2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Stackelberg Game Approach to Constrained OSNR Nash Game in WDM Optical Networks

Quanyan Zhu and Lacra Pavel

Abstract— **This paper formulates a Stackelberg $(N+1)$-person non-cooperative game framework for OSNR optimization in optical networks. We introduce cost functions with differentiated prices for users and develop a cost function for the higher level Stackelberg player. In the design, we consider the capacity constraints in optical networks, i.e., that the total optical power does not exceed the link's capacity. We formulate a novel Stackelberg framework in OSNR game and characterize its Stackelberg equilibrium. Based on it, we give a closed form solution and develop an iterative algorithm for the optical network control problem and illustrate the game with a numerical example.**

I. Introduction

Devices such as optical add/drop MUXes (OADM), optical cross connects (OXC) and dynamic gain equalizer (DGE) have provided essential building blocks for smart optical networks [1]. These technological advances have enabled a new generation of reconfigurable optical wavelength-division multiplexed (WDM) communication networks, which are dynamically evolving to respond to changes in traffic and requirements. A static network management mechanism can no longer service such networks. Therefore, intelligent network management and control systems need to be part of future network design.

This paper addresses a power control scheme at a link level, where channel optical signal-to-noise ratio (OSNR) is an important performance factor as it directly relates to the bit error rate (BER) in the transmission [2]. In recent years, research work on OSNR-based optimization is making an effort to derive iterative decentralized OSNR optimization algorithms in optical networks. Two dominant methods are commonly seen in literature. One is the centralized optimization as in [3], [4] and the other is non-cooperative game theory as in [5], [6]. The centralized approach embeds OSNR targets in constraints and indirectly minimizes the total power consumption in optical networks. It is relatively easy to find a closed form solution with this approach, however, its indirect minimization of total power consumption doesn't fully make use of the network resource for communication purposes. On the other hand, the non-cooperative game approach naturally deals with OSNR optimization in a decentralized and direct manner. However, it is a well-known fact that the resulting Nash equilibrium may not be Pareto efficient [7], [8]. In addition, under the OSNR game framework, it has been a challenge to find an analytical solution for a game with capacity constraints. Research efforts have been made to solve this problem by integrating constraints into utility functions [9], [10]. And, in particular, work has been done in [11], [12] to deal with such constraints based on classical Lagrangian duality theory. However, complexity of the theory needs to be reduced to an applicable form and it is difficult to give an analytical solution for OSNR Nash game.

In this paper, we formulate a game with an additional Stackelberg player to give a closed form solution to the constrained OSNR game. We may also use the role of Stackelberg player to achieve an efficient equilibrium under certain conditions. The Stackelberg player, in reality, can be implemented via a service channel or a transmission channel which only needs a target OSNR.

We also compare the Stackelberg equilibrium with the Nash equilibrium obtained from a similar game with a non-Stackelberg player. We propose the notion of price of leadership to quantify the price a Stackelberg player has to pay to assume the role of leadership.

This paper is organized as the following. In section II, we review a network OSNR model and give a brief introduction to unconstrained non-cooperative game approach. In section III, we establish the framework of Stackelberg game. We will characterize the Stackelberg equilibrium and discuss the achievable target OSNR of the Stackelberg player. Section IV proposes the notion of price of leadership and makes comparison with game with a fictitious player. An iterative algorithm is presented in section V and we illustrate the algorithm using a numerical example. Finally, we point out the directions of future research and conclude and conclude in section VI.

II. Background

We consider the same optical network model described in [5]. We let $\mathcal{N}$ denote the set of of channels are transmitted and u_i be the $i-$th channel input optical power (at Tx), and $\mathbf{u} = [u_1, ..., u_N]^T$ the vector of all channels' input powers. The ith channel optical OSNR is thus given as

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j \in \mathcal{N}} \Gamma_{i,j} u_j}, i \in \mathcal{N} \qquad (1)$$

where $\mathbf{\Gamma}$ is the full $n \times n$ system matrix which characterizes the coupling between channels. $n_{0,i}$ denotes the ith channel noise power at the transmitter. System matrix $\mathbf{\Gamma}$ encapsulates the basic physics present in optical fiber transmission and implements an abstraction from a network to an input-output system. This approach has been used in [6] for the wireless case to model CDMA uplink communication. Different from

Quanyan Zhu and Lacra Pavel are with Department of Electrical and Computer Engineering, University of Toronto, Ontario M5S 3L1, Canada {qzhu,pavel}@control.utoronto.ca

the system matrix used in wireless case, the matrix $\mathbf{\Gamma}$ is commonly asymmetric and is more complicatedly dependent on parameters such as spontaneous emission noise, wavelength-dependent gain, and the path channels take.

