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Delay-Dependent Robust $\mathcal{H}_\infty$ Output Feedback Control for Uncertain Discrete-Time Switched Systems with Interval Time-Varying Delay

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Abstract— **This paper revisits the problem of delay-dependent dynamic output feedback control for a class of uncertain discrete-time switched linear state-delayed systems, where the state delay is assumed to be time-varying and of an interval-like type, which means that both the lower and upper bounds of the time-varying delay are available and the parameter uncertainties are assumed to have a structured linear fractional form. The objective is to design a switched dynamic output feedback controller guaranteeing the asymptotic stability of the resulting closed-loop system with disturbance attenuation level γ. Based on a new delay-dependent switched Lyapunov-Krasovskii functional combined with Finsler's Lemma, a novel sufficient condition for robust $\mathcal{H}_\infty$ performance analysis is first derived and then the corresponding controller synthesis is developed. It is shown that the controller parameters can be obtained by solving a set of linear matrix inequalities, which are numerically efficient with commercially available software. Finally, a numerical example is provided to illustrate the advantages and less conservatism of the proposed approach in comparison with the existing approaches.**

I. INTRODUCTION

Switched systems are an important class of hybrid dynamical systems which consist of a family of subsystems described by differential or difference equations and a logical rule that orchestrates the switching mechanism between these subsystems [1], [2], [3], [4], [5]. On one hand, it is known that switched systems can be used to describe a wide range of real world processes and systems such as power systems, mechanical systems, networked control systems and so on. On the other hand, switching among different controllers for a single process can also be viewed as a switched system. It has been shown this kind of switching control strategy provides an effective means to deal with highly complex systems or to improve the transient response of adaptive control systems [6]. In addition, the study of switched systems provides a powerful tool for intelligent control design which is based on the idea of switching among different controllers.

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The last decade has witnessed increasing research activities in study of controllability [5], [7], [8], observability [7], stability [1], [2], [3], and stabilization [4], [5] of switched systems. Among these research topics, stability analysis and stabilization for switched system have attracted most of the attention. The reader can refer to the survey papers [1], [2], [4], the recent books [3], [5] and the references therein for the major advances achieved in these topics. It is well-known that the Lyapunov stability theory is a powerful and effective tool for studying the stability of various dynamical systems. It has been shown in [2] that a common Lyapunov function for all subsystems facilitates the analysis and design of switched systems under arbitrary switching rules. However, it has been well understood that the results obtained within the common Lyapunov function based framework are generally conservative in the sense there are many switched systems that do not admit a common Lyapunov function, whereas they may still be asymptotically stable. Recently, being a notable extension of the common Lyapunov function, the multiple Lyapunov functions methodology, which was first proposed by Peleties and Decarlo [9], and then developed by Branicky [10] and Ye, Michel and Hou [11] has been proven to be a more effective tool for analysis and control of switched systems, see, for example, [12], [13], [14], [15], [16], [17] and the references therein.

On the other hand, time-delay is often encountered in various engineering systems. It has been well recognized that the presence of time-delay may result in instability, chaotic mode and/or poor performance of a control system [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. It is also known that switched systems with time delay have strong engineering background in power systems [28], networked control systems [29], [30] and so on. Recently, there have appeared some results on stability analysis and controller design for switched systems with time delay in the open literature [31], [32], [33], [34], [35], [36]. The authors in [33] established some criteria on uniform stability and uniform asymptotic stability of invariant sets for hybrid dynamical systems with time delay by utilizing the method of Lyapunov functions and the Razumihin techniques. It was shown that these criteria may also be applied to conduct stability analysis for impulsive delay differential systems and nonlinear sampled-data control systems with time delay. The authors in [34] studied the stability and L_2-gain analysis for a class of switched systems with time-varying delays. In terms of a set of linear matrix inequalities, some sufficient conditions for exponential stability and weighted L_2-gain performance are developed for a class of switched signals with average dwell-time. By using a descriptor model transformation of the original system, the authors in [35] studied the delay-dependent stability analysis and state feedback control for a class of discrete-time switched linear systems with mode-dependent time delays. In a recent paper [36], by using a finite sum inequality and a switched Lyapunov functional, we studied the delay-dependent output feedback control for discrete-time switched linear systems with time-varying state delay. It was shown that the controller gains could be obtained by solving a set of linear matrix inequalities. It is known that the reasonable construction of Lyaounov functional is very crucial for deriving less conservative stability conditions. It is noted that in [36], when treating the time-varying delay and estimating the upper bound of the difference of Lyapunov functional, some useful terms such as $\sum_{m=k-\tau_2}^{k-\tau(k)-1} \bar{e}^T(m)(\bullet)\bar{e}(m)$ were neglected, which may lead to considerable conservatism [24], [25], [26]. In addition, the Lyapunov functional proposed in [36] did not essentially take into account the lower bound information of the time-varying delay, which was the second source of possible conservatism.

