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Fault Tolerant Control for a Class of Hybrid Impulsive Systems

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Abstract— **This paper investigates the fault tolerant control (FTC) problem for a class of hybrid nonlinear impulsive systems. Two kinds of faults are considered: continuous faults that affect each mode, and discrete faults that affect the impulsive switching. The FTC strategy is based on the trade-off between the frequency of switching and the decreasing rate of Lyapunov functions along the solution of the system, which maintains the stability of overall hybrid impulsive systems in spite of these two kinds of faults. A numerical example is given to illustrate the design procedure.**

Index Terms— **Hybrid impulsive systems, fault tolerant control, observer, average dwell time.**

I. INTRODUCTION

Hybrid impulsive systems (HIS) represent an important type of hybrid systems that have gained much attention in engineering, where the continuous states abruptly change due to the impulse effect at each switching instant. Examples of HIS include some biological neural networks, frequency-modulated signal processes, flying object motions [1]-[3]. However, most of the results about HIS only consider full state measurements and do not involve the on-line fault diagnosis (FD) and fault tolerant control (FTC) schemes.

Faults may lead to unacceptable system behaviors. FD is concerned, while FTC aims at guaranteeing the system goal to be achieved in spite of faults [4]-[5]. Two main kinds of faults have been defined for hybrid systems in [6]: *continuous faults* corrupt the equality constraints of the related mode, and *discrete faults* affect the switching. For the case that only partial state measurements of hybrid systems are available, the observer design is also a challenge. Until now, only a few results have been reported about observer-based FTC for non-impulsive hybrid systems [7]-[10].

In this paper, we focus on the FTC problem for hybrid nonlinear impulsive systems with both continuous and discrete faults, and without full state measurements. An observer-based FTC law is designed for each mode, and two consequent cases are considered. For the case that each mode is input to state stable (ISS) w.r.t. the estimation error as the input, an average dwell time (a.d.t.) [11] scheme is proposed such that the ISS property of the HIS is maintained in spite of faults and impulse effects. This makes the continuous states always bounded. For the case that only partial modes are ISS under the FTC law, a novel *double a.d.t. scheme* is developed to keep the overall system still ISS.

To the best of our knowledge, no ISS analysis has been reported about HIS with faults. The novelty of our approach is to stabilize the faulty HIS in the sense of ISS in general situations:

1) where all modes are individually ISS under the FTC law.
2) where some modes are ISS, and others may be not due to the fault. The individual ISS of each mode as in [13] and [15] is not necessary here.
3) without the restriction on the decay rate of the impulsive dynamics as in [1], [3], [12].

The rest of this paper is organized as follows: Section II gives some preliminaries. Section III discusses the FTC for single mode. In section IV, FTC for overall HIS is intensively analyzed. An example is given to illustrate the theoretical results in Section V, followed by some concluding remarks in Section VI.

II. PRELIMINARIES

Let $\Re$ denote the field of real numbers, $\Re^r$ the r-dimensional real vector space. $|\cdot|$ the Euclidean norm. $\|\cdot\|_{[a,b]}$ the supremum norm of a signal on the time interval $[a,b]$. Class $\mathcal{K}$ is a class of strictly increasing and continuous functions $[0,\infty) \to [0,\infty)$ which are zero at zero. Class $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ consisting of all those functions that are unbounded. $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ belongs to class $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximal and minimal eigenvalues respectively. t^- denotes the left limit time instant of t. $(\cdot)^\top$ is the transposition.

The HIS that we consider takes the form

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + G_{\sigma(t)}(x(t))\theta_{\sigma(t)}(t) + B_{\sigma(t)}u_{\sigma(t)}(t) \\ y(t) = C_{\sigma(t)}x(t), \quad t \neq t_k, k \in \{1,2,...\} \end{cases} \tag{1}$$

$$\begin{cases} x(t) = f_{\sigma(t^-),\sigma(t)}\Big(x(t^-), u_{\sigma(t^-)}(t^-), \theta^d_{\sigma(t^-),\sigma(t)}(x(t^-))\Big) \\ y(t) = C_{\sigma(t)}x(t), \quad t = t_k, k \in \{1,2,...\} \end{cases} \tag{2}$$

where $x(t) \in \Re^n$ is the non measured state which is continuous between impulses. $y(t) \in \Re^r$ is the output, $u_\sigma(t) \in \Re^m$ is the control. A_σ, B_σ and C_σ are real constant matrices of appropriate dimensions. (A_σ, B_σ) is controllable, (A_σ, C_σ) is observable. $\theta_\sigma \in \Re^j$ is a bounded parameter, $|\theta_\sigma| \leq \bar{\theta}_\sigma$ for $\bar{\theta}_\sigma > 0$. In the fault-free case,

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we have $\theta_\sigma = \theta_{H\sigma}$ with $\theta_{H\sigma}$ a known constant vector. The nonlinear term $G_\sigma(x)$ is a continuous Lipschitz function, i.e., $|G_\sigma(x_1) - G_\sigma(x_2)| \le L_\sigma |x_1 - x_2|$ for $L_\sigma > 0$. It is assumed that $G_\sigma(0) = 0$, and $|G_\sigma(x)| \le \bar{g}_\sigma$ for $\bar{g}_\sigma > 0$.

The *continuous fault* changes the parameter θ_σ unexpectedly as in [17]. In the faulty case, $\theta_\sigma = \theta_{H\sigma} + \theta_{f\sigma}$, where $\theta_{f\sigma}$ denotes the unknown constant fault vector, $|\theta_{f\sigma}| \le \bar{\theta}_{f\sigma}$, for $\bar{\theta}_{f\sigma} > 0$.