A game-theoretical model for power control in optical networks without constraints is formulated in [5]. Let's consider a game defined by a triplet $\langle \mathcal{N}, (A_i), (J_i) \rangle$, where $\mathcal{N}$ is the index set of players or channels; A_i is the strategy set $\{u_i \mid u_i \in [u_{i,\min}, u_{i,\max}]\}$; and, J_i is the cost function chosen in a way that minimizing the cost is related to maximizing OSNR level. In [5], J_i is defined as

$$J_i(u_i, u_{-i}) = \alpha_i u_i - \beta_i \ln\left(1 + a_i \frac{u_i}{X_{-i}}\right), i \in \mathcal{N} \qquad (2)$$

where α_i, β_i are channel specific parameters, that quantify the willingness to pay the price and the desire to maximize its OSNR, respectively, a_i is a channel specific parameter, X_{-i} is defined as $X_{-i} = \sum_{j\neq i} \Gamma_{i,j} u_j + n_{0,i}$. This specific choice of utility function is non-separable, nonlinear and coupled. However, J_i is strictly convex in u_i and takes a specially designed form such that its first-order derivative, the implicit best response function, takes a linear form with respect to $\mathbf{u}$.

The solution from the game approach is usually characterized by Nash equilibrium (NE). Provided that $\sum_{j\neq i}\Gamma_{i,j} \leq a_i$, the resulting NE solution is given in a closed form by

$$\widehat{\mathbf{\Gamma}}\mathbf{u}^* = \widehat{\mathbf{b}}, \qquad (3)$$

where $\hat{b}_i = \frac{a_i b_i}{\alpha_i} - n_{0,i}$ and

$$\widehat{\mathbf{\Gamma}} = \begin{pmatrix} a_1 & \Gamma_{12} & \dots & \Gamma_{1N} \\ \Gamma_{21} & a_2 & \ddots & \Gamma_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ \Gamma_{N1} & \dots & \dots & a_N \end{pmatrix}. \qquad (4)$$

Similar to the wireless case [6], we are able to construct iterative algorithms to achieve the Nash equilibrium. A simple deterministic first order parallel update algorithm can be found by $u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{X_{-i}(n)}{a_i}$, or equivalently in terms of $OSNR_i$,

$$u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(n)} - \Gamma_{i,i}\right) u_i(n). \qquad (5)$$

As proved in [5], the algorithm (5) converges to Nash equilibrium $\mathbf{u}^*$ provided that $\frac{1}{a_i}\sum_{j\neq i}\Gamma_{i,j} \leq 1, \forall i$.

III. Stackelberg Game

In optical networks, a saturation power level exists in each link of channel paths [9]. A launched power has to be below or equal to this threshold so that the nonlinear effects in the span following each amplifier are kept minimum [13]. We can easily interpret this effect as a capacity constraint on an optical link in the network. In this section, we introduce a noncooperative game with an additional Stackelberg player to deal with such capacity constraints in optical networks.

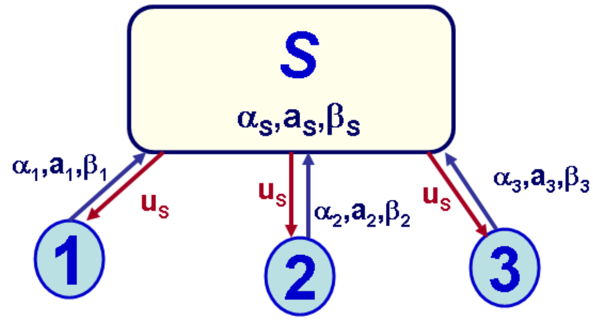

Fig. 1. Illustration of Stackelberg Game: users 1,2,3 submit their parametric information to stackelberg player S. Player S regulates the network by sending information back to the rest of the players.

Suppose we have a capacity constraint on an end-to-end link such that

$$\sum_{i\in\mathcal{N}\cup\{S\}} u_i \leq C, \qquad (6)$$

where S denotes the Stackelberg player. In the Nash game described in section 2, players behave selfishly to minimize their own cost function. They make decisions solely based on their own local information without the knowledge of other players' actions. In Stackelberg game, we treat the additional player as a leader or manager, who can make decisions based on global information and guide other users in the network as in [14],[15]. We implement this Stackelberg player by an optical service channel (OSC) as it serves as an internode communication channel for management and user data [1],[16]. We can assume that service channel can gather the information about the strategy each user employs and thus make decisions on the network resource allocation policy. It gives a structure of two-stage leader and follower extensive game [17],[18], illustrated in Figure 1. In real practice, service channel can obtain information by asking for user parameters such as α, β. Parameter α_i can be seen as an evaluation or bid on unit power, and β_i as the level of quality of service. Parameter a_i is a technical system determined parameter that controls the rate of convergence.

