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Command Filtered Backstepping

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***Abstract*— This article presents and analyzes a novel backstepping feedback control implementation approach. In practical applications, implementation of the backstepping approach becomes increasingly complex as the state order increases. The main complicating factor is computation of the command derivatives. This article presents a filtering approach that significantly simplifies the backstepping implementation, analyzes the effect of the command filtering, and derives a compensated tracking error that retains the standard stability properties of backstepping approaches.**

Keywords: *nonlinear control, backstepping, singular perturbation*

I. Introduction

One of the most commonly used approaches for nonlinear control is the backstepping control methodology [9, 10]. A typical requirement, in tracking control for an n-th order system, is that the desired output and its first n derivatives are available for use in the implementation of the control law.

In many applications, the user input device or trajectory planner only specifies a desired output signal $x_d^o(t)$. The signal $x_d^o(t)$ is constrained to be bounded, but may contain discontinuities or other features that may not be achievable by the physical system. A standard practice in applications is to treat $x_d^o(t)$ as the input to a prefilter with state space representation:

$$\begin{aligned} \dot{z}_{i-1} &= z_i \qquad \text{for } i = 1, 2 \ldots, n-1 \\ \dot{z}_n &= -a_1(z_1 - x_d^o) - a_2 z_2 - \ldots - a_n z_n \end{aligned} \tag{1}$$

where the characteristic equation $s^n + a_n s^{n-1} + \ldots + a_1 = 0$ is selected to be stable and to specify the desired bandwidth and transient response of the system, see e.g. [16, 18, 19]. Then the desired system output is $x_d(t) = z_1(t)$. The designer of the system ensures that the user input device and the prefilter are compatible in the sense that the error $x_d(t) - x_d^o(t)$ is small (i.e., that $x_d(t)$ is an accurate approximation to $x_d^o(t)$). The first $n-1$ derivatives of x_d are the states $z_2, \ldots, z_n$ which are continuous and bounded as long as $x_d^o(t)$ is bounded. The n-th derivative of x_d is specified by eqn. (1). Then the signal $x_d(t)$ and its first n derivatives are used in the implementation of the nonlinear control law, for example, by feedback linearization [8] or backstepping [10]. Alternative means for differentiating a signal are discussesd for example in [2, 12, 23]

In the backstepping control approach, the control law is designed by using states as virtual control signals. At each step in the procedure, virtual control signals, denoted by $\bar{\alpha}_i$ in Section III, and their derivatives are required. Theoretically, calculation of the virtual control signal derivatives is simple, but it can be quite complicated and tedious in applications when n is greater than three because the control signal u will include the derivative of $\bar{\alpha}_n$, which requires the second derivative of $\bar{\alpha}_{n-1}$, which requires the third derivative of $\bar{\alpha}_{n-2}$, and so on. See e.g., eqn. (44-45) in [14] or the calculation of the signal $\dot{x}_{md}$ in eqns. (3.8) and (3.10) in [11]. In certain applications, such as induction motors, the number of backstepping iterations is small and the computation is achievable, e.g. [11]. In other applications, such as the helicopter application of [13], the analytic derivation is overly cumbersome. When the analytic derivation of the virtual control signal derivatives becomes tedious, the issue has been addressed by a variety of methods. The authors in [13] approximate the command derivatives using sliding mode filters [12]. In [21] command filters are used to generate x_d and its derivatives at the input to a backstepping control system; however, certain derivatives of the virtual control signals are neglected without additional analysis while other terms are incorporated into the function approximation process. In [20], the command derivatives are modelled portions of unknown functions that are approximated during operation.

The method described herein only requires that the signals x_d and $\dot{x}_d$ to be available as inputs to the control system. If necessary, these signals can be the outputs of a command filtering of order at least one, similar to that described in the paragraph containing eqn. (1). This article will use the command filtering idea to derive and analyze a practical extension of the backstepping approach. A main motivation for this extension is simplification of the process of determining the command derivatives required for implementation of the backstepping approach. Preliminary versions of the method presented herein were applied specifically to aircraft control in [5–7, 22] without the formal proof that is now presented herein.

The benefits of the approach presented herein include:

1) Decoupling of the design of the controllers for the backstepping iterations.
2) Avoiding the tedious algebra related to computing the command signal derivatives. This computation becomes especially burdensome for scalar backstepping

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with $n > 3$ or vector backstepping.