In this paper, we revisit the problem of delay-dependent output feedback control for a class of uncertain discrete-time switched linear systems with a time-varying state delay. The uncertainties are assumed to have a structured linear fractional form. Based on a new delay-dependent switched Lyapunov-Krasovskii functional combined with Finsler's Lemma, a new sufficient condition for robust performance analysis is firstly derived and then the corresponding controller design is developed. It is shown that by using a linearization procedure incorporating a bounding inequality to handle the bilinear terms, the output feedback controller parameters can be obtained by solving a set of linear matrix inequalities, which are numerically efficient with commercial available software.

The remaining parts of this paper are organized as follows. Section II is devoted to the system model description and problem formulation. The main results for delay-dependent robust $\mathcal{H}_\infty$ output feedback controller analysis and synthesis are given in section III. Section IV demonstrates the effectiveness and less conservatism of the proposed method in comparison with the existing approaches by means of a numerical example. Finally, conclusions are given in section V.

Throughout this paper, we use the following notations. $\mathbf{I}_n$ and $\mathbf{0}_{m\times n}$ are used to denote the $n \times n$ identity matrix and $m \times n$ zero matrix, respectively. The subscripts n and $m \times n$ are omitted when the size is not relevant or can be determined from the context. He$\{A\}$ is the shorthand notation for $A + A^T$. diag$\{\cdots\}$ denotes a block-diagonal matrix. The notation $\star$ in a symmetric matrix always denotes the symmetric block in the matrix.

II. Model description and problem formulation

Consider the following state-space representation for a class of uncertain discrete-time switched linear systems Σ_S with interval-like time-varying state delay

$$\begin{cases} x(k+1) = \sum_{i=1}^{N} \delta_i(k)\,[A_i(k)x(k) + A_{di}(k)x(k-\tau(k)) \\ \qquad\qquad + B_{1i}(k)u(k) + D_{1i}(k)w(k)] \\ y(k) = \sum_{i=1}^{N} \delta_i(k)\,[C_i(k)x(k) + C_{di}(k)x(k-\tau(k)) \\ \qquad\qquad + B_{2i}(k)u(k) + D_{2i}(k)w(k)] \\ z(k) = \sum_{i=1}^{N} \delta_i(k)\,[L_i(k)x(k) + L_{di}(k)x(k-\tau(k)) \\ \qquad\qquad + B_{3i}(k)u(k) + D_{3i}(k)w(k)] \\ x(k) = \varphi(k), -\tau_2 \le k \le 0, i \in \mathcal{I} := \{1, 2, \cdots, N\} \end{cases} \tag{1}$$

where $x(k) \in \Re^n$ is the system state; $u(k) \in \Re^r$ is the control input; $y(k) \in \Re^p$ is the measured output; $z(k) \in \Re^q$ is the controlled output; $w(k) \in \Re^m$ is the disturbance input which is assumed to belong to $l_2[0, \infty)$. The vector $\delta(k) := [\delta_1(k), \delta_2(k), \cdots, \delta_N(k)]$ is called a switching signal, which specifies which subsystem will be activated at the discrete-time k, where $\delta_i(k) : \mathbb{Z}^+ \to \{0, 1\}$, $\sum_{i=1}^N \delta_i(k) = 1$, $\forall k \in \mathbb{Z}^+$. $N \ge 2$ is the number of subsystems. As in [12], [13], [15], it is also assumed here that the switching signal $\delta(k)$ is not known *a priori*, but its instantaneous value is available in real-time implementation, in other words, the information of which subsystem is activated is available. $\tau(k)$ is a positive integer function representing the time-varying delay of the system Σ_S and satisfying the following assumption