Define $\mathcal{M} = \{1, 2, \ldots, N\}$, where N is the number of modes. $\sigma(t) : [0, \infty) \to \mathcal{M}$ denotes the piecewise constant switching function [1]. At the kth switching instant t_k, the system (1) switches from mode i to mode j, where $i = \sigma(t), \forall t \in [t_{k-1}, t_k)$ and $j = \sigma(t), \forall t \in [t_k, t_{k+1})$.

The impulsive dynamics (2) is activated at each t_k. *The discrete fault* is considered as an abnormal impulse effect, which is represented by the unknown function $\theta^d_{\sigma(t^-),\sigma(t)}(x(t^-))$, and does not exist in the fault-free case.

There are quite a few practical systems that can be described by the HIS model (1)-(2), e.g., the biped walking robot [16], etc.

The objective of this work is to *design the FTC law u_σ and provide a sufficient condition on the switching frequency of σ such that the state x is always bounded in spite of faults and impulse effects.*

III. FTC FOR SINGLE MODE

In this section, we design the controller $u_{\sigma(t)}$ such that mode $\sigma(t)$ is stabilized in spite of continuous fault $\theta_{f\sigma}$.

A. Observer design

Consider the continuous mode of system (1) with $\sigma(t) = j$ for some $j \in \mathcal{M}$ starting from $t = t_k$.

$$\dot{x}(t) = A_j x(t) + G_j(x(t))\theta_j + B_j u_j(t) \tag{3}$$
$$y(t) = C_j x(t) \tag{4}$$

The work of observer design for system (3) and (4) is not only to estimate x, but also to provide the fault estimates for the fault tolerant controller design as shown later.

Assumption 1 : There exist two constant matrices $E_j, K_j \in \Re^{n\times r}$ such that $G_j(x) = E_j \bar{G}_j(x)$ and $C_j[sI - (A_j - K_jC_j)]^{-1}E_j$ is strictly positive real (SPR).

The SPR requirement is equivalent to the following: For a given matrix $Q_j \in \Re^{n\times n} > 0$, there exist a matrix $P_j \in \Re^{n\times n} > 0$ and scalar R_j such that

$$(A_j - K_jC_j)^\top P_j + P_j(A_j - K_jC_j) = -Q_j$$
$$P_jE_j = C_j^\top R_j$$

The *fault diagnosis observer* for mode j is designed as

$$\dot{\hat{x}} = A_j\hat{x} + G_j(\hat{x})\hat{\theta}_j + B_ju_j + K_j(y - \hat{y}) \tag{5}$$
$$\hat{y} = C_j\hat{x} \tag{6}$$
$$\dot{\hat{\theta}}_j = \Gamma_j G_j^\top(\hat{x}) R_j(y - \hat{y}) \tag{7}$$

where $\hat{x}, \hat{\theta}_j, \hat{y}$ are the estimates of x, θ_j, y. The weighting matrix $\Gamma_j = \Gamma_j^\top > 0$.

Remark 1 : We neither care about when the fault occurs nor design a so-called *detection observer* as in [5] and [17] to detect the fault. This fault diagnosis observer always works no matter the mode j is faulty or not (i.e., the normal condition can be treated as a special faulty case where $\theta_j = \theta_{Hj}$). ♢

Denote $e_x = x - \hat{x}$, $e_y = y - \hat{y}$, $e_\theta = \theta_j - \hat{\theta}_j$, we have the following lemma:

Lemma 1 : Under Assumption 1, the observer described by (5)-(7) can realize $\lim_{t\to\infty} e_x = 0$ and $\lim_{t\to\infty} e_\theta = 0$ if there exist two positive constants ϱ and t_0 such that for all t, the following persistent excitation condition holds:

$$\int_t^{t+t_0} \bar{G}_j^\top(x(s))\bar{G}_j(x(s))ds \ge \varrho I \tag{8}$$

Proof: The proof is similar to [17], which is omitted. □

Lemma 1 means that the observer (5)-(7) provides both the continuous state estimates $\hat{x}$ and the fault estimates $\hat{\theta}_j$, which will be used for controller design in the next section.

B. Fault tolerant Controller

Definition 1 [18]: A system $\dot{x} = f(x, u)$ is said to be *input-to-state stable* (ISS) w.r.t the input u if there exist functions $\beta \in \mathcal{KL}$, $\alpha, \gamma \in \mathcal{K}_\infty$ such that for any initial $x(0)$, we have

$$\alpha(|x(t)|) \le \beta(|x(0)|, t) + \gamma(\|u\|_{[0,t)}), \quad \forall t \ge 0$$ □

The following property has been proven in [18].

Lemma 2 : If there exist $\alpha_1, \alpha_2, \alpha_3, \gamma_1 \in \mathcal{K}_\infty$, and a smooth function $V : \Re^n \to \Re_{\ge 0}$ such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{9}$$
$$\dot{V}(x) \le -\alpha_3(|x|) + \gamma_1(|u|) \tag{10}$$

Then the system $\dot{x} = f(x, u)$ is ISS w.r.t. u. □

Recall that (A_j, B_j) is controllable. Let $W_j = W_j^T > 0$ be associated with a given symmetric positive definite matrix H_j by the Riccati equation

$$A_j^T H_j + H_jA_j - 2H_jB_jB_j^T H_j + W_j = 0 \tag{11}$$

Note that $G_j(x)$ satisfies the Lipschitz condition $|G_j(x)| \le L_j|x|$. It has been shown in [3] that there exists a positive number η_j such that

$$\theta_{Hj}^\top G_j^\top(x)H_jx \le \eta_j x^\top H_j x \tag{12}$$

The design of the proposed fault-tolerant controller makes use of the following assumption.