It is important to note that even if the designer were to derive exact analytical expressions for the command derivatives relative to the design model, these are still approximations because that model is an approximate representation of the plant. The exact command derivatives should be computed using the actual plant dynamics, which are almost never available. Therefore, the choice is not really between a correct analytic expression for or a filtered estimate of the command derivatives; instead, the choice is between two estimates of the command derivatives.

The organization of the article is as follows. Section II formally states the control problem and its related technical assumptions. Section III derives a standard backstepping controller for the problem of interest. That presentation is brief, but important, because that controller is a point of reference in Theorem 2. The command filtered back stepping approach and properties are discussed in Section IV. Section V derives dynamic equations that are require for the proofs of Theorems 1 and 2. Proofs of these theorems are presented in Section VI.

II. Problem Formulation

Consider the following class of n-th order single-input-single-output nonlinear systems

$$\dot{x}_i = f_i(w_i) + g_i(w_i)x_{i+1}, \text{ for } i = 1, \ldots, n-1 \quad (2)$$
$$\dot{x}_n = f_n(x) + g_n(x)u \quad (3)$$

where $x = [x_1, \cdots, x_n]^\top \in \Re^n$ is the state vector with initial condition $x(0) = x_0$, $w_i = [x_1, \cdots, x_i]^\top$ is introduced to simplify the notation in eqn. (2), and u is the scalar control signal. To ensure controllability, we will invoke the following assumption, which is standard in backstepping.

Assumption 1: There exists $g_o > 0$ such that for $i = 1, \ldots, n$ each function $|g_i(x)| \geq g_o$.

Since each g_i is continuous and known, Assumption 1 implies that each g_i has a constant, known sign.

Our objective is trajectory tracking. Therefore, we assume there is a desired trajectory $x_d(t) : \Re^+ \mapsto \Re$ with derivative $\dot{x}_d(t)$, each of which is available and bounded for all $t > 0$.

The objective of the control design is to specify a control signal u to steer $x_1(t)$ from any initial conditions to track the reference input $x_d(t)$ and to achieve boundedness for the states x_i for $i = 2, \ldots, n$. Note that existing approaches in the literature (e.g., [9, 10]) would require knowledge of the first n derivatives of $x_d(t)$.

III. Standard Backstepping

This section summarizes the standard backstepping design. This is necessary as the design is a point of reference in the stability proof of Section VI-B.

Define the vector of functions $\bar{\alpha} = [\bar{\alpha}_1, \ldots, \bar{\alpha}_n]^\top$ recursively:

$$\bar{\alpha}_1(w_1, x_d) = \frac{1}{g_1}\left(-k_1\bar{x}_1 + \dot{x}_d - f_1\right) \quad (4)$$
$$\bar{\alpha}_i(w_i, x_d) = \frac{1}{g_i}\left(-k_i\bar{x}_i + \dot{\bar{\alpha}}_{i-1} - f_i - g_{i-1}\bar{x}_{i-1}\right) \quad (5)$$

for $i = 2, \ldots, n$ where w_i are defined following eqn. (3) and $k_i > 0$ for $i = 1, \ldots, n$. The control variable is assigned the value $u(t) = \alpha_n(x(t), x_d(t))$. In the interest of presenting a specific formulation, in the above definition, we have cancelled the natural dynamics of the system to achieve reference input tracking. If certain nonlinearities can be considered 'beneficial', then they need not be removed.

For eqns. (4–5) to be well-defined the following technical assumption concerning $f_i(w_i),\ g_i(w_i)$, for $i = 1, \cdots, n$ must be satisfied.

Assumption 2: For each $i = 1, 2 \ldots, n$, the functions $f_i,\ g_i \in \mathcal{C}^{n-i}$ (i.e., continuously differentiable up to order $n-i$).

In addition, we require the following assumption to hold.

Assumption 3: For $t \geq 0$, for $i = 1, 2 \ldots, n-1$, the signals $x_d^{(i)}(t)$ must be continuous, bounded, and available; and, the signal $x_d^{(n)}(t)$ must be bounded and available.

The tracking error vector is defined as $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]$ with

$$\bar{x}_i = x_i - \bar{\alpha}_{i-1}$$

for $i = 1, \ldots, n$ where for convenience of notation, $\bar{\alpha}_0 = x_d$. With this change of variables, the closed-loop tracking error differential equations are

$$\dot{\bar{x}}_1 = -k_1\bar{x}_1 + g_1\bar{x}_2 \quad (6)$$
$$\dot{\bar{x}}_i = -k_i\bar{x}_i + g_i\bar{x}_{i+1} - g_{i-1}\bar{x}_{i-1} \quad (7)$$
$$\dot{\bar{x}}_n = -k_n\bar{x}_n - g_{n-1}\bar{x}_{n-1}. \quad (8)$$

for $i = 2, \ldots, n-1$ with initial conditions defined by $\bar{x}_i(0) = x_i(0) - \bar{\alpha}_{i-1}(w_i(0), x_d(0))$.