$$\tau_1 \le \tau(k) \le \tau_2 \tag{2}$$

where τ_1 and τ_2 are two constant positive integers representing the minimum and maximum time-delay respectively, for this case, $\tau(k)$ is called an *interval-like* or *range-like* time-varying delay [23], [24], [25], [26]. It is noted that this kind of time-delay describes the real situation in many practical engineering systems. For example, in the field of networked control systems, the network transmission induced delay (either from the sensor to the controller or from the controller to the plant) can be assumed to satisfy (2) without loss of generality [27]. $\varphi(k)$ is a real-valued initial condition sequence on $[-\tau_2, 0]$. $A_i(k)$, $A_{di}(k)$, $C_i(k)$, $C_{di}(k)$, $L_i(k)$, $L_{di}(k)$, $B_{\beta i}(k)$, $D_{\beta i}(k)$, $\beta \in \{1, 2, 3\}$, $i \in \mathcal{I}$ are appropriately dimensioned system matrices with time-varying parameter uncertainties, which are assumed to be of the following form

$$\begin{bmatrix} A_i(k) & A_{di}(k) & B_{1i}(k) & D_{1i}(k) \\ C_i(k) & C_{di}(k) & B_{2i}(k) & D_{2i}(k) \\ L_i(k) & L_{di}(k) & B_{3i}(k) & D_{3i}(k) \end{bmatrix}$$

$$
= \begin{bmatrix} A_i & A_{di} & B_{1i} & D_{1i} \\ C_i & C_{di} & B_{2i} & D_{2i} \\ L_i & L_{di} & B_{3i} & D_{3i} \end{bmatrix} + \begin{bmatrix} W_{1i} \\ W_{2i} \\ W_{3i} \end{bmatrix} \times \Delta(k) \begin{bmatrix} E_{1i} & E_{2i} & E_{3i} & E_{4i} \end{bmatrix} \quad (3)
$$

$$
\Delta(k) = F(k)\,[\mathbf{I} - JF(k)]^{-1} \quad (4)
$$

$$
0 < \mathbf{I} - JJ^T \quad (5)
$$

where A_i, A_{di}, C_i, C_{di}, L_i, L_{di}, $B_{\beta i}$, $D_{\beta i}$, $W_{\beta i}$, $E_{\beta i}$, E_{4i}, $\beta \in \{1, 2, 3\}$, $i \in \mathcal{I}$ and J are known real constant matrices. $F(k) : \mathbb{Z}^+ \rightarrow \Re^{s_1 \times s_2}$ is an unknown real-valued time-varying matrix function with Lesbesgue measurable elements satisfying

$$
F^T(k)F(k) \leq \mathbf{I} \quad (6)
$$

The parameter uncertainties are said to be admissible if (3)-(6) hold.

Remark 2.1. The parameter uncertainties are assumed to have a structured linear fractional form. It is noted that this kind of uncertainties has been fairly investigated in the robust control theory [37], [38], [39]. It has been shown that every rational nonlinear system possesses a linear fractional representation [39]. Notice that when $J = \mathbf{0}$, the linear fractional form uncertainties reduce to the norm-bounded ones [24], [35]. Notice also that the condition (5) guarantees that $\mathbf{I} - JF(k)$ is invertible for all $F(k)$ satisfying (6).

Now, we consider the following homogeneous switched dynamic output feedback controller Σ_C for the switched system Σ_S

$$
\begin{cases} \hat{x}(k+1) = \sum\limits_{i=1}^{N} \delta_i(k) \left[\hat{A}_i \hat{x}(k) + \hat{B}_i y(k)\right] \\ u(k) = \sum\limits_{i=1}^{N} \delta_i(k) \hat{C}_i \hat{x}(k) \end{cases} \quad (7)
$$

where $\hat{x}(k) \in \Re^n$ is the controller state; $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $i \in \mathcal{I}$ are appropriately dimensioned matrices to be determined.

By defining $\bar{x}(k) := \begin{bmatrix} x^T(k) & \hat{x}^T(k) \end{bmatrix}^T$ and applying the switched controller Σ_C to the system Σ_S result in the following closed-loop system Σ_O

$$
\begin{cases} \bar{x}(k+1) = \sum\limits_{i=1}^{N} \delta_i(k) \left[\bar{A}_i(k)\bar{x}(k) + \bar{A}_{di}(k)\bar{x}(k - \tau(k)) + \bar{B}_i(k)w(k)\right] \\ z(k) = \sum\limits_{i=1}^{N} \delta_i(k) \left[\bar{L}_i(k)\bar{x}(k) + \bar{L}_{di}(k)\bar{x}(k - \tau(k)) + D_{3i}(k)w(k)\right] \end{cases} \quad (8)
$$

