2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

On the Dynamic Programming Approach to Multi-Model Robust Optimal Control Problems

V. Azhmyakov, V.G. Boltyanski and A.S. Poznyak

Abstract— **The aim of this paper is to extend the Dynamic Programming (DP) approach to multi-model optimal control problems (OCPs). We deal with robust optimization of multi-model control systems and are particularly interested in the Hamilton-Jacobi-Bellman (HJB) equation for the above class of problems. In this paper, we study a variant of the HJB for multi-model OCPs and examine the natural relationship between the Bellman DP techniques and the Robust Maximum Principle (MP) [7], [8], [9], [18]. Moreover, we describe a concept for practical calculations in the context of multi-model LQ-problems and derive the associated Riccati-type equation.**

I. INTRODUCTION

The theory of OCPs governed by ordinary differential equations is well established since middle of the 20-th century, see e.g., [1], [3], [4], [5], [6], [10], [14], [16], [17] and the references therein. For a classical OCP, the main tools toward the contstruction of optimal trajectories, and then optimal synthesis, are the celebrated Pontryagin MP and the Bellman DP.

Recently robust optimization problems for multi-model control systems have attracted a lot of attention, thus both theoretical results and applications were developed, (see [7], [8], [9], [18], [19]). OCPs for multi-model dynamical systems arise in the control of mechanical multibody systems, electrical circuits and heterogeneous systems, where different models are coupled together. The majority of applied OCPs are problems with incomplete information on the model structure or parameters. The multi-model control systems provide useful theoretical models for some classes of dynamical systems with the above-mentioned type of uncertainties. In this case one of the most efficient approaches to the optimal design of such systems is the robust optimization technique. Optimal robust control strategies based on the minimax algorithms have found a wide use in design of complex control systems. Robust MP proposed by Boltyanski and Poznyak (see e.g., [7], [8], [9], [18]) is the basic analytical result for studying OCPs with multi-model controlled plants. This result was recently extended to some effective numerical procedures [19]. On the other hand, the Bellman DP is not far enough advanced to multi-model OCPs.

The purpose of this paper is to apply the classic DP techniques to a class of multi-model OCPs. First, we verify the Bellman *principle of optimality* for the class of problems under consideration. Second, we derive a (robust) version of the HJB equation. It should be stressed that our main result deals with a finite parametric set involved into a model description. We also apply the HJB equation to a multi-model LQ-problem (see [9], [18]) and derive a parametric Riccati equation. Moreover, the obtained theoretical facts are considered in comparison with the corresponding theorem resulting from the application of the Robust MP to multi-model LQ-problems [18]. In such a manner we establish the natural relationship between DP and the Robust MP for the given class of LQ-problems (see e.g., [11]).

The remainder of our paper is organized as follows. Section 2 contains a problem formulation, some basic concepts and preliminary results. Section 3 is devoted to the main result of this paper, namely, to a variant of the HJB equation for multi-model OCPs. Moreover, we also deal with the corresponding verification techniques. In Section 4 we apply our theoretical results to the multi-model linear quadratic problems and deduce a Riccati-formalism similar to the classic LQ-theory. Section 5 summarizes the paper.

II. PROBLEM FORMULATION AND PRELIMINARY RESULTS

Consider the following initial-value problem for a multi-model control system

$$\dot{x}(t) = f^{\alpha}(t, x(t), u(t)) \text{ a.e. on } [0, t_f],\ x(0) = x_0 \qquad (1)$$

where $u(t) \in U$ is a compact subset of $\mathbb{R}^m$, $x_0 \in \mathbb{R}^n$ is a fixed initial state and $f^{\alpha} : [0, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for every α from a finite parametric set $\mathcal{A}$. Note that parameter α indicates the corresponding "model" (or "realization") of the multi-model system under consideration (see [7], [8], [9], [18], [19]). Let us introduce the set of admissible control functions

$$\mathcal{U} := \{u(\cdot) \in \mathbb{L}^{\infty}_{m}([0, t_f]) \ : \ u(t) \in U \text{ a.e. on } [0, t_f]\}$$

Here $\mathbb{L}^{\infty}_{m}([0, t_f])$ is the standard Lebesgue space of (bounded) measurable functions $u : [0, t_f] \to \mathbb{R}^m$ such that $\operatorname{ess\,sup}_{t \in [0,t_f]} ||u(t)||_{\mathbb{R}^m} < \infty$. In addition, we assume that for each $\alpha \in \mathcal{A}$, $u(\cdot) \in \mathcal{U}$ the realized initial-value problem (1) has a unique absolutely continuous solution $x^{\alpha,u}(\cdot)$. For some constructive existence and uniqueness conditions see e.g., [5], [17], [14]. Let $u(\cdot)$ be an admissible control function. This control gives rise to the complete dynamic of the given multi-model system (1), and we can define the $(n \times |\mathcal{A}|)$-dimensional "state vector" of (1)

$$X^u(t) := \left(x^{\alpha_1,u}(t), ..., x^{\alpha_{|\mathcal{A}|},u}(t)\right)_{\alpha \in \mathcal{A}},\ t \in [0, t_f]$$

V. Azhmyakov is with Departamento de Control Automatico, CINVESTAV, A.P. 14-740, CP 07000, Mexico D.F., Mexico vazhmyakov@ctrl.cinvestav.mx (author for correspondence)

V.G. Boltyanski is with CIMAT, A.P. 402, 36000 Guanajuato, Mexico

A. Poznyak is with Departamento de Control Automatico, CINVESTAV, A.P. 14-740, CP 07000, Mexico D.F., Mexico apoznyak@ctrl.cinvestav.mx