By choosing the Lyapunov function

$$V_o = \frac{1}{2}\sum_{i=1}^{n}\bar{x}_i^2$$

and taking its time derivative, further analysis shows that $\dot{V}_o \leq -\underline{k}V_o$ where $\underline{k} = min(k_i)$. Therefore, by Theorem 4.10 in [9] the origin of the tracking error system of eqns. (6-8) is exponentially stable. In addition, for $i = 1, \ldots, n$: $\bar{x}_i \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\bar{\alpha}_i$, $x_i \in \mathcal{L}_\infty$.

Eqn. (5) has a deceptively simple form. As n increases, analytic computation of $\dot{\bar{\alpha}}_i$ for $i = 1, \ldots, n-1$ becomes increasingly complicated. For $n > 3$, computation of this term can require pages of computation. In real applications, such as that considered in [13], the computation of the feedback control algorithm using the standard backstepping procedure becomes extremely tedious. Practitioners have used various ad-hoc methods to address the issue. The ad-hoc methods are particularly problematic in adaptive backstepping approaches, because the ad-hoc approaches can result in the breakdown of the proofs of certain desirable stability properties for the closed-loop adaptive approach.

Herein, we present and analyze the command filtered backstepping approach for the nonadaptive case. The analysis is rigorous and considers the affect of the approximation on the tracking errors. The extension of the command filtering approach to the adaptive is considered in [5–7, 15] with a partial stability analysis. That article rigorously considers the stability of the parameter adaptation process, but not the command filter variables. A complete analysis of the adaptive case is beyond the scope of the present paper, but will be considered in a subsequent article.

IV. Command Filtered Backstepping

Our objective in this section is to present a modification of the backstepping approach that eliminates the analytic computation of $\dot{\bar{\alpha}}_i$ for $i = 1, \ldots, n-1$, while allowing rigorous stability analysis and allowing extension to the adaptive case. Subsection IV-A defines and discusses the definition of the signals involved in the approach. Figure 1 depicts the signal flow in block diagram form. Subsection IV-B presents two theorems that summarize the properties of the approach.

A. Design Approach

For the command filtered approach, we define pseudocontrol signals α_i of the backstepping procedure as

$$\alpha_1(w_1, x_{1,c}) = \frac{1}{g_1}\big(-k_1\tilde{x}_1 + \dot{x}_{1,c} - f_1\big) \quad (9)$$

$$\alpha_i(w_i, x_{i,c}, v_{i-1}) = \frac{1}{g_i}\big(-k_i\tilde{x}_i + \dot{x}_{i,c} - f_i - g_{i-1}v_{i-1}\big) \quad (10)$$

for $i = 2, \ldots, n$ and

$$u(t) = \alpha_n(x(t), x_{n,c}(t), v_{n-1}(t)). \quad (11)$$

The control gains, k_i, $i = 1, \cdots, n$ are designer specified positive constants as in the standard approach. The tracking error vector is defined as $\tilde{x} = [\tilde{x}_1, \cdots, \tilde{x}_n]^\top$ where

$$\tilde{x}_i = x_i - x_{i,c} \quad \text{for } i = 1, \ldots, n. \quad (12)$$

The compensated tracking error signals v_i are defined as

$$v_i = \tilde{x}_i - \xi_i, \ \text{for } i = 1, \cdots, n. \quad (13)$$

The ξ_i signals for $i = 1, \ldots, n-1$ are defined as

$$\dot{\xi}_i = -k_i\xi_i + g_i\left(x_{i+1,c} - \alpha_i\right) + g_i\xi_{i+1} \quad (14)$$

with $\xi_i(0) = 0$. For $i = n$, define $\xi_n = 0$.

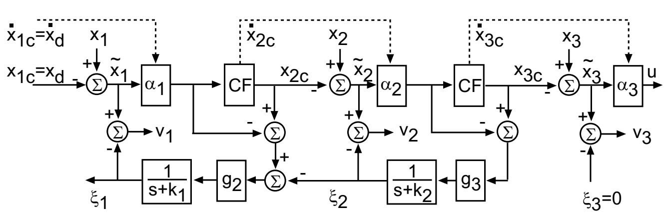

Fig. 1. Block diagram of the command filtered backstepping approach for $n = 3$. CF represents a command filter as defined in eqns. (15–16). The α_i are computed according to eqns. (9–10). The dotted lines from the command filter to the α_i represent the communication of the command derivative $\dot{x}_{i,c}$ from CF to the computation of α_i.