We have flexibility at our disposal in choosing an appropriate utility function for the Stackelberg player. One choice is to take the form like other users as a function of OSNR. However, it turns out that solving it for an analytical Stackelberg equilibrium is quite challenging. We notice that Stackelberg player may not be a network customer, but an internal network service channel. As a result, its target may not be optimizing OSNR and it can be satisfied if a certain OSNR target for this channel is met.

Therefore, in addition to N channel users, we choose a particular utility function for the Stackelberg player as in (7). It is composed of its own utility function given by $U_S = \left(C - \sum_{j\neq S} u_j\right) u_S$ and the cost function by $P_S = \frac{1}{2}\left(\omega_S + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S\right) u_S^2$. The utility is related to two factors: one is its own power usage and the other is the penalty from the constraints, or the remaining power in the optical system. The cost is specially designed such that it accounts for the entire network and gives a tractable solution. In (7), it is given in a quadratic form parameterized by ω_S.

$$\begin{aligned} J_S &= P_S(u_S) - U_S(u_S, u_{-S}) \quad (7)\\ &= \frac{1}{2}\left(\omega_S + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S\right)u_S^2 - \left(C - \sum_{j\neq S} u_j\right)u_S, \end{aligned}$$

where $\mathbf{u}_{-S} = [u_i], i \in \mathcal{N}$; $\omega_S \in \mathcal{R}$ is a network design variable, $\mathbf{g}_S$ is a vector given by $\mathbf{g}_S = [\Gamma_{1S}, \Gamma_{2S}, ..., \Gamma_{NS}]^T$ and $\widehat{\mathbf{\Gamma}}$ is defined earlier in (4). Since we are implementing the Stackelberg as OSC, $\Gamma_{iS}, i \in \mathcal{N}$ in $\mathbf{g}_S$, denotes the interactions between user channels and OSC.

Following the payoff functions (2) adopted in the unconstrained game, in the $(N+1)-$ person Stackelberg game, we choose the cost functions for user $i \in \mathcal{N}$ in a similar form given by

$$J_i(u_i, u_{-i}) = \alpha_i u_i - \beta_i \ln\left(1 + a_i \frac{u_i}{X_{-i} + \Gamma_{iS} u_S}\right), \forall i \in \mathcal{N}. \quad (8)$$

where $u_i \in A_i$, for $i = 1, \cdots, N$ and $u_S \in A_S$, for $i = S$. Such choice of payoff function includes the channel and channel coupling with OSC, or the Stackelberg player.

A. Stackelberg Equilibrium

A solution that characterizes such $(N+1)-$ person game is the Stackelberg equilibrium [19]. We provide a precise definition for the Stackelberg solution concept in (3.1), where we have one leader and multiple users, or followers.

Definition 3.1: (Stackelberg Equilibrium, [19]) Let $\Omega = \prod_{i=1}^N A_i$ and $u_i \in A_i, i \in \mathcal{N} \cup \{S\}$. A $(N+1)-$ person finite game with $(N+1)$-th person, denoted as S, as the Stackelberg leader, a strategy $\mathbf{u}^*_{-S}$, or $[u_i^*], (i = 1, \cdots, N)$, and u_S^* is called a *Stackelberg equilibrium* for the leader if

$$\begin{aligned} J_S^*(\mathbf{u}^*_{-S}, u_S^*) &:= \max_{\mathbf{u}_{-S} \in R_{-S}(u_S^*)} J_S(\mathbf{u}_{-S}, u_S^*) \quad (9)\\ &= \min_{u_S \in A_S} \max_{\mathbf{u}_{-S} \in R_{-S}(u_S)} J_S(\mathbf{u}_{-S}, u_S) \quad (10) \end{aligned}$$

where $R_{-S}(u_S) = \{\mathbf{u}'_{-S} \in \Omega : J_i(u_S, \mathbf{u}'_{-S}) \leq J_i(u_S, \mathbf{u}_{-S}),\ \forall i, \forall \mathbf{u}_{-S} \in \Omega\}$. The set $R_{-S}(u_S)$ is the optimal response of N players to the strategy $u_S \in A_S$ of the Stackelberg player. The quantity J_S^* is the Stackelberg cost of the leader.