In a similar way one can consider a "trajectory" of (1) as an absolutely continuous $(n\times|\mathcal{A}|)$-dimensional function $X^u(\cdot)$. In the following, we will use the additional notation

$$F(t,X,u) := \left(f^{\alpha_1}(t,x,u),...,f^{\alpha_{|\mathcal{A}|}}(t,x,u)\right)_{\alpha\in\mathcal{A}}$$

For every $\alpha\in\mathcal{A}, u(\cdot)\in\mathcal{U}$ we consider the cost functional

$$h(u(\cdot),x^{\alpha,u}(\cdot)) := \int_0^{t_f} f_0(t,x^{\alpha,u}(t),u(t))dt$$

where $f_0 : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ is a continuous function (the integrand of the cost functional). Clearly, functional $h(u(\cdot),x^{\alpha,u}(\cdot))$ is associated with the corresponding realized model from (1). If we assume that the realized value of the parameter α is unknown, then the *worst cost* (highest cost) can be easily defined as $J(u(\cdot)) := \max_{\alpha\in\mathcal{A}} h(u(\cdot),x^{\alpha,u}(\cdot))$. Note that the "common" cost functional J depends only on the given admissible control $u(\cdot)$. Let us now formulate the robust (minimax) OCP for a multi-model control system

$$\text{minimize } J(u(\cdot)) \text{ subject to } (1), \alpha\in\mathcal{A}, u(\cdot)\in\mathcal{U} \quad (2)$$

Roughly speaking, in the context of problem (2) we are interested in a control strategy which provides a "good" behavior for a given collection of models from (1) even in the case of the "worst" cost. A pair $(u(\cdot),X^u(\cdot))$, where $u(\cdot)\in\mathcal{U}$, is called an *admissible process* for (2). Note that we consider admissible processes defined on the (finite) time interval $[0,t_f]$.

Remark 1: Multi-model OCPs of the Bolza-type have been studied in [18]. Let us examine the Bolza cost functional associated with the multi-model system (1)

$$\tilde{h}(u(\cdot),x^{\alpha,u}(\cdot)) := \phi(x^{\alpha,u}(t_f)) + \int_0^{t_f} \tilde{f}_0(t,x^{\alpha,u}(t),u(t))dt$$

where $\phi:\mathbb{R}^n\to\mathbb{R}$ is a continuously differentiable function (a smooth terminal term) and $\tilde{f}_0$ is a continuous function. It is evident that for every $\alpha\in\mathcal{A}$ and $u(\cdot)\in\mathcal{U}$ we have

$$\begin{aligned}\tilde{h}(u(\cdot),x^{\alpha,u}(\cdot)) &= \int_0^{t_f} f_0^\alpha(t,x^{\alpha,u}(t),u(t))dt\\ &= h^\alpha(u(\cdot),x^{\alpha,u}(\cdot))\end{aligned}$$

where the new integrand f_0^α of h^α is defined as follows

$$f_0^\alpha(t,x,u) := \frac{\partial\phi(x)}{\partial x} f^\alpha(t,x,u) + \tilde{f}(t,x,u),\ \alpha\in\mathcal{A}$$

Here we put $\phi(x_0)=0$. Since the common cost functional J can also be defined as a maximum over all $h^\alpha(u(\cdot),x^{\alpha,u}(\cdot))$, $\alpha\in\mathcal{A}$, we conclude that the minimax Bolza OCP (studied in [18]) is in effect, incorporated into the modeling framework of problem (2).

Next we introduce the concept of a *local solution* to (2).

Definition 1: An admissible process $(u^{opt}(\cdot),X^{opt}(\cdot))$ is called a (local) optimal solution of (2) if there exists an $\epsilon>0$ such that $J(u^{opt}) \le J(u(\cdot))$ for all admissible processes $(u(\cdot),X^u(\cdot))$ with $||X^u(\cdot)-X^{opt}(\cdot)||_{\mathbb{C}_n([0,t_f])} < \epsilon$.

As evident, we understand a local optimal solution of the multi-model OCP (2) in the context of a *strong minimum*. We refer to [1], [10], [17] for the necessary theoretical details. In the following, we assume that the given OCP (2) has an optimal solution. Let us denote all auxiliary analytic assumptions / hypothesis formulated above as *basic assumptions*. We now present the Robust MP which establishes the necessary conditions for an admissible control to be optimal in the sense of our Definition 1.

Proposition 1: (Robust MP [18]) Assume that all basic assumptions hold. Let $(u^{opt}(\cdot),X^{opt}(\cdot))$ be an optimal solution of (2). Then there exist non-trivial constants $\lambda_\alpha^0\ge 0$, non-negative real values μ_α, where $\alpha\in\mathcal{A}$, and an absolutely continuous function $\Psi:[0,t_f]\to\mathbb{R}^{n\times|\mathcal{A}|}$ satisfying the following *adjoint system*:

$$\begin{aligned}\dot{\Psi}(t) &= -\frac{\partial H(t,X^{opt}(t),u^{opt}(t),\Psi(t),\lambda^0)}{\partial x^\alpha}\\ \Psi(t_f) &= 0\end{aligned} \quad (3)$$

and the *maximality condition*:

$$\begin{aligned}&H(t,X^{opt},u^{opt}(t),\Psi(t),\lambda^0)\\ &= \max_{u\in U} H(t,X^{opt}(t),u,\Psi(t),\lambda^0)\end{aligned} \quad (4)$$

