inverse transformation, because our method need not transform the system into the chained form.

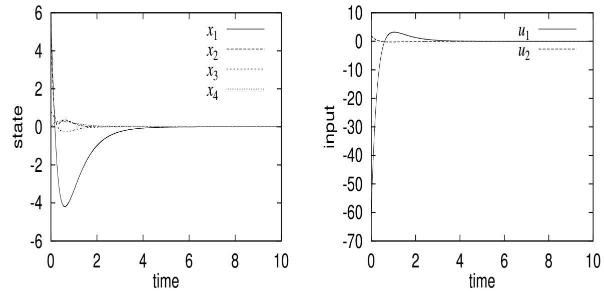
The proposed idea using an initial value of dynamic controller may be applied, for example, to a velocity type PID controller as well as DGDC. Incidentally, it was confirmed by simulation that the PID control achieved convergence to the origin for the examples studied in this paper.

The proposed method looks like a kind of path planning, because whole trajectory has been calculated when one searches the initial condition $\boldsymbol{u}_0$. However, we consider it is not path planning. In fact, the control is actually executed by the state feedback of DGDC with the best $\boldsymbol{u}_0$. We consider $\boldsymbol{u}_0$ as one parameter included in the dynamic controller (DGDC) working by the state feedback.

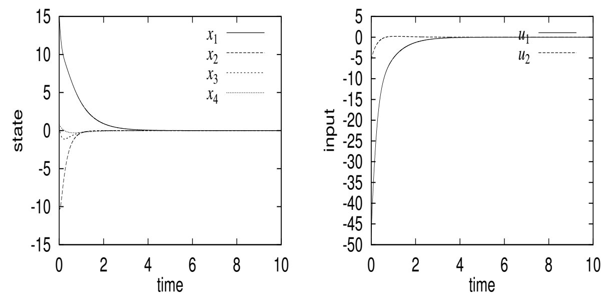
As mentioned before, an optimum $\boldsymbol{u}_0$ is not unique, in fact there exist many affective $\boldsymbol{u}_0$, and furthermore the optimum $\boldsymbol{u}_0$ is not necessarily searched at one stage. If the DGDC has made the plant converge to the undesired equilibrium with the first $\boldsymbol{u}_0$, then repeat the same process at the second stage.

Since computational time of the Nelder-Mead method is very short, our method can be practically implemented.

A similar approach for symmetric affine system is found in [17] also.

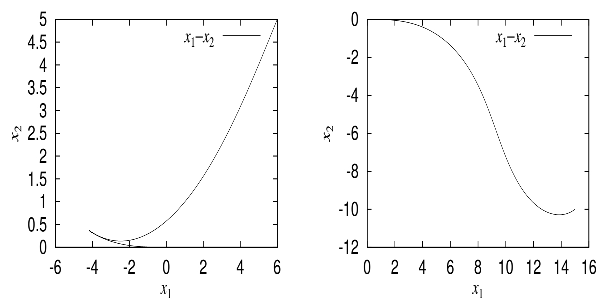


(a) $\boldsymbol{x}(0) = [6,\ 5,\ \pi/4,\ 0]^T$
$\boldsymbol{u}_0 = [-62.31,\ 2.023]^T$



(b) $\boldsymbol{x}(0) = [15,\ -10,\ \pi/6,\ \pi/4]^T$
$\boldsymbol{u}_0 = [-48.60,\ -5.442]^T$

Fig.4. The simulation result of a four-wheeled vehicle



(a) **(b)**

Fig.5. The trajectory of a four-wheeled vehicle in x-y plane

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