Remark 3.1: If $R_{-S}(u_S)$ is a singleton for each $u_S \in A_S$, then there exists a mapping $\mathbf{T}_{NS} : A_S \rightarrow \Omega$, such that $u_{-S} \in R_{-S}(u_S)$ implies $u_S = \mathbf{T}_{NS} u_S$. This corresponds to the case in which the optimal response of the followers is unique for every strategy of the leader. A solution u_S^* is a Stackelberg equilibrium if

$$J_S^*(u_S^*, \mathbf{T}_{NS} u_S^*) = \min_{u_S \in A_S} J_S(u_S, \mathbf{T}_{NS} u_S).$$

The pair $(u_S^*, \mathbf{u}^*_{-S})$ is a Stackelberg solution for the game with S as the leader, and the costs $J_i(u_S^*, \mathbf{u}^*_{-S}), i \in \mathcal{N}$ are the corresponding Stackelberg outcome.

B. Characterization of Stackelberg Equilibrium

In this section, we use Definition 3.1 to characterize the Stackelberg solution. To find the Stackelberg equilibrium $\mathbf{u}^*$, we can form a system of equations (11) from the best response function of user $i \in \mathcal{N}$. The linearity of the best response function and the non-singularity of $\widehat{\mathbf{\Gamma}}$ in (11) give rise to a singleton of set $R_{-S}(u_S)$ for each $u_S \in A_S$.

$$\widehat{\mathbf{\Gamma}}\mathbf{u}^* = \xi(u_S), \quad (11)$$

where $\widehat{\mathbf{\Gamma}}$ is defined in (4), $\xi(u_S) = \hat{\mathbf{b}} - \mathbf{g}_S u_S$, or

$$\xi(u_S) = \begin{pmatrix} \frac{a_1\beta_1}{\alpha_1} - n_{0,1} - \Gamma_{1S} u_S \\ \frac{a_2\beta_2}{\alpha_2} - n_{0,2} - \Gamma_{2S} u_S \\ \vdots \\ \frac{a_N\beta_N}{\alpha_N} - n_{0,N} - \Gamma_{NS} u_S \end{pmatrix}, \mathbf{u}_{-S} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}.$$

With the assumption of diagonal dominance, i.e., $\sum_j \Gamma_{ij} < a_i$ on $\widehat{\mathbf{\Gamma}}$, the matrix is nonsingular; the solution to (11) is uniquely determined by (12) for every u_S.

$$\mathbf{u}^*_{-S} = \widehat{\mathbf{\Gamma}}^{-1}\xi(u_S). \quad (12)$$

Substitute (12) into (7) and we can obtain J_S only in terms of u_S given by

$$\begin{aligned} J_S &= \frac{1}{2}\left(\omega_S + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S\right)u_S^2 - \left(C - \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\xi(u_S)\right)u_S.\\ &= \frac{1}{2}\left(\omega_S + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S\right)u_S^2 - Cu_S\\ &\quad + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}}u_S - \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S u_S^2. \quad (13) \end{aligned}$$

Suppose $u_{S,min}$ and $u_{S,max}$ are sufficiently small and large respectively. We can thus take the first derivative of (13) to find its minimum with respect u_S and arrive at

$$\begin{aligned} \frac{\mathbf{d}J_S}{\mathbf{d}u_S} &= \left(\omega_S + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S\right)u_S - C\\ &\quad + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}} - 2\left(\mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S u_S\right)\\ &= \omega_S u_S - C + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\left(\widehat{\mathbf{b}} - \mathbf{g}_S u_S\right) = 0 \end{aligned}$$

Therefore, a necessary condition of a minimum is that u_S^* satisfies

$$\omega_S u_S + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}(\widehat{\mathbf{b}} - \mathbf{g}_S u_S) = C. \quad (14)$$

From the second derivative of (13), we observe that a unique minimum is given by (14) provided that

$$\omega_S > \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S. \quad (15)$$

In addition, when

$$\omega_S < \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S, \quad (16)$$

the minimum is simply achieved at $u_{S,min}$.

Under the condition of (15), we observe from (14) that it is equivalent to the slacked capacity constraint

$$\omega_S u_S + \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\xi(u_S) = C.$$

or equivalently,

$$\omega_S u_S + \sum_{i\in\mathcal{N}} u_i = C. \quad (17)$$

Therefore, directly following (14), we can solve for u_S^* in a closed form given by

$$u_S^* = \frac{1}{\omega_S'}\left(C - \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}}\right), \quad (18)$$

where $\omega_S' = \omega_S - \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S$. Since u_S is positive, it is required that

$$\mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}} < C, \quad (19)$$

In summary, under (19) and (15), $\mathbf{u}_S^*$ can be obtained in a closed-form as in (18) and (12). It describes a situation in which the unconstrained allocations among N non-Stackelberg users satisfy the strict inequality (6) and the Stackelberg player gets the remaining resources. When condition (19) doesn't hold, the Stackelberg player regulates the network by allocating himself a minimum power and other users respond to this allocation according to (11). Other cases will result in unrealistic solutions.