Assumption 2 : $rank(B_j, E_j) = rank(B_j)$, which is equivalent to the existence of B_j^* such that $(I - B_jB_j^*)E_j = 0$.

The fault-tolerant controller is constructed as

$$u_j(\hat{x}) = -B_j^T H_j\hat{x} - B_j^* E_j \bar{G}_j(\hat{x})(\hat{\theta}_j - \theta_{Hj}) \tag{13}$$

Theorem 1 : Suppose that Assumptions 1-2 are satisfied, under the feedback controller (13), mode j in (3)-(4) is ISS w.r.t. e_x and e_θ, if

$$-\lambda_{\min}(W_j) + \eta_j|H_j| < 0 \tag{14}$$

Proof (sketch): Applying the control (13) to (3) results in the closed-loop dynamics

$$\dot{x} = (A_j - B_jB_j^TH_j)x + B_jB_j^TH_je_x + G(x)_j\theta_{Hj} + E_j\Big(\bar{G}_j(x)\theta_{fj} - \bar{G}_j(\hat{x})\hat{\theta}_{fj}\Big) \quad (15)$$

where $\hat{\theta}_{fj} \triangleq \hat{\theta}_j - \theta_{Hj}$. Consider a Lyapunov candidate $V_j(x) = x^TH_jx$, where $H_j > 0$ is defined by (11). Its derivative along the system (15) is

$$\dot{V}_j = -x^\top W_jx + 2x^\top H_jB_jB_j{}^\top H_je_x + 2x^\top H_jG(x)_j\theta_{Hj} + 2x^\top H_jE_j(\bar{G}_j(x)\theta_{fj} - \bar{G}_j(\hat{x})\hat{\theta}_{fj}) \quad (16)$$

Substituting (12) into (16) yields

$$\begin{aligned}\dot{V}_j \;\le\;& (-\lambda_{\min}(W_j) + \eta_j|H_j| + \epsilon_1 + \epsilon_2 + \epsilon_3)|x|^2 \\ &+ \Big(\frac{|H_jB_jB_j^TH_j|^2}{\epsilon_1} + \frac{|H_jE_jL_j|^2\bar{\theta}_{fj}^2}{\epsilon_2}\Big)|e_x|^2 \\ &+ \frac{|H_jE_j|^2\bar{g}_j^2}{\epsilon_3}|e_\theta|^2 \end{aligned} \quad (17)$$

where $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, $\bar{\theta}_{fj}$ and $\bar{g}_j$ denote the norm bounds of θ_{fj} and G_j defined in Section 2. Under the condition (14), $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ can be chosen small enough such that V_j satisfies (9) and (10), from Lemma 2, the result follows. □

If we could choose H_j and W_j such that (14) is satisfied, then each single mode is ISS w.r.t e_x and e_θ in spite of continuous faults, which, together with Lemma 1, implies that x converges to zero.

IV. FTC FOR HYBRID IMPULSIVE SYSTEMS

In this section, we first consider that all modes are ISS w.r.t. e_x and e_θ, then extend the result to the case that some modes may be not stabilized in the sense of ISS, because (14) does not hold. We will show that under some switching conditions, it is not necessary to design the stabilizing controller for each faulty mode. The stability of the overall HIS are still guaranteed.

A. All ISS modes

Consider the hybrid impulsive system (1), since all modes are ISS, it can be obtained from Theorem 1 that there exist continuously differentiable functions $V_k : \Re^n \to \Re_{\ge 0}$, $k \in \mathcal{M}$ and $\bar{\gamma}_1(\cdot)$, $\bar{\gamma}_2(\cdot) \in \mathcal{K}_\infty$, such that $\forall p \in \mathcal{M}$

$$\bar{\alpha}_1|x|^2 \;\le\; V_p(x) \le \bar{\alpha}_2|x|^2 \quad (18)$$

$$\dot{V}_p(x) \;\le\; -\lambda_0V_p(x) + \bar{\gamma}_1(|e_x|) + \bar{\gamma}_2(|e_\theta|) \quad (19)$$

where constants $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\lambda_0 > 0$.

Assumption 3 : There exist two known numbers $\xi_1, \xi_2 \ge 0$ such that the impulsive dynamic of (1) with discrete faults satisfies

$$|x(t_k)| \le \xi_1|x(t_k^-)| + \xi_2|e_x(t_k^-)|, \quad k \in \{1, 2, ...\} \quad (20)$$

Remark 2 : Assumption 3 is a mild condition due to the following aspects: 1) Since the impulsive dynamics includes x and $\hat{x}$, the discrete fault is also a function of x, the form of (20) appears naturally for the norm bound of $x(t_k)$. 2) The magnitudes of ξ_1 and ξ_2 are not restricted, and can be taken arbitrarily large. 3) Inequality (20) does not restrict the decay rate of the impulsive dynamics as in [1], [12], and has no relation with the continuous dynamics. ◊

Definition 2 [11]: Let $N_\sigma(T, t)$ denote the number of switchings of σ over the interval (t, T), if there exists a positive number τ_a such that

$$N_\sigma(T, t) \le N_0 + \frac{T - t}{\tau_a}, \quad \forall T \ge t \ge 0 \quad (21)$$

where $N_0 > 0$ denotes the chattering bound, then the positive constant τ_a is called *average dwell time*(a.d.t.) of σ over (t, T). □

Definition 2 means that there may exist some switchings separated by less than τ_a, but the average dwell period among switchings is not less than τ_a.