for a.e. $t\in[0,t_f]$, where

$$\begin{aligned}&H(t,X,u,\Psi,\lambda^0) := \langle\Psi,F(t,X,u)\rangle -\\ &-\sum_{\alpha\in\mathcal{A}} \lambda_\alpha^0 f_0(t,x^{\alpha,u}(t),u(t)),\\ &(t,u,X,\Psi,\lambda^0)\in[0,t_f]\times\mathbb{R}^m\times\mathbb{R}^{n\times|\mathcal{A}|}\times R^{n\times|\mathcal{A}|}\times R^{|\mathcal{A}|}\end{aligned}$$

is the Hamiltonian for (2) and $\lambda^0 := (\lambda_{\alpha_1}^0,...,\lambda_{\alpha_{|\mathcal{A}|}}^0)$. Moreover, for every $\alpha\in\mathcal{A}$ the following *complementarity slackness / transversality conditions* are satisfied:

$$\begin{aligned}&\mu_\alpha\big(h(u^{opt}(\cdot),x^{\alpha,opt}(\cdot)) - J(u^{opt}(\cdot))\big) = 0\\ &\lambda_\alpha^0 - \mu_\alpha = 0\end{aligned} \quad (5)$$

where $x^{\alpha,opt}(\cdot)$ is the α-components of $X^{opt}(\cdot)$.

Finally, we give an easy lower estimation of the minimal value of J in (2).

Theorem 1: Under the above-formulated basic assumptions, the following inequality is satisfied

$$\max_{\alpha\in\mathcal{A}}\min_{u(\cdot)\in\mathcal{U}} h(u(\cdot),x^{\alpha,u}(\cdot)) \le J(u^{opt}(\cdot)).$$

Proof: Clearly,

$$\min_{u(\cdot)\in\mathcal{U}} h(u(\cdot),x^{\alpha,u}(\cdot)) \le h(u(\cdot),x^{\alpha,u}(\cdot))$$

for every $\alpha\in\mathcal{A},\ u(\cdot)\in\mathcal{U}$. Hence

$$\max_{\alpha\in\mathcal{A}}\min_{u(\cdot)\in\mathcal{U}} h(u(\cdot),x^{\alpha,u}(\cdot)) \le J(u(\cdot))$$

for every $u(\cdot)\in\mathcal{U}$. Since $u^{opt}(\cdot)\in\mathcal{U}$, it follows that the last inequality is also satisfied for an optimal control $u^{opt}(\cdot)$. ■

The presented result characterizes a relation between minimal value of J in (2) and the maximum of costs computed over all of "partial" OCPs formulated for every model from the given multi-model system (1).

III. THE ROBUST VERSION OF THE HAMILTON-JACOBI-BELLMAN EQUATION

In this section we present our main results, namely, a variant of the Bellman principle of optimality and the corresponding HJB equation for the robust OCP (2).

Using the well-known (DP) techniques of the *invariant embedding* (see e.g., [12], [15]), we define a family of multi-model control systems over $[s, t_f]$

$$\dot{x}(t) = f^{\alpha}(t, x(t), u(t)) \text{ a.e. on } [s, t_f],\ x(0) = y^{\alpha} \quad (6)$$

where $(s, y^{\alpha}) \in [0, t_f) \times \mathbb{R}^n$ and $u(\cdot)$ belongs to the set

$$\mathcal{U}_s := \{u(\cdot) \in \mathbb{L}^{\infty}_m([s, t_f]) \ : \ u(t) \in U \text{ a.e. on } [0, t_f]\}$$

Let $Y := \big(y^{\alpha_1}, ..., y^{\alpha_{|\mathcal{A}|}}\big)$ and $z := (s, Y) \in [0, t_f) \times \mathbb{R}^{n\times|\mathcal{A}|}$. Similarly to Section II we also introduce the following notations $X^u_z(t) := \big(x^{\alpha_1,u}_z(t), ..., x^{\alpha_{|\mathcal{A}|},u}_z(t)\big)_{\alpha\in\mathcal{A}}$ and

$$h_z(u(\cdot), x^{\alpha,u}_z(\cdot)) := \int_s^{t_f} f_0(t, x^{\alpha,u}_z(t), u(t))dt$$
$$J_z(u(\cdot)) := \max_{\alpha\in\mathcal{A}} h(u(\cdot), x^{\alpha,u}_z(\cdot))$$

where $x^{\alpha,u}_z(\cdot)$ is a solution of (6) to a control function $u(\cdot)$ from $\mathcal{U}_s$. Moreover, we define a "trajectory" $X^u_z(\cdot)$ of (6). In parallel to (2) we also study the family of OCPs

$$\text{minimize } J_z(u(\cdot)) \text{ subject to } (6), \alpha \in \mathcal{A}, u(\cdot) \in \mathcal{U}_s \quad (7)$$

In fact, problem (7) represents OCPs parametrized by a pair $z = (s, Y) \in [0, t_f) \times \mathbb{R}^{n\times|\mathcal{A}|}$. It is evident that the initial OCP (2) is "embedded" in this family of problems for values $s = 0,\ y^{\alpha} = x_0$. Analogously to Section II we assume that every problem (7) has an optimal solution $(u^{opt}_z(\cdot), X^{opt}_z(\cdot))$ (in the sense of Definition 1). Further, we define the *value function* of (7)

$$V(s, Y) := \inf_{u(\cdot)\in\mathcal{U}_s} \big[J_z(u(\cdot))\big] \quad \forall z \in [0, t_f) \times \mathbb{R}^{n\times|\mathcal{A}|},$$
$$V(t_f, Y) = 0 \quad \forall Y \in \mathbb{R}^{n\times|\mathcal{A}|}$$

First, let us prove the Bellman optimality principle in the case of a possible "non-additive" cost functional J from (2).