Condition described by (19) can be satisfied by imposing a simplified sufficient condition described in (20) which has its implication in admission control. This is summarized into the following proposition.

$$\frac{\kappa \max_i \widehat{b}_i\sqrt{N}}{\sqrt{\rho(\widehat{\mathbf{\Gamma}}^{\mathbf{T}}\widehat{\mathbf{\Gamma}})}} \leq C. \quad (20)$$

Proposition 3.1: Let $\rho(\cdot)$ denotes the spectral radius and κ be the condition number. If $\frac{\kappa \max_i \widehat{b}_i\sqrt{N}}{\sqrt{\rho(\widehat{\mathbf{\Gamma}}^{\mathbf{T}}\widehat{\mathbf{\Gamma}})}} \leq C$, $\omega_S' > 0$ and $\omega_S \geq 1$, the Stackelberg solution $\mathbf{u}^*$ is uniquely given by (12) and (18) and satisfies the capacity constraint $\sum_{i\in\mathcal{N}\cup\{S\}} u_i \leq C$. In addition, when $\omega_S = 1$, the solution is Pareto efficient.

Proof: First of all, we need to show that (13) and the utility functions $J_i, i \in \mathcal{N}$ are convex and there exists a minimizing $\mathbf{u}^*$. It has been proved in [5] that functions (2) is convex in u_i. We just need to show the convexity of (13) in u_S. With the condition of (15), the convexity of J_S in u_S will follow because the second derivative of (13) is positive. Due to the fact that $u_i \in [u_{\min}, u_{\max}]$ gives a compact set, there exists a minimizing $\mathbf{u}_S$.

Secondly, we derive a sufficient condition for (19). Starting with the condition $\sum_{j\neq S} u_j \leq C$, we use matrix norm inequality $\|\widehat{\mathbf{\Gamma}}\|_2 \leq \sqrt{N}\|\widehat{\mathbf{\Gamma}}\|_\infty$ [20] to obtain an upper bound on $\|\mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}}\|_\infty$.

$$\|\mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}}\|_\infty \leq \|\mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\|_\infty\|\widehat{\mathbf{b}}\|_\infty \leq \frac{\kappa}{\|\widehat{\mathbf{\Gamma}}\|_\infty}\|\widehat{\mathbf{b}}\|_\infty \leq$$

$$\frac{\kappa \max_i \widehat{b}_i}{\|\widehat{\mathbf{\Gamma}}\|_\infty} \leq \frac{\kappa \max_i \widehat{b}_i\sqrt{N}}{\|\widehat{\mathbf{\Gamma}}\|_2} = \frac{\kappa \max_i \widehat{b}_i\sqrt{N}}{\sqrt{\rho(\widehat{\mathbf{\Gamma}}^{\mathbf{T}}\widehat{\mathbf{\Gamma}})}}.$$

where κ is the condition number of $\widehat{\mathbf{\Gamma}}$, given by $\kappa = \|\widehat{\mathbf{\Gamma}}\|_\infty\|\widehat{\mathbf{\Gamma}}^{-1}\|_\infty$.

If inequality $\frac{\kappa \max_i \widehat{b}_i\sqrt{N}}{\sqrt{\rho(\widehat{\mathbf{\Gamma}}^{\mathbf{T}}\widehat{\mathbf{\Gamma}})}} \leq C$ holds, then the solution to the Stackelberg player is given in (18). If we further assume that for a given $C \geq \frac{a_i b_i}{\alpha_i} - n_{0,i}, \forall i$, then it will reduce the condition (20) to $\rho(\widehat{\mathbf{\Gamma}}^{\mathbf{T}}\widehat{\mathbf{\Gamma}}) \geq \kappa^2 N$. This result alludes to the maximum number of channels to be admitted in the network for a fixed capacity.

Lastly, we prove that there exists a unique solution under the assumption of diagonal dominance of matrix $\widehat{\mathbf{\Gamma}}$. With $a_i > \sum_{j\neq i}\Gamma_{ij}$, matrix $\hat{\mathbf{\Gamma}}$ becomes diagonally dominant. From Gershgorin's Theorem [21], it follows that $\hat{\mathbf{\Gamma}}$ is non-singular and there exists a unique solution to linear system (11).