The observer-based method in Section 3 is modified for the overall system as follows:

S1: The fault diagnosis observer (5)-(6) and the controller (13) are switched according to the current mode at each switching instant t_k.

S2: The initial observer state of the current mode is chosen as the previous value $\hat{x}(t_k^-)$. The parameter estimates $\hat{\theta}_\sigma(t_k)$ are set to $\theta_{H\sigma(t_k)}$ at switching instant t_k.

We also impose another assumption:

Assumption 4 : $e_x(t_k)$ is bounded at each t_k, $k \in \{1, 2, ...\}$.

Assumption 4 is not hard to be satisfied, since Lemma 1 ensures the asymptotical stability of e_x in mode $\sigma(t_k)$ with any initial $e_x(t_k)$. Also, the impulse effect is bounded.

The following theorem provides an a.d.t. scheme such that the HIS is ISS in spite of faults.

Theorem 2 : Consider the HIS (1)-(2) that satisfies Assumption 3, and all modes are ISS w.r.t. $e_x(t)$, $e_\theta(t)$. The HIS is ISS w.r.t. $e_x(t)$, $e_\theta(t)$ in spite of any fault and any large impulse effect if the switching function σ has an a.d.t. τ_a such that

$$\tau_a > \frac{\ln \varpi}{\lambda_0} \quad (22)$$

where $\varpi \triangleq \frac{2\bar{\alpha}_2\xi_1^2}{\bar{\alpha}_1}$ and $\varpi \ge 1$.

Proof (sketch): We adopt the notations similar to that in [15]. Define $G_a^b(\lambda) = \int_a^b e^{\lambda s}\Phi ds$, where $\Phi \triangleq \bar{\gamma}_1(|(e_x|) + \bar{\gamma}_2(|(e_\theta|)$. Let $T > 0$ be an arbitrary time, denote by $t_1, \ldots, t_{N_\sigma(T,0)}$ the switching instants on the interval $(0, T)$, where $N_\sigma(T, 0)$ is defined in (21). Consider the function

$$W(s) \triangleq e^{\lambda_0 s}V_{\sigma(s)}(x(s)) \quad (23)$$

Since $\sigma(s)$ is constant on each interval $s \in [t_k, t_{k+1})$, from (19), we have $\dot{W}(s) \le e^{\lambda_0 s}\Phi, \forall s \in [t_k, t_{k+1})$. Integrating both sides of the foregoing inequality from t_k to t_{k+1}^-, we obtain

$$W(t_{k+1}^-) \le W(t_k) + G_{t_k}^{t_{k+1}^-}(\lambda_0) \quad (24)$$

Suppose $\sigma(t) = j, t \in [t_k, t_{k+1})$, and $\sigma(t) = i, t \in [t_{k-1}, t_k)$, we have

$$W(t_k) = e^{\lambda_0 t_k}V_j(x(t_k))$$

$$W(t_k^-) = e^{\lambda_0 t_k}V_i(x(t_k^-))$$

From (18) and (20), it follows that

$$V_j(x(t_k)) \le \bar{\alpha}_2|x(t_k)|^2 \le 2\bar{\alpha}_2\xi_1^2|x(t_k^-)|^2 + 2\bar{\alpha}_2\xi_2^2|e_x(t_k^-)|^2 \\ \le \varpi V_i(x(t_k^-)) + 2\bar{\alpha}_2\xi_2^2|e_x(t_k^-)|^2 \quad (25)$$

Define $\chi_k \triangleq 2e^{\lambda_0 t_k}\bar{\alpha}_2\xi_2^2|e_x(t_k^-)|^2$. Substituting (25) into (24), together with (23) leads to

$$W(T^-) \le \varpi^{N_\sigma}W(0) + \sum_{i=1}^{N_\sigma}\left(\varpi^{N_\sigma - i}\chi_i\right) \\ + \sum_{j=0}^{N_\sigma}\left(\varpi^{N_\sigma - j}G_{t_j}^{t_{j+1}^-}(\lambda_0)\right) \quad (26)$$

where $t_0 = 0$, $t_{N_\sigma(T,0)+1} = T$, and $N_\sigma \triangleq N_\sigma(T,0)$. Pick $\lambda \in (0, \lambda_0 - \frac{\ln\varpi}{\tau_a})$, we have $\tau_a \ge \frac{\ln\varpi}{(\lambda_0 - \lambda)}$. Based on (21), we have

$$\varpi^{N_\sigma - j} \le \varpi^{N_0 + \frac{T}{\tau_a} - j + 1 - 1} \\ \le \varpi^{1+N_0}e^{(\lambda_0-\lambda)(T - t_{j+1})} \quad (27)$$

and

$$G_{t_j}^{t_{j+1}^-}(\lambda_0) = \int_{t_j}^{t_{j+1}^-} e^{\lambda_0 s}\Phi ds \le e^{(\lambda_0 - \lambda)t_{j+1}}G_{t_j}^{t_{j+1}^-}(\lambda) \quad (28)$$