Eqns. (9–10) use the signals $x_{i,c}$ and $\dot{x}_{i,c}$ for $i = 1, 2\ldots, n$ that are defined in this paragraph. For $i = 1$, $x_{1,c} = x_d = \bar{\alpha}_0$ and $\dot{x}_{1,c} = \dot{x}_d = \dot{\bar{\alpha}}_0$. For $i = 1, \ldots, n-1$ define state space implementation of the command filters as

$$\dot{z}_{i,1} = \omega_n z_{i,2} \quad (15)$$

$$\dot{z}_{i,2} = -2\zeta\omega_n z_{i,2} - \omega_n\left(z_{i,1} - \alpha_i\right) \quad (16)$$

with $x_{i+1,c}(t) = z_{i,1}$ and $\dot{x}_{i+1,c}(t) = \omega_n z_{i,2}$ as the outputs of each filter. The filter initial conditions are $z_{i,1}(0) = \alpha_i(w_i(0), x_{i,c}(0), v_{i-1}(0))$ and $z_{i,2}(0) = 0$. The filter design parameters are $\omega_n > 0$ and $\zeta \in (0, 1]$. Each command filter is designed to compute $x_{i+1,c}$ and $\dot{x}_{i+1,c}$ without differentiation. The transfer functions corresponding to eqns. (15-16) are

$$\frac{\begin{bmatrix} \omega_n^2 \\ s\omega_n^2 \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2.}$$

Therefore, the natural frequency of the command filter is equal to the parameter ω_n; the filter has unit DC gain to the first output; and the second output is the derivative of the first output. The designer would typically select $\omega_n > k_{i+1}$ for all i so that $x_{i+1,c}$ and $\dot{x}_{i+1,c}$ will accurately track α_i and $\dot{\alpha}_i$, respectively. The effect of the errors $(x_{i+1,c} - \alpha_i)$ and $(\dot{x}_{i+1,c} - \dot{\alpha}_i)$ is a crucial issue to be analyzed in the stability of this approach.

B. Summary of Properties

The two theorems of this section summarize the properties of the command filtered backstepping approach. The theorems will be proved in the subsequent sections of the article.

Prior to stating the theorems we state the following assumptions.

Assumption 4: For each $i = 1, 2\ldots, n$, the following conditions hold on any compact set $D_i \subset \Re^i$

- f_i and g_i and their first partial derivatives are continuous and bounded; and,
- f_i is (locally) Lipshitz in w_i.

Assumption 5: For $t \geq 0$, for $i = 0, 1$, the signals $x_d^{(i)}(t)$ must be continuous, bounded, and available.

Whereas the backstepping approach of Section III required Assumptions 2 and 3, the command filtered approach will invoke Assumptions 2 and 5.

In comparison: for $i = n$, Assumption 4 is more stringent that Assumption 2; for $i = n-1$, Assumption 4 somewhat equivalent to Assumption 2; and, for $i = 1, 2\ldots, n-1$, Assumption 4 is less stringent that Assumption 2. Assumptions 5 is always less stringent that Assumption 3.

Theorem 1: For the system described by eqns. (2)-(3) that satisfies Assumptions 1, 4 and 5 with the feedback control law defined in eqn. (11), we have that $\tilde{x}_n$ and v_i for $i = 1, 2\ldots, n$ converge to zero exponentially. Therefore, $\tilde{x}_n$ and $v_i \in \mathcal{L}_\infty \bigcap \mathcal{L}_2$.

Theorem 1 states that the compensated tracking errors of the command filtered backstepping approach (i.e., v) have the same properties as the tracking errors of the standard backstepping approach (i.e., $\bar{x}$). The proof of Theorem 1 is in Section VI-A and mainly uses the equations derived in Section V-B. Theorem 1 leaves open the question of the properties of the signals $\tilde{x}_i$ and ξ_i for $i = 1, 2 \ldots, n-1$, which are discussed in Theorem 2. In Theorem 2, we will use the notation $y(t,\epsilon) = \mathcal{O}(\epsilon)$ which is defined as follows [9].