Theorem 2: Let all basic assumptions from Section II hold. Then for any $z \in [0, t_f) \times \mathbb{R}^{n\times|\mathcal{A}|}$,

$$\begin{aligned} V(s, Y) = \inf_{u(\cdot)\in\mathcal{U}_s} \big[\max_{\alpha\in\mathcal{A}} \int_s^{\hat{s}} f_0(t, x^{\alpha,u}_z(t), u(t))dt \\ + V(\hat{s}, X^u_z(\hat{s}))\big] \quad \forall 0 \le s \le \hat{s} \le t_f \end{aligned} \quad (8)$$

Proof: Using the definition of the value function, we deduce that

$$\begin{aligned} V(s, Y) &\le \max_{\alpha\in\mathcal{A}} \big[\int_s^{\hat{s}} f_0(t, x^{\alpha,u}_z(t), u(t))dt \\ &+ \int_{\hat{s}}^{t_f} f_0(t, x^{opt}_z(t), u^{opt}_z(t))dt\big] \\ &\le \max_{\alpha\in\mathcal{A}} \big[\int_s^{\hat{s}} f_0(t, x^{\alpha,u}_z(t), u(t))dt\big] + V(\hat{s}, X^u_z(\hat{s})) \end{aligned} \quad (9)$$

for every control $u_z(t) := \mathbf{1}_{[s,\hat{s})}(t)u(t) + \mathbf{1}_{[\hat{s},t_f]}(t)u^{opt}_z(\cdot)$ from $\mathcal{U}_s$. Here $x^{opt}_z(\cdot)$ is the α-component of $X^{opt}_z(\cdot)$ and $\mathbf{1}_{\tau}(\cdot)$ is a characteristic function of a time-interval τ. From (9) we obtain

$$\begin{aligned} V(s, Y) \le \inf_{u(\cdot)\in\mathcal{U}_s} \big[\max_{\alpha\in\mathcal{A}} \int_s^{\hat{s}} f_0(t, x^{\alpha,u}_z(t), u(t))dt+ \\ + V(\hat{s}, X^u_z(\hat{s}))\big] \end{aligned} \quad (10)$$

On the other hand there exists a control function $u_{\delta}(\cdot) \in \mathcal{U}_s$ with the following property (see e.g., [12])

$$\begin{aligned} V(s, Y) + \delta &\ge \max_{\alpha\in\mathcal{A}} \big[\int_s^{\hat{s}} f_0(t, x^{\alpha,\delta}_z(t), u_{\delta}(t))dt \\ &+ V(\hat{s}, X^{\delta}_z(\hat{s}))\big] \ge \inf_{u(\cdot)\in\mathcal{U}_s} \big[V(\hat{s}, X^u_z(\hat{s})) \\ &+ \max_{\alpha\in\mathcal{A}} \int_s^{\hat{s}} f_0(t, x^{\alpha,u}_z(t), u(t))dt\big] \end{aligned} \quad (11)$$

where $x^{\alpha,\delta}_z(\cdot)$ is the α-component of $X^{\delta}_z(\cdot)$ (the set of solutions to (6) corresponding to the control u_{δ}). Combining (10) and (11) we obtain (8). ■

We now turn back to the initial minimax OCP (2). Let $(\partial V(t, X)/\partial x^{\alpha})_{\mathcal{A}}$ be the gradient of V with respect to the vector $\mathbf{x} := (x^{\alpha_1}, ..., x^{\alpha_{|\mathcal{A}|}})$. Using the above result, we are able to establish the robust version of the HJB equation for the multi-model OCP (2).

Theorem 3: Let in addition to the basic assumptions from Section II all functions f^{α}, $\alpha \in \mathcal{A}$ and f_0 be uniformly continuous. Moreover, assume that f^{α}, $\alpha \in \mathcal{A}$ are Lipschitz with respect to x uniformly in $(t \times u) \in [0, t_f] \times U$. Suppose that V is a continuous differentiable function. Then there exists a vector $\lambda^0 := (\lambda^0_{\alpha_1}, ..., \lambda^0_{\alpha_{|\mathcal{A}|}}) \in S_{\mathcal{A}}$ such that V is a solution of the following boundary value problem for the (HJB) partial differential equation

$$\begin{aligned} &-\frac{\partial V_t(t, X)}{\partial t} \\ &+ \max_{u\in U} H\Big(t, X, u, -\Big(\frac{\partial V(t, X)}{\partial x^{\alpha}}\Big)_{\mathcal{A}}, \lambda^0\Big) = 0 \\ &\forall (t, X) \in [0, t_f) \times \mathbb{R}^{n\times|\mathcal{A}|},\ V(t_f, X) = 0 \end{aligned} \quad (12)$$

where H is the Hamiltonian from Proposition 1 and

$$S_{\mathcal{A}} := \{\lambda \in \mathbb{R}^{|\mathcal{A}|} \ : \ \lambda_\alpha \geq 0 \ \ \forall \alpha \in \mathcal{A}, \ \sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1\}$$

is a barycentric system.