where

$$
\begin{cases} \bar{A}_i(k) := \begin{bmatrix} A_i(k) & B_{1i}(k)\hat{C}_i \\ \hat{B}_i C_i(k) & \hat{A}_i + \hat{B}_i B_{2i}(k)\hat{C}_i \end{bmatrix}, \\ \bar{A}_{di}(k) := \begin{bmatrix} A_{di}(k) & \mathbf{0} \\ \hat{B}_i C_{di}(k) & \mathbf{0} \end{bmatrix}, \\ \bar{B}_i(k) := \begin{bmatrix} D_{1i}(k) \\ \hat{B}_i D_{2i}(k) \end{bmatrix}, \\ \bar{L}_i(k) := \begin{bmatrix} L_i(k) & B_{3i}(k)\hat{C}_i \end{bmatrix}, \\ \bar{L}_{di}(k) := \begin{bmatrix} L_{di}(k) & \mathbf{0} \end{bmatrix}. \end{cases} \quad (9)
$$

The robust $\mathcal{H}_\infty$ control problem to be investigated in this paper is formulated as follows.

Robust $\mathcal{H}_\infty$ Control Problem. Given the switched system Σ_S and for a prescribed disturbance attenuation level $\gamma > 0$, design a switched controller Σ_C such that the closed-loop system Σ_O is robustly asymptotically stable and the induced l_2-norm of the operator from w to the controlled output z is less than γ under zero initial conditions,

$$
\mathcal{H}_\infty^{zw} := \sup \frac{\|z\|_2}{\|w\|_2} < \gamma \quad (10)
$$

for any nonzero $w \in l_2[0, \infty)$ and all admissible uncertainties.

Before ending this section, we introduce the following well-known lemmas, which will be used in the derivation of our main results.

Lemma 2.1 (S-procedure). [37], [41] Suppose that $\Delta(k)$ is given by (3)-(6) with matrices $M = M^T$, S and N of appropriate dimensions, the inequality

$$
M + S\Delta(k)N + N^T\Delta^T(k)S^T < 0
$$

holds if and only if for some positive scalar $\varepsilon > 0$,

$$
M + \begin{bmatrix} \varepsilon^{-1}N \\ \varepsilon S^T \end{bmatrix}^T \begin{bmatrix} \mathbf{I} & -J \\ -J^T & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon^{-1}N \\ \varepsilon S^T \end{bmatrix} < 0.
$$

Lemma 2.2 (Schur Complement Lemma). [41] For matrices A, B and C with compatible dimensions, the following two inequalities:

$$
A > 0, \ C - B^T A^{-1} B > 0
$$

are equivalent to the inequality

$$
\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0.
$$

Lemma 2.3 (Finsler's Lemma). [40] Let $x \in \Re^n$, $P = P^T \in \Re^{n \times n}$, and $H \in \Re^{m \times n}$ such that $rank(H) = r < n$. The following two statements are equivalent:

$$
\begin{aligned} &1) x^T P x < 0, \forall Hx = 0, x \neq 0. \\ &2) \exists N \in \Re^{n \times m} : P + \mathrm{He}\{NH\} < 0. \end{aligned}
$$

III. Main results

In this section, based on a new delay-dependent switched Lyapunov-Krasovskii functional combined with Finsler's Lemma, an improved delay-dependent approach will be developed to solve the robust $\mathcal{H}_\infty$ control problem formulated in section II. We first present a robust performance analysis result for the closed-loop system Σ_O and then give a parameterized representation of the controller gains in terms of the feasible solutions to a set of linear matrix inequalities.

Theorem 3.1. Given the switched system Σ_S and controller Σ_C, the resulting closed-loop system Σ_O is robustly asymptotically stable with disturbance attenuation level γ if there exist matrices $P_i = P_i^T > 0$, X_{ij}, Y_{ij}, M_{ij}, N_{ij}, R_{ij}, $\tilde{G}_i$, $Q_\nu = Q_\nu^T \geq 0$, $\nu \in \{1, 2, 3\}$, $Z_\mu = Z_\mu^T > 0$, $\mu \in \{1, 2\}$ such that for all $(i, j) \in \mathcal{I} \times \mathcal{I}$ the following matrix inequalities are satisfied

$$
\Theta_{ij} + \Psi_{ij} + \mathrm{He}\{\Phi_{ij} + \tilde{G}_i \tilde{A}_i(k)\} < 0 \quad (11)
$$

$$\begin{bmatrix} X_{ij}+Y_{ij} & M_{ij} \\ \star & Z_1+Z_2 \end{bmatrix} \geq 0 \tag{12}$$