Definition 1: $y(t,\epsilon) = \mathcal{O}(\epsilon)$ if there exists positive constants k and c such that $|y(t,\epsilon)| \leq k|\epsilon|$, $\forall |\epsilon| < c$ and $t \geq 0$.

Theorem 2: For the system described by eqns. (2)-(3) that satisfies Assumptions 1, 2, 3, and 4 with the feedback control law defined in eqn. (11), we have the following properties:

1) $\tilde{x}(t,\epsilon) - \bar{x}(t) = \mathcal{O}(\epsilon)$;
2) $z_{i,1}(t,\epsilon) - \bar{\alpha}_i = \mathcal{O}(\epsilon)$; and,
3) $z_{i,2}(t,\epsilon) - \dot{\bar{\alpha}}_i = \mathcal{O}(\epsilon)$

for $i = 1, 2 \ldots, n-1$ where $\bar{\alpha}_i$ is defined in eqn. (5), $\bar{x}(t)$ is the standard backstepping tracking error solution to eqns. (6–8), $[z_{i,1}, z_{i,2}]^\top$ is the solution to eqns. (15–16), the notation $\tilde{x}(t,\epsilon)$ represents the tracking error of eqn. (12) for a specific choice of the command filter parameter ω_n, and $\epsilon = \frac{1}{\omega_n}$. Theorem 2 shows that, by increasing the command filter natural frequency ω_n, the solution to the command filtered backstepping closed-loop system can be made arbitrarily close to the backstepping solution that relies on analytic derivatives. The proof of Theorem 2 is in Section VI-B. The proof uses the command filtered tracking error differential equations derived in Section V-A and Tikhonov's theorem (Theorem 11.2 in [9]).

Remark 1: Because Theorem 2 compares the solutions of the command filtered and standard backstepping approaches, Assumptions 1, 2, 3, and 4 are all required for its proof. However, implementation of the command filtered backstepping controller only requires Assumptions 1, 4, and 5. In fact, Assumption 4 is stronger than necessary for implementation.

V. Error Dynamics

The stability analysis of subsequent sections will utilize the dynamics of the tracking error $\tilde{x}_i$ and the dynamics of the compensated tracking error v_i. These equations are derived in this section.

The analysis of Section VI-B will use Tikhonov's theorem which requires analysis of the dependence of the initial conditions of the system on the parameter ϵ. Therefore, we explicitly state the dependence in the following subsections.

A. Tracking Error

This subsection uses the control approach defined in Section IV to derive the differential equations for the tracking error. This analysis can be divided into three cases.

1) For $i = 1$:

$$\begin{aligned}\dot{\tilde{x}}_1 &= f_1 + g_1 x_2 - \dot{x}_{1,c} \\ &= f_1 + g_1\alpha_1 - \dot{x}_{1,c} \\ &\quad + g_1\left(x_{2,c} - \alpha_1\right) + g_1(x_2 - x_{2,c}) \\ &= -k_1\tilde{x}_1 + g_1\left(x_{2,c} - \alpha_1\right) + g_1(x_2 - x_{2,c}) \\ &= -k_1\tilde{x}_1 + g_1\left(x_{2,c} - \alpha_1\right) + g_1\tilde{x}_2. \qquad (17)\end{aligned}$$

2) Similarly, for $i = 2, 3, \ldots, n-1$:

$$\begin{aligned}\dot{\tilde{x}}_i &= f_i + g_i x_{i+1} - \dot{x}_{i,c} \\ &= f_i + g_i\alpha_i - \dot{x}_{i,c} \\ &\quad + g_i\left(x_{i+1,c} - \alpha_i\right) + g_i\left(x_{i+1} - x_{i+1,c}\right) \\ &= -k_i\tilde{x}_i - g_{i-1}v_{i-1} \\ &\quad + g_i\left(x_{i+1,c} - \alpha_i\right) + g_i\tilde{x}_{i+1}. \qquad (18)\end{aligned}$$

3) For $i = n$:

$$\begin{aligned}\dot{\tilde{x}}_n &= f_n + g_n u - \dot{x}_{n,c} \\ &= -k_n\tilde{x}_n - g_{n-1}v_{n-1}. \qquad (19)\end{aligned}$$

The initial conditions for the tracking error differential eqns. (17–19) are

$$\tilde{x}_i(0) = x_i(0) - x_{i,c}(0) \qquad (20)$$

which are independent of ϵ. The equations of this section will be used in the following subsection to derive the differential equations for the compensated tracking errors defined in (13). They will also be used in Section VI-B to analyze the error between the command filtered and standard backstepping implementations.