Proof: Let $u(t) = u \in U$ and $X_z^u(\cdot)$ be the corresponding trajectory of (6). From (8) we deduce

$$\frac{1}{\hat{s}-s}\big[-\big(V(\hat{s}, X_z^u(\hat{s})) - V(s,Y)\big) - \max_{\alpha \in \mathcal{A}} \int_s^{\hat{s}} f_0(t, x^{\alpha,u}(t), u)dt\big] \leq 0 \tag{13}$$

Using (13) and a representation of a finite maximization problem by an equivalent linear programm over a system of barycentric coordinates, we obtain

$$\begin{aligned} &\frac{1}{\hat{s}-s}\big[-\big(V(\hat{s}, X_z^u(\hat{s})) - V(s,Y)\big) \\ &- \max_{\lambda(s,\hat{s}) \in S_{\mathcal{A}}} \sum_{\alpha \in \mathcal{A}} \lambda_\alpha(s,\hat{s}) \int_s^{\hat{s}} f_0(t, x^{\alpha,u}(t), u)dt\big] \\ &= \frac{1}{\hat{s}-s}\big[-\big(V(\hat{s}, X_z^u(\hat{s})) - V(s,Y)\big) - \\ &- \max_{\lambda(s,\hat{s}) \in S_{\mathcal{A}}} \sum_{\alpha \in \mathcal{A}} \lambda_\alpha(s,\hat{s}) \int_s^{\hat{s}} f_0(t, x^{\alpha,u}(t), u)dt\big] \\ &\leq 0 \end{aligned} \tag{14}$$

where $\lambda(s,\hat{s}) \in S_{\mathcal{A}}$ for every $0 \leq s \leq \hat{s} \leq t_f$. We now take the limit as $\hat{s} \to s$ in (14). Since $\{\lambda(s,\cdot)\}$ is a bounded sequence, there exists at least one accumulation point $\lambda^1(s) \in S_{\mathcal{A}}$ of this sequence (see e.g.,[20]). Using the continuity/differentiability properties of f_0 and V, we obtain the inequality

$$\begin{aligned} &-\frac{\partial V(s,Y)}{\partial t} - \Big\langle \Big(\frac{\partial V(s,Y)}{\partial x^\alpha}\Big)_{\mathcal{A}}, F(s,Y,u) \Big\rangle \\ &- \sum_{\alpha \in \mathcal{A}} \hat{\lambda}^1_\alpha(s) f_0(s, y^\alpha, u) \leq 0 \end{aligned} \tag{15}$$

for all $u \in U$. Here $\hat{\lambda}^1(s) \in S_{\mathcal{A}}$ is a solution of the following linear programm $\max_{\lambda(s) \in S_{\mathcal{A}}} \sum_{\alpha \in \mathcal{A}} \lambda_\alpha(s) f_0(s, y^\alpha, u)$ for every $s \in [0, t_f]$. Note that this linear programm is a consequence of the limiting process (as $\hat{s} \to s$) applying to the maximization-procedure in (14). Since we deal with a linear programm over the barycentric set $S_{\mathcal{A}}$, the value $\hat{\lambda}^1_\alpha(s)$ for all $s \in [0, t_f]$ belongs to the set of vertexes of $S_{\mathcal{A}}$. Recall that the existence of an optimal solution to every (7) is assumed. Since (15) is satisfied for all $u \in U$, we deduce the next inequality

$$\begin{aligned} &-\frac{\partial V(s,Y)}{\partial t} \\ &+ \max_{u \in U} H\Big(s, Y, u, -\Big(\frac{\partial V(s,Y)}{\partial x^\alpha}\Big)_{\mathcal{A}}, \hat{\lambda}^1(s)\Big) \leq 0 \end{aligned} \tag{16}$$

Alternatively, for any $\delta > 0$ and for a small $(\hat{s} - s)$, there exists a control function $u_\delta(\cdot) \in \mathcal{U}_s$ such that (see e.g., [12])

$$\begin{aligned} &V(s,Y) + \delta(\hat{s}-s) \geq \max_{\alpha \in \mathcal{A}} \int_s^{\hat{s}} f_0(t, x_z^{\alpha,\delta}(t), u_\delta(t))dt \\ &+ V(\hat{s}, X_z^\delta(\hat{s})) \end{aligned}$$

where $x_z^{\alpha,\delta}(\cdot)$ is the α-component of $X_z^\delta(\cdot)$ (the set of solutions to (6) corresponding to the control u_δ). Applying the above-mentioned representation of a finite maximization problem by an equivalent linear programm over a system of barycentric coordinates, we get

$$\begin{aligned} &-\delta \leq \frac{1}{\hat{s}-s}\big[-\big(V(\hat{s}, X_z^\delta(\hat{s})) - V(s,Y)\big) \\ &- \max_{\alpha \in \mathcal{A}} \int_s^{\hat{s}} f_0(t, x_z^{\alpha,\delta}(t), u_\delta(t))dt\big] \\ &= \frac{1}{\hat{s}-s}\big[-\big(V(\hat{s}, X_z^\delta(\hat{s})) - V(s,Y)\big) \\ &- \max_{\lambda(s,\hat{s}) \in S_{\mathcal{A}}} \sum_{\alpha \in \mathcal{A}} \lambda_\alpha(s,\hat{s}) \int_s^{\hat{s}} f_0(t, x_z^{\alpha,\delta}(t), u_\delta(t))dt\big] \end{aligned} \tag{17}$$