$$\begin{bmatrix} X_{ij} & N_{ij} \\ \star & Z_1 \end{bmatrix} \geq 0 \tag{13}$$

$$\begin{bmatrix} Y_{ij} & R_{ij} \\ \star & Z_2 \end{bmatrix} \geq 0 \tag{14}$$

where

$$\Theta_{ij} := \begin{bmatrix} P_j+\Pi_1 & \mathbf{0} & -\Pi_1 & \mathbf{0} \\ \star & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & -P_i+\Pi_1+\Pi_2 & \mathbf{0} \\ \star & \star & \star & \Pi_3 \end{bmatrix},$$

$$\Pi_1 := \tau_2 Z_1 + (\tau_2-\tau_1)Z_2,$$
$$\Pi_2 := Q_1+Q_2+(\tau_2-\tau_1+1)Q_3,$$
$$\Pi_3 := \text{diag}\left\{-Q_3, -Q_1-Q_3, -Q_2-Q_3, -\gamma^2\mathbf{I}\right\},$$
$$\Psi_{ij} := \tau_2 X_{ij} + (\tau_2-\tau_1)Y_{ij},$$
$$\Phi_{ij} := \begin{bmatrix} \mathbf{0} & N_{ij} & M_{ij}-N_{ij}-R_{ij} & R_{ij} & -M_{ij} & \mathbf{0} \end{bmatrix},$$
$$\bar{A}_i(k) := \begin{bmatrix} -\mathbf{I} & \mathbf{0} & \bar{A}_i(k) & \bar{A}_{di}(k) & \mathbf{0} & \mathbf{0} & \bar{B}_i(k) \\ \mathbf{0} & -\mathbf{I} & \bar{L}_i(k) & \bar{L}_{di}(k) & \mathbf{0} & \mathbf{0} & D_{3i}(k) \end{bmatrix}.$$

Proof. The proof is omitted here due to the space limitation.

Now, we are in a position to present the main result in this section.

Theorem 3.2. Consider the uncertain switched system Σ_S, a full-order dynamic output feedback controller in the form of Σ_C exists such that the closed-loop system Σ_O is robustly asymptotically stable with disturbance attenuation level γ if there exist matrices $\bar{P}_i = \bar{P}_i^T > 0$, $\bar{X}_{ij}$, $\bar{Y}_{ij}$, $\bar{M}_{ij}$, $\bar{N}_{ij}$, $\bar{R}_{ij}$, $\vec{A}_i$, $\vec{B}_i$, $\vec{C}_i$, S, U_1, V_1, $\bar{Q}_\nu = \bar{Q}_\nu^T \geq 0$, $\nu \in \{1,2,3\}$, $\bar{Z}_\mu = \bar{Z}_\mu^T > 0$, $\mu \in \{1,2\}$ and a set of positive scalars $\varepsilon_{ij} > 0$ such that for all $(i,j) \in \mathcal{I}\times\mathcal{I}$ the following linear matrix inequalities are satisfied

$$\begin{bmatrix} \Sigma_{ij} & \varepsilon_{ij}\bar{E}_i^T & \bar{W}_i & H_1 & H_2 \\ \star & -\varepsilon_{ij}\mathbf{I}_{s_2} & \varepsilon_{ij}J & \mathbf{0} & H_3 \\ \star & \star & -\varepsilon_{ij}\mathbf{I}_{s_1} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\mathbf{I}_{3n} & \mathbf{0} \\ \star & \star & \star & \star & -\mathbf{I}_{3n} \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} \bar{X}_{ij}+\bar{Y}_{ij} & \bar{M}_{ij} \\ \star & \bar{Z}_1+\bar{Z}_2 \end{bmatrix} \geq 0 \tag{16}$$

$$\begin{bmatrix} \bar{X}_{ij} & \bar{N}_{ij} \\ \star & \bar{Z}_1 \end{bmatrix} \geq 0 \tag{17}$$

$$\begin{bmatrix} \bar{Y}_{ij} & \bar{R}_{ij} \\ \star & \bar{Z}_2 \end{bmatrix} \geq 0 \tag{18}$$

where

$$\Sigma_{ij} := \bar{\Theta}_{ij} + \bar{\Psi}_{ij} + \text{He}\{\bar{\Phi}_{ij} + \Xi_i\},$$