B. Compensated Tracking Error

The variables $\xi_i,\ i = 1, \cdots, n-1$ as defined in eqn. (14) are produced by filtering $(x_{i+1,c} - \alpha_i)$, which will be referred to as the unachieved portion of α_i. The variables v_i are referred as *compensated tracking errors* and are obtained by removing the filtered unachieved portion of α_i, as represented by ξ_i, from the tracking error (see eqn. (13)). The dynamics of the compensated tracking errors are derived below.

1) For $i = 1$

$$\begin{aligned}\dot{v}_1 &= \dot{\tilde{x}}_1 - \dot{\xi}_1 \\ &= -k_1 v_1 + g_1 v_2. \qquad (21)\end{aligned}$$

2) For $i = 2, 3, \ldots, n-1$:

$$\begin{aligned}\dot{v}_i &= \dot{\tilde{x}}_i - \dot{\xi}_i \\ &= -k_i v_i - g_{i-1}v_{i-1} + g_i v_{i+1}. \qquad (22)\end{aligned}$$

3) For $i = n$ ($v_n = \tilde{x}_n$):

$$\begin{aligned}\dot{v}_n &= \dot{\tilde{x}}_n \\ &= -k_n v_n - g_{n-1}v_{n-1}. \qquad (23)\end{aligned}$$

The initial condition for differential eqns. (21–23) are $v_i(0) = \tilde{x}_i(0)$ which are defined in eqn. (20) and are independent of ϵ.

VI. Stability Analysis

Section IV-B presented two theorems that summarized the properties of the command filtered backstepping approach. These theorems are proved in the following two subsections.

A. Proof of Theorem 1

The properties of the vector v are analyzed by considering the following Lyapunov function

$$V = \sum_{i=1}^{n} V_i(v_i) \tag{24}$$

where $V_i = \frac{1}{2}v_i^2$. With this definition, $V = \frac{1}{2}\|v\|_2^2$, where $v = [v_1, v_2, \ldots, v_n]^\top \in \Re^n$.

The time derivative of the V is $\dot{V} = \sum_{i=1}^{n} \dot{V}_i$, and $\dot{V}_i$ along solutions of eqns. (21 - 23) are:

1) For $i = 1$,

$$\begin{aligned} \dot{V}_1 &= v_1\Big[-k_1 v_1 + g_1 v_2\Big] \\ &= -k_1 v_1^2 + v_1 g_1 v_2. \end{aligned} \tag{25}$$

2) For $i = 2, 3, \ldots, n-1$,

$$\begin{aligned} \dot{V}_i &= v_i\Big[-k_i v_i - g_{i-1}v_{i-1} + g_i v_{i+1}\Big] \\ &= -k_i v_i^2 - v_{i-1}g_{i-1}v_i + v_i g_i v_{i+1}. \end{aligned} \tag{26}$$

3) For $i = n$,

$$\begin{aligned} \dot{V}_n &= v_n\Big[-k_n v_n - g_{n-1}v_{n-1}\Big] \\ &= -k_n v_n^2 - v_{n-1}g_{n-1}v_n. \end{aligned} \tag{27}$$

Therefore, the derivative of $V(t)$ satisfies

$$\dot{V} \leq -\underline{k}\|v\|_2^2 = -2\underline{k}V \tag{28}$$

where $\underline{k} = \min_i(k_i)$. Therefore, by Theorem 4.10 in [9], the equilibrium $v = 0$ of eqns. (21 - 23) is globally exponentially stable. The state $\tilde{x}_n$ converges exponentially to zero, because $\tilde{x}_n = v_n$. Also, by integration of $\dot{V} \leq -\underline{k}\|v\|_2^2$, it is straightforward to show that $v \in \mathcal{L}_2$.

Note that the structure of the command filtered system is intentionally designed so that the above proof will parallel a standard backstepping proof [10]. However, this proof shows the exponential stability of the compensated tracking error v, not $\tilde{x}$. The properties of $\tilde{x}$, z, and $\tilde{x} - \bar{x}$ are addressed by Theorem 2.

B. Proof of Theorem 2

This proof uses singular perturbation theory. In particular, the proof shows that all preconditions of Theorem 11.2 in [9] are satisfied, so that the theorem can be applied. The proof uses the compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$ where $\mathcal{D}_{\hat{x}} \subset \Re^{2n-1}$ and $\mathcal{D}_{\hat{z}} \subset \Re^{2n-2}$ are compact sets that contain the origin.