This implies the following

$$\begin{aligned} &-\delta \leq \frac{1}{\hat{s}-s} \int_s^{\hat{s}} \Big[-\frac{\partial V(t, X_z^\delta(t))}{\partial t} \\ &- \Big\langle \Big(\frac{\partial V(t, X_z^\delta(t))}{\partial x^\alpha}\Big)_{\mathcal{A}}, F(t, X_z^\delta(t), u(t)) \Big\rangle \\ &- \sum_{\alpha \in \mathcal{A}} \lambda^2_\alpha(s,\hat{s}) f_0(t, x_z^{\alpha,\delta}(t), u_\delta(t))\Big]dt \\ &= \frac{1}{\hat{s}-s} \int_s^{\hat{s}} \Big[-\frac{\partial V(t, X_z^\delta(t))}{\partial t} \\ &+ H\Big(t, X_z^\delta(t), u_\delta(t), -\Big(\frac{\partial V(t, X_z^\delta(t))}{\partial x^\alpha}\Big)_{\mathcal{A}}, \lambda^2(s,\hat{s})\Big)\Big]dt \end{aligned}$$

where $\lambda^2_\alpha(s,\hat{s})$ is the α-component of the vector function $\lambda^2(s,\hat{s}) \in S_{\mathcal{A}}$ which is a solution of the linear programm in (17). Since we deal with a linear programm over the barycentric set $S_{\mathcal{A}}$, the value $\lambda^2(s,\hat{s})$ belongs to the set of vertexes of $S_{\mathcal{A}}$. Consequently,

$$\begin{aligned} &-\delta \leq \frac{1}{\hat{s}-s} \int_s^{\hat{s}} \Big[-\frac{\partial V(t, X_z^\delta(t))}{\partial t} \\ &+ \max_{u \in U} H\Big(t, X_z^u(t), u, -\Big(\frac{\partial V(t, X_z^u(t))}{\partial x^\alpha}\Big)_{\mathcal{A}}, \lambda^2(s,\hat{s})\Big)\Big]dt \end{aligned}$$

Using the uniform continuity of the functions f^α, $\alpha \in \mathcal{A}$, f_0 and the boundness of the function $\lambda^2(s,\hat{s})$ we can define the accumulation points in the right hand side of the previous inequality as $\hat{s} \to s$. Thus, we deduce the inequality

$$0 \leq -\frac{\partial V(s,Y)}{\partial t} + \max_{u\in U} H\Big(s, Y, u, -\Big(\frac{\partial V(s,Y)}{\partial x^\alpha}\Big)_{\mathcal{A}}, \hat{\lambda}^2(s)\Big)$$

We now consider the procedure $\min_{t\in[0,t_f]} \min_{\mathrm{ver}(S_{\mathcal{A}})}$ for the left hand side of (16) and the procedure $\max_{t\in[0,t_f]} \max_{\mathrm{ver}(S_{\mathcal{A}})}$ for the right hand side of the last inequality, where $\mathrm{ver}(S_{\mathcal{A}})$ is the set of all vertexes of the system $S_{\mathcal{A}}$. Note that the Hamiltonian is a continuous function of time and the above procedures are consistent in the sense of the minimization with respect to the time value. Let now $\hat{\lambda}^{01}, \hat{\lambda}^{02} \in \mathrm{ver} S_{\mathcal{A}}$ are solutions of the corresponding minimization and maximization procedures. This implies that

$$-\frac{\partial V(s,Y)}{\partial t} + \max_{u\in U} H\Big(s, Y, u, -\Big(\frac{\partial V(s,Y)}{\partial x^\alpha}\Big)_{\mathcal{A}}, \hat{\lambda}^{01}\Big) \leq 0$$
$$0 \leq -\frac{\partial V(s,Y)}{\partial t} + \max_{u\in U} H\Big(s, Y, u, -\Big(\frac{\partial V(s,Y)}{\partial x^\alpha}\Big)_{\mathcal{A}}, \hat{\lambda}^{02}\Big)$$

for some $\hat{\lambda}^{01}, \hat{\lambda}^{02} \in \mathrm{ver}(S_{\mathcal{A}})$ and all $s \in [0, t_f]$. The Hamiltonian H is a continuous (linear) function with respect to the last variable. Therefore, from the last two inequalities and from the generalization of the Bolzano Theorem we deduce the existence of a constant vector $\lambda^0 \in S_{\mathcal{A}}$ such that the HJB in (12) is satisfied. ∎

Using some techniques of the classic Bellman DP, one can establish the relationship between Robust MP and the HJB from Theorem 3. This relationship is analogous to the standard result from the classic optimal control theory (see e.g., [3], [4], [15])).

Theorem 4: Let in addition to the assumptions of Theorem 3 all functions f^α, $\alpha \in \mathcal{A}$ and f_0 be continuously differentiable with respect to x and the derivatives are Lipschitz with respect to x uniformly in $(t \times u)$ from $[0, t_f] \times U$. Suppose that V is a continuous differentiable function and that its derivative $\partial V(t, \cdot)/\partial t$ is continuous differentiable. Let $\Psi(\cdot)$ be the adjoint variable from Proposition 1 and $(u^{opt}(\cdot), X^{opt}(\cdot))$ be an optimal solution of (2). Then $-(\partial V(t, X^{opt}(t))/\partial x^\alpha)_{\mathcal{A}} = \Psi(t)$ for all $t \in [0, t_f]$.

The main motivation of the introducing DP is that one might be able to compute an optimal control strategy via the value function. Recall that the classic result which gives a way to construct an optimal control is called *verification theorem*. Let us now formulate the corresponding result for the multi-model OCP (2).

Theorem 5: Let all assumptions of Theorem 3 hold. Suppose that there exists a vector $\lambda^0 \in S_{\mathcal{A}}$ such that v (a verification function) is a continuously differentiable solution of (12). An admissible process $(u^{opt}(\cdot), X^{opt}(\cdot))$ is an optimal solution of (2) if and only if

$$\begin{aligned}
&\frac{\partial v_t(t, X^{opt}(t))}{\partial t} \\
&= \max_{u\in U} H\Big(t, X^{opt}(t), u, -\Big(\frac{\partial v(t, X^{opt}(t))}{\partial x^\alpha}\Big)_{\mathcal{A}}, \lambda^0\Big) \\
&= H\Big(t, X^{opt}(t), u^{opt}(t), -\Big(\frac{\partial v(t, X^{opt}(t))}{\partial x^\alpha}\Big)_{\mathcal{A}}, \lambda^0\Big)
\end{aligned}$$

for a.e. $t \in [0, t_f]$.