$$\bar{\Theta}_{ij} := \begin{bmatrix} \bar{P}_j+\bar{\Pi}_1 & \mathbf{0} & -\bar{\Pi}_1 & \mathbf{0} \\ \star & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\bar{P}_i+\bar{\Pi}_1+\bar{\Pi}_2 & \mathbf{0} \\ \star & \star & \star & \bar{\Pi}_3 \end{bmatrix},$$

$$\bar{\Pi}_1 := \tau_2\bar{Z}_1 + (\tau_2-\tau_1)\bar{Z}_2,$$
$$\bar{\Pi}_2 := \bar{Q}_1+\bar{Q}_2+(\tau_2-\tau_1+1)\bar{Q}_3,$$
$$\bar{\Pi}_3 := \text{diag}\{-\bar{Q}_3, -\bar{Q}_1-\bar{Q}_3, -\bar{Q}_2-\bar{Q}_3, -\gamma^2\mathbf{I}\},$$
$$\bar{\Psi}_{ij} := \tau_2\bar{X}_{ij} + (\tau_2-\tau_1)\bar{Y}_{ij},$$
$$\bar{\Phi}_{ij} := \begin{bmatrix} \mathbf{0} & \bar{N}_{ij} & \bar{M}_{ij}-\bar{N}_{ij}-\bar{R}_{ij} & \bar{R}_{ij} & -\bar{M}_{ij} & \mathbf{0} \end{bmatrix},$$

$$\Xi_i :=$$
$$\left[\begin{matrix} -U_1 & -\mathbf{I} & \mathbf{0} & A_iU_1+B_{1i}\vec{C}_i & A_i \\ -S^T & -V_1^T & \mathbf{0} & \vec{A}_i & V_1^TA_i+\vec{B}_iC_i \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & L_iU_1+B_{3i}\vec{C}_i & L_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix}\right.$$
$$\left.\begin{matrix} A_{di}U_1 & A_{di} & \mathbf{0} & D_{1i} \\ \mathbf{0} & V_1^TA_{di}+\vec{B}_iC_{di} & \mathbf{0} & V_1^TD_{1i}+\vec{B}_iD_{2i} \\ L_{di}U_1 & L_{di} & \mathbf{0} & D_{3i} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix}\right],$$

$$\begin{bmatrix} H_1 \mid H_2 \end{bmatrix} :=$$
$$\left[\begin{array}{ccc|ccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ V_1^TA_{di}+\vec{B}_iC_{di} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon_{ij}\mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \varepsilon_{ij}\mathbf{I}_n & U_1^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right],$$

$$H_3 := \begin{bmatrix} \mathbf{0} & E_{1i}U_1+E_{3i}\vec{C}_i & E_{2i}U_1 \end{bmatrix},$$
$$\bar{E}_i := \begin{bmatrix} \mathbf{0}_{s_2\times(3n+q)} & E_{1i} & \mathbf{0}_{s_2\times n} & E_{2i} & \mathbf{0}_{s_2\times 4n} & E_{4i} \end{bmatrix},$$
$$\bar{W}_i := \begin{bmatrix} W_{1i}^T & W_{1i}^TV_1+W_{2i}^T\vec{B}_i^T & W_{3i}^T & \mathbf{0}_{s_1\times(8n+m)} \end{bmatrix}^T.$$

Moreover, an admissible switched dynamic output feedback controller in the form of Σ_C can be constructed as

$$\begin{cases} \hat{A}_i = V_2^{-T}\left(\vec{A}_i - V_1^TA_iU_1 - \vec{B}_iC_iU_1 \right. \\ \qquad\qquad \left. - V_1^TB_{1i}\vec{C}_i - \vec{B}_iB_{2i}\vec{C}_i\right)U_2^{-1}, \\ \hat{B}_i = V_2^{-T}\vec{B}_i, \hat{C}_i = \vec{C}_iU_2^{-1}, i\in\mathcal{I}. \end{cases} \tag{19}$$

with $U_2^TV_2 = \mathtt{S.V.D}\left(S - U_1^TV_1\right)$.

Proof. The proof is omitted here due to the space limitation.

It is noted that the $\mathcal{H}_\infty$ performance index γ described in Theorem 3.2 can also be optimized by the following convex optimization algorithm [41].

Algorithm 3.1. min γ, subject to LMIs (15)-(18).

IV. AN ILLUSTRATIVE EXAMPLE

In this section, we demonstrate the advantages and less conservatism of the proposed approach in this paper via a numerical example.