Remark 2: Theorem 11.2 in [9] is too long to allow its direct inclusion herein. To allow straightforward interpretation of the results of this section in terms of that theorem, eqns. (29–30) are in the form of eqns. (11.6-11.7) in [9]. In addition, the terminology of this section and the technical statements following each numbered equation correspond to the requirements of Theorem 11.2.

Define the vectors $\hat{x} = [\tilde{x}_1, \ldots, \tilde{x}_n, \xi_1, \ldots, \xi_{n-1}]^\top \in \Re^{2n-1}$ and $\hat{z} = [z_{1,1}, z_{1,2}, \ldots, z_{n-1,1}, z_{n-1,2}] \in \Re^{2n-2}$. The differential equations for these vectors are

$$\begin{aligned} \dot{\hat{x}} &= \hat{f}(t, \hat{x}, \hat{z}, \epsilon), && (29) \\ \epsilon\dot{\hat{z}} &= \hat{g}(t, \hat{x}, \hat{z}, \epsilon), && (30) \end{aligned}$$

where $\hat{f}$ and $\hat{g}$ are defined below. The initial conditions are $\hat{x}(0) = [\tilde{x}_1(0), \ldots, \tilde{x}_n(0), 0, \ldots, 0]^\top$ and $\hat{z}(0) = 0$ which are independent of ϵ.

The vector field $\hat{f}$, as derived based on eqns. (14) and (17–19), is

$$\left.\begin{aligned} \hat{f}_1(t, \hat{x}, \hat{z}, \epsilon) &= -k_1\tilde{x}_1 + g_1(z_{1,1} - \alpha_1) + g_1\tilde{x}_2 \\ \hat{f}_i(t, \hat{x}, \hat{z}, \epsilon) &= -k_i\tilde{x}_i - g_{i-1}(\tilde{x}_{i-1} - \xi_{i-1}) \\ &\quad + g_i(z_{i,1} - \alpha_i) + g_i\tilde{x}_{i+1} \\ &\qquad \text{for } i = 2, 3 \ldots, n-1 \\ \hat{f}_n(t, \hat{x}, \hat{z}, \epsilon) &= -k_n\tilde{x}_n - g_{n-1}(\tilde{x}_{n-1} - \xi_{n-1}) \\ \hat{f}_{i+n}(t, \hat{x}, \hat{z}, \epsilon) &= -k_i\xi_i + g_i(z_{i,1} - \alpha_i) + g_i\xi_{i+1} \\ &\qquad \text{for } i = 1, 2 \ldots, n-1. \end{aligned}\right\} \tag{31}$$

Note that $\hat{f}$ is independent of ϵ. Therefore, on any compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$, with Assumption 4: the function $\hat{f}$ and its first partial derivatives with respect to $(\hat{x}, \hat{z}, \epsilon)$ are continuous and bounded; and, $\frac{\partial \hat{f}}{\partial t}$ is Lipschitz in $\hat{x}$ uniformly in t.

Note that $\hat{z}$ is just the concatenation of the states of each of the command filters defined in eqns. (15-16). Therefore, $\hat{g}$ is the concatenation of these same eqns. For $i = 1, 2 \ldots, n-1$ the elements of the vector field g are determined from eqns. (15–16) as

$$\left.\begin{aligned} \hat{g}_{2i}(t, \hat{x}, \hat{z}, \epsilon) &= z_{i,2} \\ \hat{g}_{2i+1}(t, \hat{x}, \hat{z}, \epsilon) &= -2\zeta z_{i,2} - (z_{i,1} - \alpha_i) \end{aligned}\right\} \tag{32}$$

which shows that $\hat{g}$ is independent of ϵ. Therefore, on any compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$, with Assumptions 4 and 1: the function $\hat{g}$ and its first partial derivatives with respect to $(\hat{x}, \hat{z}, \epsilon)$ are continuous and bounded; the first partial of $\hat{g}$ with respect to t is continuous and bounded; and, $\frac{\partial \hat{g}(t,\hat{x},\hat{z},0)}{\partial \hat{z}}$ has bounded first partial derivatives with respect to it arguments.

For $\epsilon = 0$ the unique solution to eqn. (30) is defined by $z_{i,1} = \alpha_i$ and $z_{i,2} = 0$ which in vector form will be denoted by $\hat{z} = \hat{h}(t, \hat{x})$ where for $i = 1, 2 \ldots, n-1$

$$\left.\begin{aligned} \hat{h}_{2i}(t, \hat{x}) &= \alpha_i \\ \hat{h}_{2i+1}(t, \hat{x}) &= 0 \end{aligned}\right\} \tag{33}$$

With Assumptions 4 and 1, on any compact set $\mathcal{D}_{\hat{x}}$, the function $\hat{h}(t, \hat{x})$ has bounded first partial derivatives with respect to its arguments.