When $\omega_S \geq 1$, any solution from (17) satisfies the capacity constraint $\sum_{i\in\mathcal{N}\cup\{S\}} u_i \leq C$ due to the fact

$$\sum_{i\in\mathcal{N}} u_i + u_S \leq \omega_S u_S + \sum_{i\in\mathcal{N}} u_i = C, \forall \omega_S \geq 1,$$

where the equality comes from (17) and the inequality comes from $\omega \geq 1$. In addition, since we have $\omega_S > \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}_S$, this ensures that $u_S > 0$. It is obvious that when $\omega_S = 1$, $\mathbf{u}^S$ will be Pareto efficient. ■

IV. Price of Leadership

In this section, we make a comparison between constrained OSNR Stackelberg game (GSP) and the game with fictitious player (GFP). We define the notion of price of leadership which quantifies the price that a Stackelberg player has to pay to assume his role of leadership and obtain the knowledge of his teammates.

A. Game with a Fictitious Player

Following the same idea described in section 3, we use an additional player to formulate the game with fictitious player (GFP) to tackle the game in section 2 with capacity constraints. Instead of having a Stackelberg player, we let the additional player be a fictitious player, denoted as F, who optimizes his own utility without the knowledge of his peers. We choose the cost function of the fictitious player as

$$J_F = P_F(u_F) - U_F(u_F) = \omega_F u_F - \left(C - \sum_{j\neq F} u_j\right)\ln u_F,$$

where $P_F = \omega_F u_F$ is the cost term for the player F's power usage, $U_F = \left(C - \sum_{j\neq F} u_j\right)\ln u_F$ is the utility of player F, and $\omega_F \in \mathcal{R}$ is a user parameter. It has been shown in [22] that the solution to the GFP is given by

$$u_{-F} = \widehat{\mathbf{\Gamma}}^{-1}\xi(u_F),$$
$$u_F = \frac{1}{\omega_F}\left(C - \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}}\right),$$

when $\mathbf{g}_F = [\Gamma_{1F}, \Gamma_{2F}, \cdots, \Gamma_{NF}]^T = \mathbf{0}$.

It is easy to observe that the solutions to GSP and GFP are in the same form. Suppose $\mathbf{g}_S = \mathbf{g}_F = 0$. In this case, $u_{-S} = u_{-F}$; however, u_F and u_S are related by

$$\eta = \frac{u_S}{u_F} = \frac{\omega_F}{\omega_S'}.$$

B. GSP vs. GFP

To compare GSP with GFP, we let $\mathbf{g}_S = \mathbf{g}_F = \mathbf{0}$, and $\eta = 1$, i.e., $u_F = u_S = u$, $\omega_F = \omega'_S = \omega_S - \mathbf{1}^T\widehat{\mathbf{\Gamma}}\mathbf{g} = \omega$. Under these assumptions, the solutions from GSP and GFP become identical, but the the payoff values for player S and player F are different. We define η_P and η_U as follows.

$$\eta_P = \frac{P_F}{P_S} = \frac{2}{u}, \eta_U = \frac{U_F}{U_S} = \frac{\ln(u)}{u}.$$

It is obvious that η_P and η_U measure the cost and utility ratios between F and S in GFP and GSP, respectively. With full information, the Stackelberg player obtains more utility than being a fictitious player. On other hand, it costs less for the Stackelberg player when his power consumption doesn't exceed 2 units.

To characterize the total payoff, we define the payoff difference between GSP and GFP as the price of leadership, given by $\Delta J_{FS} = J_F - J_S$=$-\frac{1}{2}\omega u^2 - du^2 + (t+\omega)u - t\ln u$. where $d = \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\mathbf{g}$, $t = C - \sum_{j\neq F,S} u_j$.

Theorem 4.1: Suppose $\omega > t$, $u \in [0, u_{max}]$. An upper bound and a lower bound exist on ΔJ_{FS}. They are given by

$$\omega u_{\max} \le \Delta J_{FS} \le t + \frac{1}{2}\omega$$

Proof: To show an upper bound of ΔJ_{FS}, we find an inequality as follows.

$$\Delta J_{FS} \le \min_u -\frac{1}{2}\omega u^2 + (t+\omega)u - t\ln(u).$$

The optimal solution to the right-side inequality is achieved at u^* such that

$$\frac{\mathrm{d}R(u)}{\mathrm{d}u} = -\omega u + (t+\omega) - t/u = 0.$$

Therefore, $u = t/\omega$ or $u = 1$. From the second derivative

$$\frac{\mathrm{d}^2 R(u)}{\mathrm{d}^2 u} = -\omega + tu^{-2},$$

we can conclude that $u = 1$ is an optimal solution when $\omega > t$, and $u_{max} > 1$. Therefore, we obtain an upper bound $\Delta J_{FS} \le t + \frac{1}{2}\omega$.