Consider the following two-mode uncertain discrete-time switched delay system of the form Σ_S with parameters as follows

$$A_1 = \begin{bmatrix} 0.4725 & 0.3675 \\ 0.105 & 1.05 \end{bmatrix}, A_2 = \begin{bmatrix} 0.483 & -0.42 \\ -0.21 & 1.05 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -0.0525 & 0.105 \\ 0 & 0 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.0525 & -0.21 \\ 0 & 0 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, B_{12} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, D_{11} = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix},$$
$$D_{12} = \begin{bmatrix} 0.5 & 0 \end{bmatrix}^T, W_{11} = W_{12} = \begin{bmatrix} 0.002 & 0 \end{bmatrix}^T,$$
$$C_1 = \begin{bmatrix} 0.6 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & -0.6 \end{bmatrix}, C_{d1} = \begin{bmatrix} 0.1 & 0 \end{bmatrix},$$
$$C_{d2} = \begin{bmatrix} 0 & -0.1 \end{bmatrix}, B_{21} = B_{22} = D_{21} = D_{22} = 0.1,$$
$$L_1 = -L_2 = \begin{bmatrix} 0.6 & 0 \end{bmatrix}, L_{d1} = L_{d2} = \begin{bmatrix} 0.02 & 0 \end{bmatrix},$$
$$B_{31} = B_{32} = D_{31} = D_{32} = 0, W_{21} = W_{22} = 0.001,$$
$$W_{31} = W_{32} = 0, E_{11} = E_{12} = \begin{bmatrix} 0.0015 & 0 \end{bmatrix},$$
$$E_{21} = E_{22} = \begin{bmatrix} 0 & 0.001 \end{bmatrix}, E_{31} = E_{32} = 0.002,$$
$$E_{41} = E_{42} = 0, J = 0.2, F(k) = \sin(k).$$