Let $\bar{x}(t)$ denote the solution of the reduced order problem (see eqn. (11.5) p. 424 in [9])

$$\dot{\bar{x}} = \hat{f}(t, \bar{x}, \hat{h}(t, \hat{x}), 0) \tag{34}$$

with $\hat{x}(0) = [\tilde{x}_1(0), \ldots, \tilde{x}_n(0), 0, \ldots, 0]^\top$. Because of the initial condition and the fact that $z_{i,1} = \alpha_i$, the solution of the reduced order problem has $\bar{x}_i(t) = 0$ for $i = n+1, n+2, \ldots, 2n-1$ for all $t > 0$. Given the facts in the previous sentence, the solution for states $\bar{x}_i$ for $i = 1, 2 \ldots, n$, for the same initial conditions, is the solution to the standard backstepping problem presented in Section III which is exponentially stable. Therefore, for the reduced order problem, the states $\bar{x}_i$ with $i = 1, 2 \ldots, n$ converge exponentially to zero. Because $\bar{x}_i$ converge exponentially to zero for $i = 1, 2 \ldots, n$ and $\bar{x}_i(t) = 0$ for all $t > 0$ for $i = n+1, n+2, \ldots, 2n-1$, the origin is an exponentially stable equilibrium of the reduced order system.

By defining $\hat{y} = \hat{z} - \hat{h}(t, \hat{x})$, the boundary layer model $\frac{dy}{d\tau} = \hat{g}(t, \hat{x}, \hat{y} + \hat{h}(t, \hat{x}))$ with $(t, \hat{x})$ considered fixed and $\tau = \frac{t}{\epsilon}$ (see eqn. (11.14) p. 433 in [9]) is

$$\frac{d\hat{y}}{d\tau} = A\hat{y} \tag{35}$$

where A is a block diagonal matrix with $(n-1)$ blocks each defined by

$$J_i = \left[\begin{array}{cc} 0 & 1 \\ -1 & -2\zeta \end{array} \right].$$

The boundary layer model is independent of $\hat{x}$. The matrix A is Hurwitz. Therefore, the origin is a globally exponentially stable equilibrium of the boundary layer model.

All conditions of Theorem 11.2 in [9] hold on any compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$. If we denote the solutions to eqns. (29), (30), and (34) respectively, as $\hat{x}(t, \epsilon)$, $\hat{z}(t, \epsilon)$, and $\bar{x}(t)$ then for all $t > 0$,

$$\begin{aligned} \hat{x}(t, \epsilon) - \bar{x}(t) &= \mathcal{O}(\epsilon) &(36)\\ \hat{z}(t, \epsilon) - \hat{h}(t, \bar{x}) &= \mathcal{O}(\epsilon) &(37) \end{aligned}$$

which proves Theorem 2.

VII. Conclusions

This article has presented a practical extension of the backstepping nonlinear control approach. A main motivation was facilitation of backstepping implementation by offering a means to determine the time derivatives of the virtual control signals, denoted herein by $\bar{\alpha}$, that is feasible even when the number of iterations of the backstepping method is large (i.e., greater than three). The required derivatives are determined by a method referred to as *command filtering*. The method is described in Section IV-A. Its properties are summarized in the Theorems 1 and 2 of Section IV-B. In particular, Theorem 2 states that by increasing the bandwidth of the command filters, the performance of the command filtered backstepping approach can be made arbitrarily close to that of the standard backstepping approach that uses analytic calculation of derivatives. Because the approach produces the command derivatives through low pass filters involving only integrative processes, the command filtered approach does not increase and may decrease the effects of measurement noise, relative to the standard backstepping approach. Additional benefits of the command filtered approach presented herein are that it is applicable to a wider class of systems that standard backstepping (see Remark 1) and the command filters can also be used to enforce constraints on the state trajectories [5, 7, 22].

This article has consider backstepping for a system with a known model. The command filtering approach is extendable to the adaptive case using the signals v discussed in Theorem 1, but that analysis is beyond the scope of the present article. Adaptive applications are presented in, for example [5, 15]. The approach also allows various physical constraints to be enforced as discussed in specific applications in [5, 15].

Due to space limitations, an application example could not be included herein. An application example using a land vehicle is contained in the companion paper [3].

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