On the other hand, since $u > \ln(u)$, we have the inequality of

$$\Delta J_{FS} \ge \max_u \ln(u).$$

The optimality of the right-hand inequality is achieved at $u = \ln(u_{max})$. Therefore, $\Delta J_{FS} \ge \ln(u_{max})$. ■

V. Iterative Algorithm and Example

We can improve the first-order algorithm in (5) by including the Stackelberg player. It is described as follows.

Algorithm 5.1:

Stackelberg player:

$$u_S = \max\left(u_{S,min}, \frac{1}{\omega'}(C - \mathbf{1}^T\widehat{\mathbf{\Gamma}}^{-1}\widehat{\mathbf{b}})\right) \qquad (21)$$

User algorithm:

$$\begin{aligned} u_i(n+1) &= \frac{\beta_i}{\alpha_i} - \frac{1}{a_i}[(\frac{1}{OSNR_i(n)} - \Gamma_{i,i})u_i(n) \\ &+ \Gamma_{iS}u_S], \forall i \in \mathcal{N}; \end{aligned} \qquad (22)$$

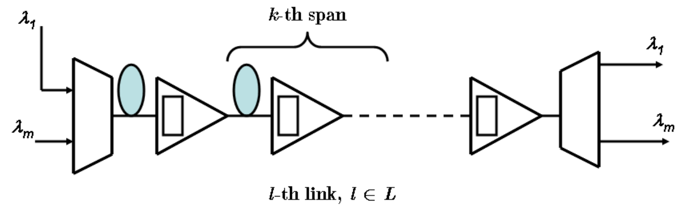

Fig. 2. A Typical Optical Link in DWMW Optical Networks

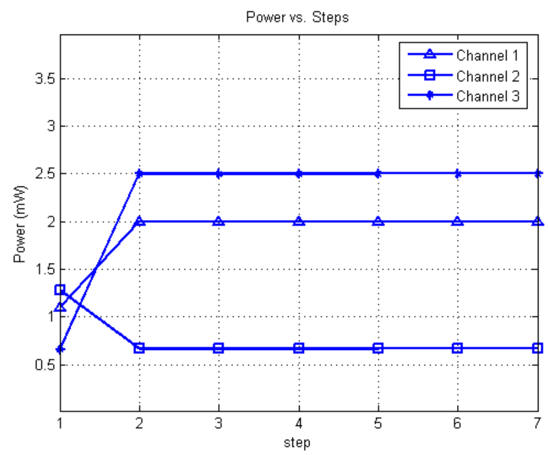

Fig. 3. Channel Power in Time Steps

where $OSNR_i$ measures the OSNR level of each channel as in (1). The Stackelberg player firstly sets his own power in (21) and then passes its value to users who respond to u_S in (22).

We illustrate the linear OSNR Nash game by a MATLAB simulation. We consider an end-to-end link described in Figure (2) with 5 amplified spans. We assume 3 channels are transmitted at wavelengths distributed from 1554nm to 1556nm with channel separation of 1nm. Suppose input noise power is 0.5 percent of the input signal power and the total power constraint is C=7.0mW. The gain profile for each amplifier is identically assumed to be parabolic and gives $G_1 = 29.2$dB,$G_2 = 30.0$dB, and $G_3 = 29.2$dB, respectively. The 3-by-3 $\mathbf{\Gamma}$ matrix is thus given as

$$\mathbf{\Gamma} = \begin{bmatrix} 6.187 \times 10^{-4} & 1.094 \times 10^{-4} & 2.732 \times 10^{-4} \\ 4.063 \times 10^{-4} & 6.786 \times 10^{-4} & 2.206 \times 10^{-4} \\ 2.728 \times 10^{-4} & 3.752 \times 10^{-4} & 2.728 \times 10^{-4} \end{bmatrix}$$

In Figure 3, we show the convergence of channel power evolution in steps with a Stackelberg channel, which takes 1.8338mW of the remaining power from the capacity and manifests a power allocation efficiency of 73.8% for the link with respect to the capacity. Figure 4 demonstrates the resulting OSNR level in the game with a Stackelberg player.