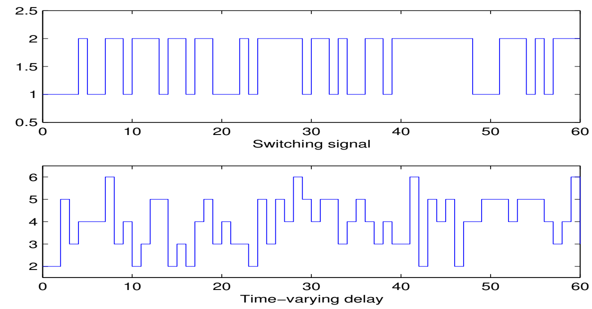

Fig. 1. Switching signal and time-varying delay

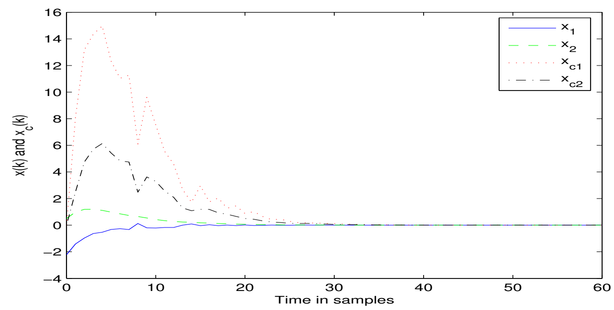

Fig. 3. State response of the closed-loop system

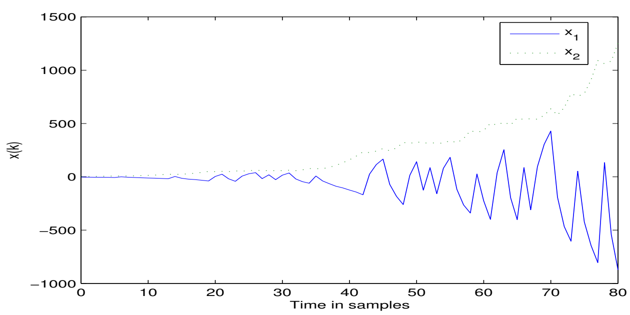

Fig. 2. State response of the open-loop system

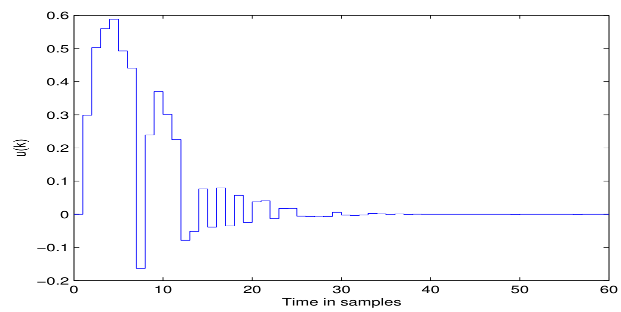

Fig. 4. Time response of the control input $u(k)$

The time-varying delay $\tau(k)$ is assumed to satisfy (2) with $\tau_1 = 2$, $\tau_2 = 6$ and the initial condition sequence is given by $\varphi(k) = \begin{bmatrix} -0.3\mathrm{e}^{(k+5)/3} & 0.5 \end{bmatrix}$. $-\tau_2 \le \tau(k) \le 0$. The exogenous disturbance input $w(k) \in l_2[0,\infty)$ is given by $w(k) = 0.001\mathrm{e}^{-0.003k}\sin(0.002\pi k)$.

The objective is to design a switched dynamic output feedback controller such that the resulting closed-loop system is robustly asymptotically stable with a minimized disturbance attenuation level $\gamma_{\min}$. It is found that there is no feasible solution based on either the delay-independent method or the delay-dependent method in [22], [36]. However, by applying Algorithm 3.1, one indeed obtains a feasible solution with the optimal $\mathcal{H}_\infty$ performance index $\gamma_{\min} = 5.5378$ and controller parameters as follows

$$\hat{A}_1 = \begin{bmatrix} 0.3617 & 0.6429 \\ 0.4779 & 0.0236 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} 0.0785 & -0.6596 \\ 0.0351 & 0.0329 \end{bmatrix},$$
$$\hat{B}_1 = \begin{bmatrix} 3.3008 & 9.6869 \end{bmatrix}^T, \hat{C}_1 = \begin{bmatrix} 0.0175 & -0.076 \end{bmatrix},$$
$$\hat{B}_2 = \begin{bmatrix} 23.0965 & 7.1425 \end{bmatrix}^T, \hat{C}_2 = \begin{bmatrix} 0.0297 & 0.0233 \end{bmatrix}.$$

With the above initial condition and randomly generated switching signal and time-varying delay between $\tau_1 = 2$ and $\tau_2 = 6$ shown in Fig. 1, state responses of the open-loop system and the closed-loop system are shown in Fig. 2 and Fig. 3 respectively, while Fig. 4 represents time response of the control input. It can be observed that the performance of the resulting closed-loop system is satisfactory.

A more detailed comparison between the minimum $\mathcal{H}_\infty$ performance indexes $\gamma_{\min}$ obtained based on different methods for different cases is summarized in Table I. It is shown from this table that $\gamma_{\min}$ increases as τ_2 increases when the delay lower bound τ_1 is fixed and $\gamma_{\min}$ decreases as τ_1 increases when the delay upper bound τ_2 is fixed. The results clearly demonstrate that the delay-dependent switched Lyapunov-Krasovskii functional proposed in this paper is a much richer class of Lyapunov function candidates and the corresponding controller design approach produces much less conservative results.

V. Conclusions

In this paper, based on a new delay-dependent switched Lyapunov-Krasovskii functional combined with Finsler's

TABLE I

COMPARISON OF MINIMUM $\mathcal{H}_\infty$ PERFORMANCES FOR DIFFERENT CASES

Case I: for different τ_2 with given $\tau_1 = 2$				
method	$\tau_2 = 3$	$\tau_2 = 4$	$\tau_2 = 5$	$\tau_2 = 6$
algorithm 3.1	1.0846	1.7125	2.6283	5.5378
[36]	1.4639	2.1777	3.5770	∞
delay-independent method	4.9679	∞	∞	∞
Case II: for different τ_1 with given $\tau_2 = 6$				
method	$\tau_1 = 3$	$\tau_1 = 4$	$\tau_1 = 5$	$\tau_1 = 6$
algorithm 3.1	3.0194	1.9966	1.2575	0.7287
[36]	4.7483	2.7692	1.8337	1.1472
delay-independent method	∞	∞	4.9679	1.4837

Lemma, an improved approach has been developed to study the delay-dependent robust $\mathcal{H}_\infty$ output feedback control for a class of uncertain discrete-time switched linear systems with interval-like time-varying state delay. It is shown that the controller parameters can be obtained by solving a set of linear matrix inequalities. A numerical example is presented to demonstrate the effectiveness and less conservatism of the proposed approach in comparison with the existing approaches. It is also noted that the proposed method can be easily extended to solve some other issues on discrete-time switched delay systems such as $\mathcal{H}_\infty$ filtering or mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problems.

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