Substituting (27), (28) into (26) yields

$$W(T^-) \le \varpi^{1+N_0}e^{-\lambda T}\Big(e^{\lambda_0 T}W(0) + \sum_{j=0}^{N_\sigma(T,0)} e^{\lambda_0 T}G_{t_j}^{t_{j+1}^-}(\lambda)\Big) \\ + \sum_{i=1}^{N_\sigma}\left(2e^{\lambda_0 T}\bar{\alpha}_2\xi_2^2\varpi^{N_0}|e_x(t_i^-)|^2\right)$$

Note that there exists a function $\bar{\gamma}_3 \in \mathcal{K}_\infty$ such that

$$\bar{\gamma}_3(\|e_x(t_i^-)\|_{[t_1, t_{N_\sigma}]}) = \sum_{i=1}^{N_\sigma}\left(2\bar{\alpha}_2\xi_2^2\varpi^{N_0}|e_x(t_i^-)|^2\right)$$

It follows that

$$\bar{\alpha}_1|x(T)|^2 \le \varpi^{1+N_0}e^{-\lambda T}(\bar{\alpha}_2|x(0)|^2 + G_0^T(\lambda)) \\ + \bar{\gamma}_3(\|e_x(t_i^-)\|_{[t_1, t_{N_\sigma}]}) \\ \le \beta_a(|x(0)|, t) + \gamma_{ex}(\|e_x\|_{[0,T)}) + \gamma_{e\theta}(\|e_\theta\|_{[0,T)})$$

where $\beta_a \in \mathcal{KL}$, γ_{ex}, $\gamma_{e\theta} \in \mathcal{K}_\infty$. □

Roughly speaking, Theorem 2 shows that, under a low switching frequency, the overall HIS is ISS w.r.t. e_x, e_θ. This result, together with S1-S2 and Assumption 4, guarantees the global boundness of x in spite of faults and impulse effects.

Remark 3 : The global convergence of x to zero can be achieved if the estimation errors also converge to zero globally, which is satisfied under more restrict conditions. Some related work can be seen in [10]. ♢

Remark 4 : The discrete fault is hard to be detected since it appears and vanishes instantly, unless the impulsive dynamics satisfies some special structures such that the fault can be detected rapidly from outputs as in [9]. Theorem 2 shows that the discrete fault detection and diagnosis is not necessary to keep the HIS stable. ♢

Remark 5 : Note that if $\varpi \le 1$, i.e., the impulsive dynamics decreases the norm bound of x, then the HIS can switch at any time without affecting the ISS, due to the fact that $N_\sigma \to \infty \Rightarrow \varpi_\sigma^N \to 0$. This property is unavailable for general non-impulsive hybrid systems [15]. ♢

B. Partial ISS modes

Now consider the case that some modes are ISS while others may be not. Define two subsets of $\mathcal{M}$ as $\mathcal{M} = \mathcal{M}_s \cup \mathcal{M}_{us}$, where $\mathcal{M}_s$ ($\mathcal{M}_{us}$) denotes the set of modes that are (not) ISS.

The following two inequalities are considered instead of inequality (19)

$$\begin{cases} \dot{V}_p(x) \le -\lambda_0 V_p(x) + \bar{\gamma}_1(|e_x|) + \bar{\gamma}_2(|e_\theta|) & \forall p \in \mathcal{M}_s \\ \dot{V}_q(x) \le \lambda_1 V_q(x) + \bar{\gamma}_1(|e_x|) + \bar{\gamma}_2(|e_\theta|) & \forall p \in \mathcal{M}_{us} \end{cases} \quad (29)$$

where $0 < \lambda_1 \triangleq \max_{j \in \mathcal{M}_{us}}\{-\lambda_{\min}(W_j) + \eta_j|H_j|\}$. In this case, the continuous flow in mode $p \in \mathcal{M}_{us}$ can potentially destroy ISS.

Define T_s (T_{us}) the dwell period of ISS (non-ISS) modes in $[t, T)$. Then we define the *double a.d.t.* as follows, which generalizes Definition 2 and provides two a.d.t. scales for the HIS with both ISS and non-ISS modes.

Definition 3 : Let $N_\sigma^s(T,t)$ ($N_\sigma^{us}(T,t)$) denote the number of switchings of σ during the period T_s (T_{us}), if there exists two positive numbers τ_s and τ_{us} such that

$$N_\sigma^s(T,t) \le N_0 + \frac{T_s}{\tau_s}, \quad N_\sigma^{us}(T,t) \le N_0 + \frac{T_{us}}{\tau_{us}}, \quad \forall T \ge t \ge 0 \quad (30)$$

where $N_0 > 0$, then τ_s and τ_{us} are called *double a.d.t.* of σ over (t, T). □

Definition 3 generalizes Definition 2 and provides two a.d.t. scales for the HIS with both ISS and non-ISS modes.

Consider the time interval $[0, T)$ for $T > 0$, for the sake of simplicity, in the following, we divide $[0, T) = [0, T_c^-) \cup [T_c, T)$ and focus on two cases: Case 1, $T_{us} = T_c, T_s = (T - T_c)$, i.e., non-ISS modes work in $[0, T_c^-)$ and ISS ones work in $[T_c, T)$. Case 2, $T_s = T_c, T_{us} = T - T_c$, i.e., ISS modes work in $[0, T_c^-)$ and non-ISS ones work in $[T_c, T)$. The results can be extended to the more general case. It is still assumed that $\varpi \ge 1$.