Finally, note that Theorem 5 is an immediate consequence of Theorem 3 and some standard techniques from the classic Bellman DP (see e.g., [3], [4], [15]).

IV. DYNAMIC PROGRAMMING APPROACH TO MULTI-MODEL LQ-TYPE PROBLEMS

In this section we apply the obtained theoretic results, namely, Theorem 3 and Theorem 5 to a multi-model LQ-problem. Let us consider the following special case of (2) (see [18], [19] for details)

$$\begin{aligned}
&\text{minimize } J(u(\cdot)) \text{ subject to} \\
&\dot{x}^\alpha(t) = A^\alpha(t)x(t) + B^\alpha(t)u(t) + d^\alpha(t),\ x(0) = x_0
\end{aligned} \tag{18}$$

where $d(t) \in \mathbb{R}^n$, $A^\alpha(t) \in \mathbb{R}^{n\times n}$, $B^\alpha(t) \in \mathbb{R}^{n\times m}$ for all $\alpha \in \mathcal{A}$, $t \in [0, t_f]$. The control region U in (18) coincides with the full space $\mathbb{R}^m$ and the admissible control functions are assumed to be square-integrable. We also assume that functions $A^\alpha(\cdot)$ and $B^\alpha(\cdot)$ are continuous and introduce the quadratic cost functional

$$h(u(\cdot), x^{\alpha,u}(\cdot)) := \frac{1}{2}\int_0^{t_f} x^\alpha(t)^T Q x^\alpha(t) + u(t)^T R u(t) dt$$

where Q is a symmetric positively semidefinite matrix and R is symmetric positively definite matrix. Note that the general LQ-problem of the Bolza type for a linear multi-model system can be reduced to the OCP (18) in the sense of the above Remark 1.

For (18) we can rewrite the HJB equation as follows

$$\begin{aligned}
-\frac{\partial v_t(t,X)}{\partial t} &= \max_{u\in U}\Big[\Big\langle \Big(\frac{\partial v(t,X)}{\partial x^\alpha}\Big)_{\mathcal{A}}, F(t,X,u)\Big\rangle \\
&+ \frac{1}{2}\sum_{\alpha\in\mathcal{A}} \lambda^0_\alpha\big(x^\alpha(t)^T Q x^\alpha(t)\big) + u(t)^T R u(t)\Big]
\end{aligned}$$

where $v(t_f, X) = 0$ and v is a (smooth) verification function. We now define the robust optimal control

$$u^{opt}(t) = -R^{-1}\sum_{\alpha\in\mathcal{A}} \lambda^0_\alpha B^\alpha(t)^T \frac{\partial v(t,X)}{\partial x^\alpha}$$

where $\partial v(t,X)/\partial x^{\alpha}$ is the n-dimensional α-component of the full $n \times |\mathcal{A}|$-dimensional vector $(\partial v(t,X)/\partial x^{\alpha})_{\mathcal{A}}$. Let

$$\mathbf{A}(t) := \mathrm{diag}\{A^{\alpha}(t)\}_{\mathcal{A}},\ \mathbf{B}(t) := \mathrm{diag}\{B^{\alpha}(t)\}_{\mathcal{A}},$$
$$\mathbf{Q} := \mathrm{diag}\{Q\}_{\mathcal{A}},\ \mathbf{R} := \mathrm{diag}\{R\}_{\mathcal{A}}, \mathbf{\Lambda}^{0} := \mathrm{diag}\{\lambda^{0}_{\alpha}\}_{\mathcal{A}}$$

be blocked diagonal matrices for all $t \in [0, t_f]$ and moreover, $\mathbf{d} := \mathrm{diag}(\{d^{\alpha}(t)\}_{\mathcal{A}})$. Replacing u by u^{opt} in the above equation, we obtain

$$\begin{aligned} &-\frac{\partial v_t(t,X)}{\partial t} = \Big(\frac{\partial v(t,X)}{\partial x^{\alpha}}\Big)^{T}_{\mathcal{A}} \mathbf{A}(t)\mathbf{x} + \frac{1}{2}\mathbf{\Lambda}\mathbf{x}^{T}\mathbf{Q}\mathbf{x} \\ &-\frac{1}{2}\Big(\frac{\partial v(t,X)}{\partial x^{\alpha}}\Big)^{T}_{\mathcal{A}} \mathbf{B}(t)^{T}\mathbf{R}^{-1}\mathbf{B}(t)^{T}\Big(\frac{\partial v(t,X)}{\partial x^{\alpha}}\Big)_{\mathcal{A}} \\ &+\Big(\frac{\partial v(t,X)}{\partial x^{\alpha}}\Big)_{\mathcal{A}}\mathbf{d}(t) \end{aligned} \tag{19}$$

where $\mathbf{x} := (x^{\alpha_1}, ..., x^{\alpha_{|\mathcal{A}|}})$, . This is the HJB equation for the LQ-type multi-model OCP (18). Following Dreyfus [13], we conclude that for a given vector λ^0 the solution to (19) as a quadratic function $v(t,x) = \frac{1}{2}\mathbf{x}^{T}\mathbf{P}_{\lambda^0}(t)\mathbf{x} + \mathbf{p}_{\lambda^0}(t)^{T}\mathbf{x}$, where $\mathbf{P}(t)$ is a symmetric positively definite matrix and $\mathbf{p}(t)$ is a "shifting" vector for all $t \in [0, t_f]$. Applying this verification function v to (19) we obtain the main theorem for the LQ-type multi-model OCP (18).