VI. Conclusion

In this paper, we introduced an additional player into the unconstrained OSNR Nash game to formulate Stackelberg game. It can be implemented via an optical service channel

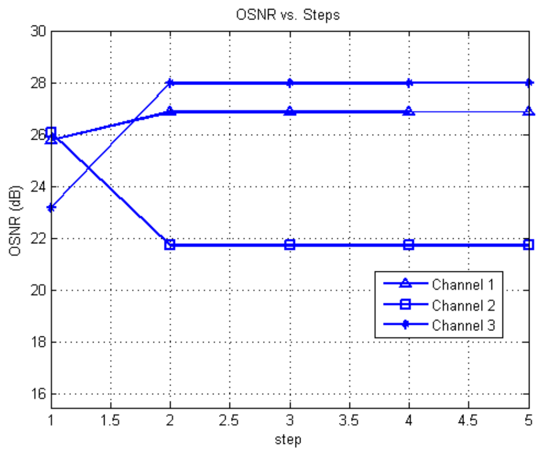

Fig. 4. Channel OSNR in Time Steps

which serves as a manager that regulates network performance. By choosing a cost function for the Stackelberg player, we can design an arbitrating mechanism to solve the corresponding game with capacity constraints in optical networks. We gave a closed form of the resulting Stackelberg equilibrium, from which we derived an iterative algorithm. In addition, we discussed the attainable OSNR levels within this framework.

References

[1] R. Ramaswami and K. Sivarajan, *Optical Networks: A Practical Perspective*. Academic Press, 2002.

[2] G. Agrawal, *Lightwave Technology*. Wiley-Interscience, 2005.

[3] L. Pavel, "OSNR optimization in optical networks: Modeling and distributed algorithms via a central cost approach," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 4, pp. 54–65, April 2006.

[4] T. Alpcan, X. Fan, T. Basar, M. Arcak, and J.T.Wen, "Power control for multicell CDMA wireless networks: A team optimization approach," *Third International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks*, pp. 379–388, 2005.

[5] L. Pavel, "A noncooperative game approach to OSNR optimization in optical networks," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 848–852, May 2006.

[6] R. Srikant, E. Altman, T. Alpcan, and T. Basar, "CDMA uplink power control as noncooperative game," *Wireless Networks*, vol. 8, p. 659 690, 2002.

[7] P. Dubey, "Inefficiency of Nash equilibria," *Mathematics of Operations Research*, vol. 11, no. 1, February 1986.

[8] T. Roughgarden and E. Tardos, "Bounding the inefficiency of equilibria in nonatomic congestion games," *Games and Economic Behavior*, vol. 47, pp. 389–403, 2004.

[9] Y. Pan and L. Pavel, "OSNR optimization in optical networks: Extension for capacity constraints," *Proceedings of 2005 American Control Conference*, pp. 2379–2385, June 2005.

[10] ——, "Global convergence of an iterative gradient algorithm for the Nash equilibrium in an extended OSNR game," *Proc. IEEE INFOCOM*, May 2007.

[11] L. Pavel, "An extension of duality to a game-theoretic tramework," *Automatica*, vol. 43, no. 2, pp. 226–237, 2007.

[12] ——, "Hierarchical iterative algorithm for a coupled constrained OSNR Nash game," *Proceedings of IEEE GLOBECOM*, November 2006.

[13] R. T. F. Forghieri and D. Favin, "Simple model of optical amplifier chains to evaluate penalties in WDM systems," *Journal of Lightwave Technology*, vol. 16, no. 9, pp. 1570–1576, 1998.

[14] H. Shen and T. Basar, "Hierarchical network games with various types of public and private information," *Proc. of 45th IEEE Conf. on Decision and Control*, pp. 2825–2830, 2006.

[15] T. Basar and R. Srikant, "A Stackelberg network game with a large number of followers," *J. Optimization Theory and Applications*, vol. 115, no. 3, pp. 479–490, December 2002.

[16] J. Walrand and P. Varaiya, *High-Performance Communication Networks*, 2nd ed. Morgan Kaufmann, 2000.

[17] M. Osborne and A. Rubinstein, *Bargaining and Market*. Elsevier, 1990.

[18] M. Osborne, *Introduction to Game Theory*. MIT Press, 1995.

[19] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed. Society for Industrial Mathematics, 1987.

[20] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press, 1990.

[21] A. Berman and R. Plemmons, *Nonnegative Matrices in Mathematical Sciences*. SIAM, 1994.

[22] Q. Zhu and L. Pavel, "Constrained OSNR optimization in optical networks with a fictitious player," *IEEE Proc. of Broadnets*, 2007.