Theorem 3 : Consider the HIS (1)-(2) that satisfies Assumption 3, the ISS and non-ISS modes work respectively in $[0, T_c^-)$ and $[T_c, T)$. The HIS is ISS w.r.t. $e_x(t)$, $e_\theta(t)$ in spite of any fault and any large impulse effect if the switching function σ has the double a.d.t. τ_s, τ_{us} such that

$$\lambda_0\tau_s > \ln\varpi, \quad T_{us} = T_c,\ T_s = (T - T_c) > 0 \quad (31)$$

$$\lambda_0\tau_s > \max\left\{\ln\varpi, \ln\varpi\frac{T_{us}}{\tau_{us}} + \lambda_1 T_{us}\right\} \\ T_s = T_c > 0,\ T_{us} = T - T_c \quad (32)$$

where $T > 0$ is an arbitrary time.

Before proving Theorem 3, we provide some insight into the condition (31)-(32): If the HIS is ended at the ISS mode, then (31) is equivalent to (22) in Theorem 1. If the HIS is ended at the non-ISS mode, then

- The larger (smaller) λ_1 is, the longer (shorter) a.d.t. of ISS modes is needed.
- The larger (smaller) λ_0 is, the shorter (longer) a.d.t. of ISS modes is needed.
- With the frequent switching of non-ISS modes, long a.d.t. of ISS modes is needed.
- With the long dwell period of non-ISS modes, long a.d.t. of ISS modes is needed.

Proof of Theorem 3 (sketch): Modify the function $W(s)$ as

$$W(s) = \begin{cases} e^{\lambda_0 s} V_{\sigma(s)}(x(s)) & \forall \sigma(s) \in \mathcal{M}_s \\ e^{-\lambda_1 s} V_{\sigma(s)}(x(s)) & \forall \sigma(s) \in \mathcal{M}_{us} \end{cases} \tag{33}$$

Then we have $\dot{W}(s) \leq e^{\lambda_0 s}\Phi, \forall s \in T_s$, and $\dot{W}(s) \leq e^{-\lambda_1 s}\Phi, \forall s \in T_{us}$. Denote by $t_1^{us}, \dots, t_{N_\sigma^{us}}^{us}$, and $t_1^s, \dots, t_{N_\sigma^s}^s$ the switching instants on the interval T_{us} and T_s respectively.

Case 1 : $T_{us} = [0, T_c^-), T_s = [T_c, T)$.

We first consider the time interval $[T_c, T)$, following the results of Theorem 2 and (33), one has

$$\begin{aligned} W(T^-) \leq\ & \varpi^{N_\sigma^s} e^{(\lambda_1+\lambda_0)T_c^-} W(T_c^-) + \sum_{i=1}^{N_\sigma^s} \Big(\varpi^{N_\sigma^s - i} \chi_i^s\Big) \\ & + \sum_{j=1}^{N_\sigma^s} \Big(\varpi^{N_\sigma^s - j} G_{t_j^s}^{t_{j+1}^{s-}}(\lambda_0)\Big) \end{aligned} \tag{34}$$

where $\chi_k^s \triangleq 2e^{\lambda_0 t_k^s} \bar{\alpha}_2 \xi_2^2 |e_x(t_k^{s-})|^2$, $t_{N_\sigma+1}^s = T$. We further obtain

$$\begin{aligned} W(T_c^-) \leq\ & \varpi^{N_\sigma^{us}} W(0) + \sum_{i=1}^{N_\sigma^{us}} \Big(\varpi^{N_\sigma^{us} - i} \chi_i^{us}\Big) \\ & + \sum_{j=0}^{N_\sigma^{us}} \Big(\varpi^{N_\sigma^{us} - j} G_{t_j}^{t_{j+1}^-}(-\lambda_1)\Big) \end{aligned} \tag{35}$$

where $\chi_k^{us} \triangleq 2e^{-\lambda_1 t_k^{us}} \bar{\alpha}_2 \xi_2^2 |e_x(t_k^{us-})|^2$, Combining (34) and (35) leads to

$$\begin{aligned} W(T^-) \leq\ & \varpi^{N_\sigma^s + N_\sigma^{us}} e^{(\lambda_1+\lambda_0)T_c^-} W(0) \\ & + e^{(\lambda_1+\lambda_0)T_c^-} \sum_{i=1}^{N_\sigma^{us}} \Big(\varpi^{N_\sigma^{us} + N_\sigma^s - i} \chi_i^{us}\Big) \\ & + e^{(\lambda_1+\lambda_0)T_c^-} \sum_{j=0}^{N_\sigma^{us}} \Big(\varpi^{N_\sigma^{us} + N_\sigma^s - j} G_{t_j^{us}}^{t_{j+1}^{us-}}(-\lambda_1)\Big) \\ & + \sum_{i=1}^{N_\sigma^s} \Big(\varpi^{N_\sigma^s - i} \chi_i^s\Big) + \sum_{j=1}^{N_\sigma^s} \Big(\varpi^{N_\sigma^s - j} G_{t_j^s}^{t_{j+1}^{s-}}(\lambda_0)\Big) \end{aligned} \tag{36}$$

From the condition (31), choose a number $\lambda < \lambda_0 - \frac{\ln \varpi}{\tau_s}$, one has the following inequalities

$$\begin{aligned} \varpi^{N_\sigma^s + N_\sigma^{us}} e^{(\lambda_1+\lambda_0)T_c^-} \leq\ & \varpi^{2N_0} e^{(\lambda_0 - \lambda)T} e^{\ln \varpi \frac{T_c}{\tau_{us}} + (\lambda_1+\lambda_0)T_c} \\ \leq\ & \varpi^{2N_0} \Delta(\tau_{us}, T_c) e^{(\lambda_0 - \lambda)T} \end{aligned} \tag{37}$$

where $\Delta(\tau_{us}, T_c) \triangleq e^{\ln \varpi \frac{T_c}{\tau_{us}} + (\lambda_1+\lambda_0)T_c}$ is a positive number.