Theorem 6: The robust optimal feedback control for the multi-model LQ-problem (18) has the following linear form $u(x) = -\mathbf{R}^{-1}\mathbf{B}^{T}\big(\mathbf{P}_{\lambda^0}\mathbf{x} + \mathbf{p}_{\lambda^0}\big)$, where the (Riccati) matrix $\mathbf{P}_{\lambda^0}$ satisfies the parametric boundary value problem

$$\begin{aligned} &\dot{\mathbf{P}}_{\lambda^0} + \mathbf{P}_{\lambda^0}\mathbf{A} + \mathbf{A}^{T}\mathbf{P}_{\lambda^0} - \mathbf{P}_{\lambda^0}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{T}\mathbf{P}_{\lambda^0} + \\ &+ \mathbf{\Lambda}\mathbf{Q} = 0,\ \mathbf{P}_{\lambda^0}(t_f) = 0 \end{aligned} \tag{20}$$

Moreover, the shifting vector $\mathbf{p}_{\lambda^0}$ is also a solution of the boundary value problem

$$\begin{aligned} &\dot{\mathbf{p}}_{\lambda^0} + \mathbf{A}^{T}\mathbf{p}_{\lambda^0} - \mathbf{P}_{\lambda^0}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{T}\mathbf{P}_{\lambda^0} \\ &+ \mathbf{P}_{\lambda_0}\mathbf{d} = 0,\ \mathbf{p}_{\lambda^0}(t_f) = 0 \end{aligned} \tag{21}$$

Clearly, Theorem 6 is a variant of the verification Theorem 5 and coincides with the corresponding result from [18]. The parametric equations from (20) and (21) provide a basis for an effective numerical treatment of the LQ-type multi-model OCPs (18) (see [19] for details).

V. CONCLUSIONS

This paper deals with DP techniques and with a robust variant of the HJB equation for multi-model OCPs. We establish the relationship between the Robust MP and the obtained variant of the HJB equation for multi-model OCPs. In particular, for the LQ-type OCPs we deduce the Riccati-formalism and show that the results obtained using the robust HJB equation coincide with consequences of the applications of the Robust MP to the multi-model LQ-problems.

Finally, note that the main results presented in this paper are based on the assumption that the value function and the verification function are smooth. It is well known that this assumption does not necessarily hold and that the *viscosity solution* theory provides an excellent framework to deal with the above problem [2]. Evidently, a generalization of the viscosity concepts to the multi-model OCPs is a challenging problem which provides a new perspective in the optimal control of multi-model systems.

REFERENCES

[1] V.M. Alekseev, V.M. Tichomirov, and S.V. Fomin, *Optimal Control*, Plenum Publishing Co., New York, NY; 1987.

[2] M. Bardi and I. Cappuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equation, Birkhäuser, New York, NY; 2008.

[3] R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, NJ; 1957.

[4] R. Bellman and S. Dreyfus *Applied Dynamic Programming*, Princeton University Press, Princeton, NJ; 1962.

[5] L.D. Berkovitz, *Optimal Control Theory*, Springer, New York, NY; 1974.

[6] V. Boltyanskii, R. Gamkrelidze, E. Mishenko and L.S. Pontryagin, *The Mathematical Theory of Optimal Processes*, Interscience, New York, NY; 1962

[7] V.G. Boltyanski and A.S. Poznyak, Robust Maximum Principle in Minimax Control *International Journal of Control*, vol. 72, 1999, pp 305-314.

[8] V.G. Boltyanski and A.S. Poznyak, Robust Maximum Principle for a Measured Space as Uncertainty Set, *Dynamic Systems and Applications*, vol. 11, 2002, pp 277-292.

[9] V. Boltyanski and A. Poznyak, Linear Multi-Model Time - Optimization *Optimal Control Applications and Methods*, vol. 23, 2002, pp 141-161.

[10] L. Cesari, *Optimization Theory and Applications*, Springer, New York, NY; 1983.

[11] F.H. Clarke, The Relationship between the Maximum Principle and Dynamic Programming, *SIAM Journal on Control and Optimization*, vol. 25, 1987, pp 1291-1311.

[12] F.H. Clarke, Y.S. Ledyaev, R.J. Stern and P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York, NY; 1998.

[13] S. Dreyfus, Control problems with linear dynamics, quadratic criterion and linear terminal constraints. *IEEE Transactions on Automatic Control*, vol. 12, 1967, pp 323-324.

[14] H.O. Fattorini, *Infinite-Dimensional Optimization and Control Theory*, Cambridge University Press, Cambridge, UK; 1999.

[15] W.H. Fleming and R.W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, Berlin, Germany; 1975.

[16] M.R. Hestenes, *Calculus of Variations and Optimal control Theory*, John Wiley, New York, NY; 1966.

[17] A.D. Ioffe and V.M. Tihomirov, *Theory of Extremal Problems*, North Holland, Amsterdam, Holland; 1979.

[18] A.S. Poznyak, T.E. Duncan, B. Pasik-Duncan and V.G. Boltyanski, Robust Maximum Principle for Multi-Model LQ-Problem, *International Journal of Control*, vol. 75, 2002, pp 1170-1177.

[19] A. Poznyak, F.J. Bejarano and L. Fridman, Numerical Method for Weights Adjument in Minimax Multi-model LQ-Control, *Optimal Control Applications and Methods*, vol. 28, 2007, 289-300.

[20] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, NY; 1966.