$$\begin{aligned} & e^{(\lambda_1+\lambda_0)T_c^-} \varpi^{N_\sigma^{us} + N_\sigma^s - i} \chi_i^{us} \\ & \leq 2\varpi^{2N_0} \Delta(\tau_{us}, T_c) \bar{\alpha}_2 \xi_2^2 |e_x(t_i^-)|^2 e^{\lambda_0 T} \end{aligned} \tag{38}$$

$$\begin{aligned} & e^{(\lambda_1+\lambda_0)T_c^-} \varpi^{N_\sigma^{us} + N_\sigma^s - j} G_{t_j^{us}}^{t_{j+1}^{us-}}(-\lambda_1) \\ & \leq \varpi^{2N_0} \Delta(\tau_{us}, T_c) e^{(\lambda_0 - \lambda)T} G_{t_j^{us}}^{t_{j+1}^{us-}}(\lambda) \end{aligned} \tag{39}$$

Substituting (37)-(39) into (36), together with the results of Theorem 1, yields

$$\begin{aligned} \bar{\alpha}_1 |x(T)|^2 \leq\ & \varpi^{2N_0} \Delta(\tau_{us}, T_c) e^{-\lambda T} (\bar{\alpha}_2 |x(0)|^2 + G_0^T(\lambda)) \\ & + \bar{\gamma}_4 (\|e_x(t_i^-)\|_{[t_1, t_{N_\sigma}]} \end{aligned} \tag{40}$$

where the function $\bar{\gamma}_4 \in \mathcal{K}_\infty$. The ISS result can be obtained straight from Theorem 1.

Case 2 : $T_s = [0, T_c^-), T_{us} = [T_c, T)$.

Similar to (34),(35), we can obtain

$$\begin{aligned} W(T^-) \leq\ & \varpi^{N_\sigma^s + N_\sigma^{us}} e^{-(\lambda_1+\lambda_0)T_c^-} W(0) \\ & + e^{-(\lambda_1+\lambda_0)T_c^-} \sum_{i=1}^{N_\sigma^s} \Big(\varpi^{N_\sigma^{us} + N_\sigma^s - i} \chi_i\Big) \\ & + e^{(\lambda_1+\lambda_0)T_c^-} \sum_{j=0}^{N_\sigma^s} \Big(\varpi^{N_\sigma^{us} + N_\sigma^s - j} G_{t_j^s}^{t_{j+1}^{s-}}(-\lambda_1)\Big) \\ & + \sum_{i=1}^{N_\sigma^{us}} \Big(\varpi^{N_\sigma^{us} - i} \chi_i^{us}\Big) + \sum_{j=1}^{N_\sigma^{us}} \Big(\varpi^{N_\sigma^{us} - j} G_{t_j^{us}}^{t_{j+1}^{us-}}(\lambda_0)\Big) \end{aligned} \tag{41}$$

From the condition (32), choose a number λ satisfying

$$\lambda < \min\Big\{\lambda_0 - \frac{\ln \varpi}{\tau_s}, \lambda_0 - \ln \varpi \frac{T_{us}}{\tau_{us} \cdot \tau_s} - \frac{\lambda_1 T_{us}}{\tau_s}\Big\}$$

The following inequalities can be obtained

$$\varpi^{N_\sigma^s + N_\sigma^{us}} e^{-(\lambda_1+\lambda_0)T_c^-} \leq \varpi^{2N_0+1} e^{-\lambda_1 T} e^{-\lambda \tau_s} \tag{42}$$

Since $\lambda > 0$, there exists a $\lambda^* > 0$ such that $\lambda^* T = \lambda \tau_s$.

$$\begin{aligned} & e^{-(\lambda_1+\lambda_0)T_c^-} \varpi^{N_\sigma^{us} - i} \chi_i^{us} \\ & \leq 2\varpi^{2N_0} \bar{\alpha}_2 \xi_2^2 |e_x(t_i^-)|^2 e^{\lambda_0 \tau_s} e^{-\lambda_1 T} \end{aligned} \tag{43}$$

$$\begin{aligned} & e^{-(\lambda_1+\lambda_0)T_c^-} \varpi^{N_\sigma^{us} - j} G_{t_j^{us}}^{t_{j+1}^{us-}}(-\lambda_1) \\ & \leq \varpi^{2N_0} e^{\lambda_0 \tau_s} e^{-\lambda^* T} e^{-\lambda_1 T} G_{t_j^{us}}^{t_{j+1}^{us-}}(\lambda) \end{aligned} \tag{44}$$

Substituting (42)-(44) into (41), together with the results of Theorem 1 and Case 1, yields

$$\begin{aligned} \bar{\alpha}_1 |x(T)|^2 \leq\ & \varpi^{2N_0+1} e^{-\lambda^* T} \bar{\alpha}_2 |x(0)|^2 \\ & + \bar{\gamma}_5(\|e_x\|_{[0,T)}) + \bar{\gamma}_6(\|e_\theta\|_{[0,T)}) \end{aligned} \tag{45}$$

where the function $\bar{\gamma}_5, \bar{\gamma}_6 \in \mathcal{K}_\infty$. □

Theorem 3 relaxes the condition that all modes are required to be made ISS, the overall HIS in the presence of faults can still be ISS with partial ISS modes. This result is very useful for stabilization of HIS and non-impulsive hybrid sytems with unstable modes due to faults.

Remark 6 : Consider the worst case that no mode is ISS. Three alterative methods could be applied: 1) Redesign the continuous controller u_σ guaranteeing that each mode is ISS; 2) Impose some conditions on the decay rate of impulsive dynamics; 3) Apply the so-called *impulsive controller* at each switching instants. ◊

V. AN EXAMPLE

An example borrowed from [1] is given to illustrate the theoretical results. Consider a HIS with two modes as

$$\text{mode 1: } \begin{cases} \dot{x}_1 = \frac{1}{8}x_1 - x_2 \\ \dot{x}_2 = x_1 + \frac{1}{8}x_2 + (\sin^2 x_1 + \sin x_1)\theta_1 + u_1 \\ y = x_1 - x_2 \end{cases}$$

$$\text{mode 2: } \begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = x_1 - 3x_2 + (\sin^2 x_1)\theta_2 + u_2 \\ y = x_1 - x_2 \end{cases}$$

$$f_{1,2}: \begin{cases} x_1 = \frac{2}{3}x_1 + \theta^d_{1,2}(x) \\ x_2 = \frac{1}{3}x_1 + \frac{2}{3}x_2 \end{cases}, \quad f_{2,1}: \begin{cases} \dot{x}_1 = x_1 + \theta^d_{2,1}(x) \\ \dot{x}_2 = \frac{1}{2}x_1 + x_2 \end{cases}$$

where $\theta_{H1} = \frac{1}{8}$, $\theta_{H2} = 1$, the bound of faulty parameters are assumed $\bar{\theta}_{f1} = \frac{1}{8}$, $\bar{\theta}_{f2} = 1$, and $\bar{\theta}_1 = \frac{1}{4}$, $\bar{\theta}_2 = 2$. It can be seen that $E_1 = E_2 = [0 \quad 1]^\top$, $L_1 = 3$, $L_2 = 2$, $\bar{g}_1 = 2$, $\bar{g}_2 = 1$.

As for mode 1, the matrix K_1 and Q_1 are chosen as $K_1 = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$, $Q_1 = \begin{bmatrix} 0.9993 & -0.5788 \\ -0.5788 & 1.9412 \end{bmatrix}$, we can obtain $R_1 = -0.3376$ and $P_1 = \begin{bmatrix} 1.3564 & -0.3376 \\ -0.3376 & 0.3376 \end{bmatrix}$. Note that Assumption 1 holds, which implies that the fault diagnosis observer works well.

On the other hand, by choosing $W_1 = I_{2\times 2}$, we obtain the matrix H_1 from (11) as $H_1 = \begin{bmatrix} 2.0048 & -0.5003 \\ -0.5003 & 1.0646 \end{bmatrix}$. Simple calculation leads to that $\eta_1 = 0.6366$ in (12), it can be checked that $-\lambda_{\min}(W_1) + \eta_1|H_1| = 0.5136$, which means mode 1 is not ISS.

As for mode 2, K_2 and Q_2 are chosen as $K_2 = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 10.8180 & -0.7843 \\ -0.7843 & 1.0912 \end{bmatrix}$, one has $R_2 = -0.0341$ and $P_2 = \begin{bmatrix} 1.7348 & -0.0341 \\ -0.0341 & 0.0682 \end{bmatrix}$. Assumption 1 also holds.

By choosing $W_2 = I_{2\times 2}$, we obtain $H_2 = \begin{bmatrix} 0.1350 & 0.0417 \\ 0.0417 & 0.1708 \end{bmatrix}$, and $\eta_2 = 2.8497$, it follows that $-\lambda_{\min}(W_2) + \eta_2|H_1| = -0.3572$, which implies mode 2 is ISS w.r.t. e_x, e_θ.

From above calculations, we get $\bar{\alpha}_1 = 0.1076$, $\bar{\alpha}_2 = 2.2212$ in (18), $\lambda_0 = 0.3572$, $\lambda_1 = 0.5136$ in (29).

Now consider the impulsive dynamics, assume $\theta^d_{1,2} = \frac{1}{3}x_1$, $\theta^d_{2,1} = x_1$, we have $\xi_1 = 1.8028$ in (20), it follows that $\ln \varpi = 4.8992$.

Now we illustrate the results of Theorem 3, we consider two cases: the HIS is initialized at mode 1 then switches to mode 2, and the converse. For the former case, the HIS is ended at ISS mode 2, from the condition (31), if the dwell time of mode 2 is larger than $\frac{\ln \varpi}{\lambda_0} = 13.6876s$, then HIS is ISS w.r.t. e_x, e_θ. For the latter case, the HIS is ended at non-ISS mode 1, provided that the dwell time of mode 1 is $10s$, i.e., $T_{us} = 10s$, from the condition (32), if the dwell time of mode 2 is larger than $28.0751s$, then HIS is still ISS w.r.t. e_x, e_θ.

VI. CONCLUSION

In this paper, an observer-FTC method for HIS is proposed, which is based on a.d.t and guarantees that the HIS is ISS w.r.t the convergent estimation error of observer, no matter whether all modes are ISS or only partial modes are ISS. This result is useful for stabilization of HIS and non-impulsive hybrid systems with unstable submodes.

Future work will be focused on the extension of the a.d.t and double a.d.t schemes to other kind of impulsive systems with some real applications.

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